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NON-TRANSITIVE POINTS AND POROSITY

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Abstract. We establish that for a fairly general class of topologically transitive dynamical systems, the set of non-transitive points is very small when the rate of transitivity is very high. The notion of smallness that we consider here is that of σ-porosity, and in particular we show that the set of non-transitive points is σ-porous for any subshift that is a factor of a transitive subshift of finite type, and for the tent map of $[0, 1]$. The result extends to some finite-to-one factor systems. We also show that for a family of piecewise monotonic transitive interval maps, the set of non-transitive points is σ-polynomially porous. We indicate how similar methods can be used to give sufficient conditions for the set of non-recurrent points and the set of distal pairs of a dynamical system to be very small.

1. Introduction. In a complete separable metric space, the commonly used notion of smallness for a subset is that of being first category. There is another stronger notion of smallness called *σ-porosity*: σ-porous subsets are always of first category, but by [ZA2] every complete metric space without isolated points contains nowhere dense closed subsets that are not σ-porous. The notion of σ-porosity was introduced by Dolzhenko [DOL] in 1967, and there has been a lot of work on it since then; see the survey articles of Zajíček [ZA1, ZA3]. One classical result about σ-porosity is that for a real-valued convex function defined on an open subset of a Banach space having a separable dual, the set of points of non-Fréchet differentiability is σ-porous (see Theorem 4.19 of [BEL]). In $\mathbb{R}^n$, every σ-porous set has Lebesgue measure zero as a consequence of the Lebesgue density theorem, but the converse is not true; see [ZA1, ZA3] for more information on σ-porous sets.

In this article, we examine how porosity and a more general notion that we name *polynomial porosity* appear in the study of three important notions from topological dynamics: transitivity, recurrence, and proximality. For us, a *dynamical system* is a pair (X, f), where X is a complete separable metric space and $f : X \to X$ is a continuous map. We give fairly general conditions

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under which the set of non-transitive points, the set of non-recurrent points, and the set of non-proximal pairs become σ-(polynomially) porous.

Our main results may be summarized as follows. We show that if the rate of transitivity in a topologically transitive dynamical system is exponential in a certain sense, then the set of non-transitive points is σ-(polynomially) porous (Theorem 3.2). Consequently, the set of non-transitive points is σ-porous for any subshift that is a factor of a transitive subshift of finite type, and for the tent map of $[0, 1]$ (Theorem 4.1). The result extends to some finite-to-one factor systems (Theorem 3.3). We also show that for a family of piecewise monotonic transitive maps of $[0, 1]$, the set of non-transitive points is σ-polynomially porous (Theorem 4.5). We indicate how similar methods can be used to give sufficient conditions for the set of non-recurrent points and the set of non-proximal pairs of a dynamical system to be σ-(polynomially) porous (Theorems 5.2, 5.4, 5.5).

2. Preliminaries. Let (X, d) be a complete separable metric space and $f : X \to X$ be continuous. In the dynamical system (X, f), the f*-orbit* of a point $x \in X$ is the set $O(f, x) := \{f^n(x) : n = 0, 1, 2, \ldots\}$. Let

$$\mathrm{Recu}(f) = \Big\{x \in X : \liminf_{n\to\infty} d(x, f^n(x)) = 0\Big\}$$

be the set of *recurrent points* of the map f. Let $\mathrm{Trans}(f)$ denote the set of *transitive points* of f, i.e., points $x \in X$ with $\overline{O(f, x)} = X$. For $U, V \subset X$, let

$$N_f(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}.$$

We say f is *transitive* if $N_f(U, V) \neq \emptyset$ for any two non-empty open sets $U, V \subset X$. When X has no isolated points, it is easy to see with a Baire category argument that f is transitive $\Leftrightarrow$ $\mathrm{Trans}(f) \neq \emptyset \Leftrightarrow \mathrm{Trans}(f)$ is a dense G_δ subset of X. We say f is *weakly mixing* if $f \times f$ is transitive, and *mixing* if $N_f(U, V)$ is cofinite for any two non-empty open sets $U, V \subset X$.

Let

$$\mathrm{Prox}(f) = \Big\{(x, y) \in X^2 : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\Big\}$$

be the collection of *proximal pairs* for f. Any $(x, y) \in X^2 \setminus \mathrm{Prox}(f)$ is called a *distal pair* for f. Observe that when X has no isolated points, we have $\mathrm{Trans}(f) \subset \mathrm{Recu}(f)$ and $\mathrm{Trans}(f\times f) \subset \mathrm{Prox}(f)$. A dynamical system (Y, g) is said to be a *factor* of (X, f) if there is a continuous surjection $h : X \to Y$ with $h \circ f = g \circ h$, and in this case h is called a *factor map*. The reader who wishes to have a quick introduction to Topological Dynamics may refer to the initial chapters of [BRS].

For $\lambda > 1$, we say $f : X \to X$ is λ*-Lipschitz* if $d(f(x), f(y)) \leq \lambda d(x, y)$ for every $x, y \in X$. For $\lambda > 1$ and $0 < \alpha \leq 1$, we say f is (λ, α)*-Hölder* *continuous* if $d(f(x), f(y)) \leq \lambda d(x, y)^\alpha$ for every $x, y \in X$. Note that a λ-Lipschitz map is same as a $(\lambda, 1)$-Hölder continuous map.

For $x \in X$ and $r > 0$ let $B(x, r) = \{y \in X : d(x, y) < r\}$, which we may also denote as $B_d(x, r)$ if it is necessary to emphasize the metric d. A subset $E \subset X$ is said to be *porous* if there is $0 < \theta < 1$ having the following property: for any $x \in E$ there is a sequence (x_k) in $X \setminus \{x\}$ with $\varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$ and $E \cap B_d(x_k, \theta\varepsilon_k) = \emptyset$ for every $k \in \mathbb{N}$. If this holds, we say E is porous of *type* θ. We remark that our definition of a porous set is stronger than what is usually given since the same θ works uniformly for all $x \in E$ in our definition whereas usually θ is allowed to vary with $x \in E$. We say $E \subset X$ is *σ-porous* if E can be written as $E = \bigcup_{n=1}^\infty E_n$, where each E_n is porous of type θ_n for some $\theta_n \in (0, 1)$.

We say a set $E \subset X$ is *polynomially porous* if there exist $0 < \theta < 1$ and a real number $\beta \geq 1$ with the property that for every $x \in E$ there is a sequence (x_k) in $X \setminus \{x\}$ with $\varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$ and $E \cap B_d(x_k, \theta\varepsilon_k^\beta) = \emptyset$ for every $k \in \mathbb{N}$. If this holds, we say E is polynomially porous of *type* θ and *degree* β. When $\beta = 1$, we recover the notion of porosity as a special case. For $\beta \geq 1$, we say $E \subset X$ is *σ-polynomially porous* of *degree* β if E can be written as $E = \bigcup_{n=1}^\infty E_n$, where each E_n is polynomially porous of type θ_n and degree β for some $\theta_n \in (0, 1)$. Note that if E is σ-polynomially porous of degree β, then E is σ-polynomially porous of degree β' for every $\beta' \geq \beta$.

Polynomial porosity (and in particular, porosity) is a metric-dependent notion, but it is preserved if we change the original metric to an equivalent one:

PROPOSITION 2.1. *Let (X, d) be a complete metric space and $E \subset X$ be a polynomially porous set of type θ and degree β. Let $\widetilde{d}$ be an equivalent metric on X in the following sense: there are constants $0 < C_1 \leq 1 \leq C_2$ such that $C_1 d(x, y) \leq \widetilde{d}(x, y) \leq C_2 d(x, y)$ for every $x, y \in X$. Then E is polynomially porous of type $\widetilde{\theta} := C_1 C_2^{-\beta}\theta$ and degree β with respect to $\widetilde{d}$.*

Proof. Let $x \in E$ and (x_k) be a sequence in $X \setminus \{x\}$ with $\varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$ and $E \cap B_d(x_k, \theta\varepsilon_k^\beta) = \emptyset$ for every $k \in \mathbb{N}$. Then $\delta_k := \widetilde{d}(x, x_k) \leq C_2\varepsilon_k \to 0$. Moreover for any $y_k \in B_{\widetilde{d}}(x_k, \widetilde{\theta}\delta_k^\beta)$, we have

$$d(x_k, y_k) \leq C_1^{-1}\widetilde{d}(x_k, y_k) < C_1^{-1}\widetilde{\theta}\delta_k^\beta \leq C^{-1}\widetilde{\theta}C_2^\beta\varepsilon_k^\beta = \theta\varepsilon_k^\beta$$

so that $B_{\widetilde{d}}(x_k, \widetilde{\theta}\delta_k^\beta) \subset B_d(x_k, \theta\varepsilon_k^\beta)$, and therefore $E \cap B_{\widetilde{d}}(x_k, \widetilde{\theta}\delta_k^\beta) = \emptyset$. ■

Clearly, polynomially porous sets are nowhere dense, and therefore σ-polynomially porous sets are of first category. The writing of this article is justified by the following proposition which says that the collection of σ-polynomially porous sets is strictly smaller than the collection of first category sets.

Proposition 2.2. *Let X be a complete metric space without isolated points. Then there is a nowhere dense closed set $Y \subset X$ such that it is not possible to write Y as a countable union of polynomially porous sets even if we are allowed to choose different types and different degrees.*

Proof. For the sake of this proof, let us define $E \subset X$ to be *exponentially porous* if for every $x \in E$ there is a sequence (x_k) in $X \setminus \{x\}$ with $\varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$ and $E \cap B(x_k, e^{-1/\varepsilon_k}) = \emptyset$ for every $k \in \mathbb{N}$. It is clear that for any $\theta \in (0,1)$ and $\beta \geq 1$, we have $B(x_k, e^{-1/\varepsilon_k}) \subset B(x_k, \theta\varepsilon_k^\beta)$ for all large $k \in \mathbb{N}$ if (ε_k) is a sequence of positive reals converging to 0. Therefore any polynomially porous set is exponentially porous. If we can write a subset $Y \subset X$ as $Y = \bigcup_{n=1}^\infty Y_n$, where Y_n is polynomially porous of type θ_n and degree β_n, then it follows that Y is σ-exponentially porous, i.e., a countable union of exponentially porous sets. On the other hand—and this is the real hard part of the proof—by (a special case of) Theorem 1 of [ZA2], every complete metric space without isolated points has a nowhere dense closed subset Y that is not σ-exponentially porous. ∎

Subshifts play a significant role in the theory of dynamical systems. A basic reference for the theory of subshifts is [LIM]. Let A be a finite discrete space (called an *alphabet*) with at least two elements in it. A metric d on $A^\mathbb{Z}$ inducing the product topology is given as follows: $d(x,x) = 0$, and $d(x,y) = 2^{-n}$ for $x \neq y$ if $n \geq 0$ is the smallest such that $x_n \neq y_n$ or $x_{-n} \neq y_{-n}$. This metric is an *ultrametric*, which means it satisfies the stronger version of triangle inequality that $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for every $x, y, z \in A^\mathbb{Z}$. The (*two-sided*) *shift map* $\psi : A^\mathbb{Z} \to A^\mathbb{Z}$ is given by $(\psi(x))_n = x_{n+1}$ for $x = (x_n) \in A^\mathbb{Z}$ and $n \in \mathbb{Z}$. The shift ψ is mixing. If $X \subset A^\mathbb{Z}$ is a non-empty, ψ-invariant closed subset, then the dynamical system (X, ψ) is called a (*two-sided*) *subshift*. Similarly one can define *one-sided subshifts* by considering $A^\mathbb{N}$ in place of $A^\mathbb{Z}$. A member w of A^n is called a *word* of length n over A, and we write $|w| = n$. Let $A^+ = \bigcup_{n=1}^\infty A^n$, the collection of all non-empty words over A. For a subshift (X, ψ), its *language* $L(X)$ is defined as

$$L(X) = \{w \in A^+ : w \text{ appears in some } x \in X\}.$$

Next we describe *subshifts of finite type* and their subshift factors called *sofic subshifts*. Elements of these subshifts can always be represented as infinite paths on some finite directed graph in the manner described below, after a suitable change of alphabet; see Chapters 2 and 3 of [LIM] and Section 3.7 of [BRS]. A change of alphabet results in changing the metric to an equivalent metric so that this is alright for our investigations about (polynomially) porous sets because of Proposition 2.1. For those readers who have some familiarity with subshifts, we remark that we are going to consider subshifts factors of subshifts of finite type specified by *forbidden words* of length 2. Let A, B be alphabets with $2 \le |B| \le |A|$, let $\pi : A \to B$ be a surjection, and let G be a directed graph with A as the vertex set. Denote the edge set of G by E_G, and we will write $ij \in E_G$ to mean there is an edge from vertex i to vertex j in the directed graph G, where $i, j \in A$. Let

$$X_G = \{x \in A^{\mathbb{Z}} : x_n x_{n+1} \in E_G \text{ for every } n \in \mathbb{Z}\},$$
$$Y_G^\pi = \{(\pi(x_n)) \in B^{\mathbb{Z}} : x \in X_G\}.$$

The subshift (X_G, ψ) is called a *subshift of finite type*, and the subshift (Y_G^π, ψ) is called a *sofic shift*. Note that $h : X_G \to Y_G$ given by $h(x) = (\pi(x_n))$ is a factor map from (X_G, ψ) to (Y_G^π, ψ).

The standard *tent map* $f : [0,1] \to [0,1]$ is defined as $f(x) = 2x$ for $0 \le x \le 1/2$ and $f(x) = 2-2x$ for $1/2 \le x \le 1$. The tent map is a prototype while studying transitivity and *chaos* of interval maps. More details about the dynamics of interval maps can be found in [BLC].

3. Exponential rate of transitivity, and porosity. Our basic idea is to show that in a transitive dynamical system (X, f), the set of non-transitive points is very small when *the rate of transitivity is exponential*, where the phrase in italics means roughly the following: given any $x \in X$ and any non-empty open set $V \subset X$, there exists a sequence (ε_k) of positive reals converging to 0 such that if $n_k \in \mathbb{N}$ is the smallest with $f^{n_k}(B(x, \varepsilon_k)) \cap V \neq \emptyset$, then e^{n_k} is not much larger than $1/\varepsilon_k$ (equivalently, n_k is not much larger than $\log(1/\varepsilon_k)$).

First we will formulate some abstract results giving sufficient conditions for the set of non-transitive points to be σ-(polynomially) porous. Applications to some familiar systems will be presented in the next section. We start with a lemma whose proof contains the key idea that generated this article.

LEMMA 3.1. *Let (X, f) be a dynamical system, and suppose f is λ-Lipschitz for some $\lambda > 1$. Let $V = B(z, r) \subset X$ be an open ball with $0 < r < 1$ and $P \subset X$ be an arbitrary subset. Assume that there are constants $\beta, C \ge 1$ so that for any $x \in P$ the following is satisfied:*

There is a sequence (x_k) in $X \setminus \{x\}$ with $1 > \varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$, and there is an increasing sequence (n_k) of natural numbers such that

$$f^{n_k}(x_k) \in B(z, r/2) \quad \textit{and} \quad n_k \le \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \textit{for every } k \in \mathbb{N}.$$

Then the set $E := \{x \in P : f^n(x) \notin V \text{ for every } n \in \mathbb{N}\}$ is polynomially porous of type $\theta := 2^{-1} C^{-\beta} r$ and degree β in X.

Proof. Fix $x \in E$, and let $(x_k), (n_k)$ be as in the hypothesis. Note that $\lambda^{n_k} = e^{n_k \log \lambda} \leq e^{\beta \log(C/\varepsilon_k)} = (C/\varepsilon_k)^\beta$. Also, $C^\beta \theta = r/2$. Therefore, for any $y_k \in B(x_k, \theta\varepsilon_k^\beta)$, we have

$$d(f^{n_k}(x_k), f^{n_k}(y_k)) \leq \lambda^{n_k} d(x_k, y_k) < \lambda^{n_k} \theta \varepsilon_k^\beta \leq (C/\varepsilon_k)^\beta \theta \varepsilon_k^\beta = r/2,$$

which implies $d(z, f^{n_k}(y_k)) \leq d(z, f^{n_k}(x_k)) + d(f^{n_k}(x_k), f^{n_k}(y_k)) < r/2 + r/2 = r$, and thus $f^{n_k}(y_k) \in V$. This shows that $E \cap B(x_k, \theta\varepsilon_k^\beta) = \emptyset$ for every $k \in \mathbb{N}$. ∎

By a *cover* of X we mean a collection of subsets (not necessarily open) whose union contains X. From Lemma 3.1 we deduce that:

THEOREM 3.2. *Let (X, f) be a dynamical system, and suppose f is λ-Lipschitz for some $\lambda > 1$. Assume that there exist a constant $\beta \geq 1$, a countable collection $\mathcal{P}$ of subsets of X covering X, and a countable base $\mathcal{V}$ of open balls with radius < 1 for the topology of X such that the following condition is satisfied:*

For each $P \in \mathcal{P}$ and $V = B(z, r) \in \mathcal{V}$, there is a constant $C = C_{P,V} \geq 1$ so that for any $x \in P$ there is a sequence (x_k) in $X \setminus \{x\}$ with $1 > \varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$, and there is an increasing sequence (n_k) of natural numbers such that

$$f^{n_k}(x_k) \in B(z, r/2) \quad \text{and} \quad n_k \leq \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \text{for every } k \in \mathbb{N}.$$

Then $X \setminus \mathrm{Trans}(f)$ is σ-polynomially porous of degree β in X. In particular, if $\beta = 1$, then $X \setminus \mathrm{Trans}(f)$ is σ-porous.

Proof. For $(P, V) \in \mathcal{P} \times \mathcal{V}$, the set $E_{P,V} := \{x \in P : f^n(x) \notin V \text{ for every } n \in \mathbb{N}\}$ is polynomially porous of type some $\theta_{P,V} \in (0, 1)$ and degree β in X by Lemma 3.1. And $X \setminus \mathrm{Trans}(f) = \bigcup_{(P,V) \in \mathcal{P} \times \mathcal{V}} E_{P,V}$, which is a countable union. ∎

Theorem 3.2 has the flexibility that we can transfer the conclusion to some special factor dynamical systems also:

THEOREM 3.3. *Let (X, f), (Y, g) be dynamical systems. Suppose that:*

(i) *f is λ_1-Lipschitz and g is λ_2-Lipschitz for some $\lambda_2 \geq \lambda_1 > 1$.*
(ii) *There is a factor map $h : (X, f) \to (Y, g)$ such that $h^{-1}(y)$ is a finite set for each $y \in Y$, and h is (λ, α)-Hölder continuous for some $\lambda > 1$ and $0 < \alpha \leq 1$.*
(iii) *(X, f) satisfies the hypothesis of Theorem 3.2 with constant $\beta_1 \geq 1$.*

Then $Y \setminus \mathrm{Trans}(g)$ is σ-polynomially porous of degree $\beta_2 := \frac{\beta_1 \log \lambda_2}{\alpha \log \lambda_1}$ in Y.

Proof. Let the constant $\beta_1 \geq 1$, cover $\mathcal{P}_1$ of X, and countable base $\mathcal{V}_1$ of X be as in the hypothesis of Theorem 3.2. Let $\mathcal{P}_2 = \{h(P) : P \in \mathcal{P}_1\}$, and $\mathcal{V}_2$ be any countable base of open balls of radius < 1 for Y. Fix $P_2 = h(P_1) \in \mathcal{P}_2$ and $V_2 = B(z_2, r_2) \in \mathcal{V}_2$. Let $V_1 = B(z_1, r_1) \in \mathcal{V}_1$ be such that $h(B(z_1, r_1/2)) \subset B(z_2, r_2/2)$. Let $C_2 = C_1\lambda$, where $C_1 = C_{P_1,V_1}$. We will now show that (Y, g) satisfies the hypothesis of Theorem 3.2 with constant $\beta_2 = (\beta_1 \log \lambda_2)/(\alpha \log \lambda_1)$.

Let d_1, d_2 be the metrics on X and Y respectively. Consider $y \in P_2$ and choose $x \in P_1$ with $h(x) = y$. Since (X, f) satisfies the hypothesis of Theorem 3.2, there is a sequence (x_k) in $X\backslash\{x\}$ with $1 > \varepsilon_k := d_1(x, x_k) \to 0$ as $k \to \infty$, and there is an increasing sequence (n_k) of natural numbers such that

$$f^{n_k}(x_k) \in B(z_1, r_1/2) \quad \text{and} \quad n_k \leq \frac{\beta_1 \log(C_1/\varepsilon_k)}{\log \lambda_1} \quad \text{for every } k \in \mathbb{N}.$$

Since $h^{-1}(y)$ is a finite set, without loss of generality we may assume $y_k := h(x_k) \neq y$ for every $k \in \mathbb{N}$. Observe that $\delta_k := d_2(y, y_k) \leq \lambda\varepsilon_k^\alpha \to 0$ as $k \to \infty$, and $g^{n_k}(y_k) = g^{n_k}(h(x_k)) = h(f^{n_k}(x_k)) \in h(B(z_1, r_1/2)) \subset B(z_2, r_2/2)$ for every $k \in \mathbb{N}$. Since $\delta_k \leq \lambda\varepsilon_k^\alpha$ and $C_1^\alpha \leq C_1$, we have $(C_1/\varepsilon_k)^\alpha \leq C_1\lambda/\delta_k = C_2/\delta_k$ and thus $\log(C_1/\varepsilon_k) \leq \alpha^{-1} \log(C_2/\delta_k)$. Also recall that $\beta_2 = (\beta_1 \log \lambda_2)/(\alpha \log \lambda_1)$. Therefore,

$$n_k \leq \frac{\beta_1 \log(C_1/\varepsilon_k)}{\log \lambda_1} \leq \frac{\beta_1 \log(C_2/\delta_k)}{\alpha \log \lambda_1} = \frac{\beta_2 \log(C_2/\delta_k)}{\log \lambda_2}.$$

Thus (Y, g) satisfies the hypothesis of Theorem 3.2 with constant β_2, and we are through. ∎

4. Applications. First we apply Theorem 3.2 to subshift factors of transitive subshifts of finite type, and to the tent map. For the subshifts mentioned in Theorem 4.1 below, the metric under consideration is the ultrametric defined earlier.

THEOREM 4.1. *For the following dynamical systems, the set of non-transitive points is a σ-porous subset of the domain of the dynamical system:*

(i) *Any subshift that is a factor of a transitive subshift of finite type.*
(ii) *The tent map of $[0, 1]$ when $[0, 1]$ is equipped with the Euclidean metric.*
(iii) *The map $z \mapsto z^k$ of the unit circle for any integer $k \geq 2$ when the unit circle is equipped with either the arc-length metric or the Euclidean metric.*

Proof. In all the three cases, we will show that the system satisfies the hypothesis of Theorem 3.2 with the constant $\beta = 1$.

(i) We will work with two-sided subshifts. The proof for one-sided subshifts is similar. Following our description in Section 2 involving a finite directed graph G, we may assume the following:

(1) There are alphabets A, B with $2 \leq |B| \leq A$, and there is a surjection $\pi : A \to B$.
(2) The given sofic subshift (Y_G^π, ψ) is a factor of a transitive subshift of finite type (X_G, ψ) via the factor map $h : X_G \to Y_G^\pi$ given by $h(x) = (\pi(x_n))_{n \in \mathbb{Z}}$.
(3) Here, $X_G = \{x \in A^{\mathbb{Z}} : x_n x_{n+1} \in E_G \text{ for every } n \in \mathbb{Z}\}$.

From now onwards, write X for X_G and Y for Y_G^π. Without loss of generality we assume each letter of A appears in some $x \in X$. If Y has an isolated point, (Y, ψ) reduces to a periodic orbit, and there is nothing to prove. So we assume Y (and similarly X) is infinite with no isolated points. Let $L_0(X) = \{w \in L(X) : |w| \text{ is odd}\}$, and for $w \in L_0(X)$, let $U_w = \{x \in X : x_{[-s,s]} = w\}$, where $2s + 1 = |w|$ and $x_{[-s,\, s]} = x_{-s} \cdots x_0 \cdots x_s$. Note that $\{U_w : w \in L_0(X)\}$ is a countable base for X. Since our plan is to arrange things so that Theorem 3.2 becomes applicable to (Y, ψ), we will now try to use the notations from Theorem 3.2.

Let $\mathcal{P} = \{Y\}$ and $\mathcal{V}$ be any countable base of open balls of radius < 1 for Y. Fix an open ball $V = B(z, r) \in \mathcal{V}$ and choose a word $w \in L_0(X)$ such that $h(U_w) \subset B(z, r/2)$. Let $s \in \mathbb{N}$ with $|w| = 2s + 1$. Since (X, ψ) is transitive, there is $t \in \mathbb{N}$ such that for every $a, a' \in A$ there is a word $u \in L(X)$ of length $\leq t$ such that u starts with a and ends with a'. We will show that the hypothesis of Theorem 3.2 is satisfied for (Y, ψ) with constants $\beta = 1$ and $C = C_{Y,V} = 2^{t+s}$.

Consider $y \in Y$ and let $x \in X$ with $h(x) = y$. By the choice of t, for each $k \in \mathbb{N}$ there exist $k < j_k \leq k + t$ and $x(k) \in X$ such that $x(k)_{[-k,\, k]} = x_{[-k,\, k]}$ and $x(k)_{[j_k,\, j_k+2s]} = w$. We claim that the choice of $x(k)$ can be made in such a way that we also have $h(x(k)) \neq y$ for every $k \in \mathbb{N}$. We argue as follows. Observe that since there are conditions only on finitely many coordinates of $x(k)$, we may choose $x(k)$ from any dense subset of X. If y is a transitive point for (Y, ψ), we may assume each $x(k)$ is a periodic point in (X, ψ) since it is well-known and easy to show that a transitive subshift of finite type has a dense set of periodic points. Then $h(x(k)) \neq y$, for otherwise y will become a periodic point, forcing (Y, ψ) to be a finite periodic orbit, contradicting the earlier assumption that Y has no isolated points. Next, if y is a non-transitive point, then we may assume each $x(k)$ is a transitive point in (X, ψ), which will ensure $h(x(k)) \neq y$. This proves our claim.

Let $y(k) = h(x(k))$ for $k \in \mathbb{N}$. Since $y(k)_{[-k,\, k]} = y_{[-k,k]}$, we get $\varepsilon_k := d(y, y(k)) < 2^{-k} \to 0$ as $k \to \infty$. Let $n_k = j_k + s$. Then

$$\psi^{n_k}(y(k)) = h(\psi^{n_k}(x(k))) \in h(U_w) \subset B(z, r/2).$$

Since $\varepsilon_k < 2^{-k}$, we obtain $k < \log(1/\varepsilon_k)/\log 2$, and therefore

$$n_k = j_k + s \leq k + t + s < \frac{\log(1/\varepsilon_k)}{\log 2} + t + s = \frac{\log(C/\varepsilon_k)}{\log 2}$$

by our choice of C. Finally note that the shift map ψ is 2-Lipschitz with respect to the ultrametric d. Thus (Y, ψ) satisfies the hypothesis of Theorem 3.2 with constant $\beta = 1$, and therefore $Y \setminus \mathrm{Trans}(\psi)$ is σ-porous in Y.

(ii) Let $\mathcal{P} = \{[0,1]\}$, and $\mathcal{V}$ be any countable base of open balls of radius < 1 for $[0,1]$. Consider $V = B(z,r) \in \mathcal{V}$. We will show that the hypothesis of Theorem 3.2 is satisfied with constants $\beta = 1$ and $C = C_{[0,1],V} = 1$. Consider $x \in [0,1]$. Let $I(k,j) = [(j-1)/2^k, j/2^k]$ for $1 \leq j \leq 2^k$ and $k \in \mathbb{N}$. The tent map has the property that $f^k(I(k,j)) = [0,1]$ for $1 \leq j \leq 2^k$ and $k \in \mathbb{N}$. Hence we can find a sequence (x_k) in $[0,1]\setminus\{x\}$ such that $\varepsilon_k := |x - x_k| \leq 2^{-k}$ and $f^k(x_k) \in B(z, r/2)$ for every $k \in \mathbb{N}$. We have $k \leq \log(1/\varepsilon_k)/\log 2$. Since f is 2-Lipschitz, we are done by Theorem 3.2.

(iii) is similar to the case of the tent map, and is left to the reader. ■

Remark 4.2. Another commonly used metric $\widetilde{d}$ on $A^{\mathbb{Z}}$ is given by $\widetilde{d}(x,y) := \sum_{n\in\mathbb{Z}} 2^{-|n|}\rho(x_n, y_n)$, where ρ is the discrete metric on A. We have $d(x,y) \leq \widetilde{d}(x,y) \leq 4d(x,y)$, where d is the ultrametric. Hence by Proposition 2.1, the assertion about sofic subshifts in Theorem 4.1 remains true with respect to the metric $\widetilde{d}$ also.

Remark 4.3. There is a class of well-studied dynamical systems called *topologically Anosov* (*TA*) homeomorphisms. By Theorem 4.3.6 of [AOH] (first proved by Bowen), any mixing TA homeomorphism is a finite-to-one factor of a two-sided mixing subshift of finite type. If $g : Y \to Y$ is a mixing TA homeomorphism, and if the hypothesis of Theorem 3.3 is satisfied with the corresponding mixing subshift of finite type in place of (X, f), then we can conclude that $Y \setminus \mathrm{Trans}(g)$ is σ-(polynomially) porous in Y.

Remark 4.4. Let $f : [0,1] \to [0,1]$ be the tent map. Since $E := [0,1] \setminus \mathrm{Trans}(f)$ is an uncountable F_σ set, we can find a Cantor set $K \subset (0,1) \cap E$. By a re-scaling of the open intervals in $[0,1] \setminus K$, we can find a homeomorphism $h : [0,1] \to [0,1]$ such that the Cantor set $h(K)$ has positive Lebesgue measure. Let $g : [0,1] \to [0,1]$ be $g = h \circ f \circ h^{-1}$. Then g is *conjugate* to the tent map f, but $[0,1] \setminus \mathrm{Trans}(g)$ is not σ-porous since it has a subset of positive Lebesgue measure, namely $h(K)$.

As another application of Theorem 3.2, we will show that for a family of piecewise monotonic transitive maps on $[0,1]$, the set of non-transitive points is σ-polynomially porous. This generalizes the case of the tent map with a weaker conclusion, and we do not know whether the conclusion can be improved to σ-porosity in general.

THEOREM 4.5. *Let $f : [0,1] \to [0,1]$ be λ-Lipschitz for some $\lambda > 1$. Assume that f satisfies the following two conditions:*

(i) *There exist $\eta \in (1, \lambda]$ and a partition $0 = a_0 < a_1 < \cdots < a_{q-1} < a_q = 1$ of $[0,1]$ for some integer $q \geq 2$ with the property that f is differentiable on $\bigcup_{j=1}^{q} U_j$, where $U_j := (a_{j-1}, a_j)$, and $|f'(x)| \geq 2\eta$ for every $x \in \bigcup_{j=1}^{q} U_j$.*

(ii) *For every $i, j \in \{1, \ldots, q\}$, there is $n \in \mathbb{N}$ such that $U_j \subset f^n(U_i)$.*

Then $[0,1] \setminus \mathrm{Trans}(f)$ is σ-polynomially porous of degree $\beta := \log\lambda/\log\eta$ in $[0,1]$.

Proof. Let $t \in \mathbb{N}$ be such that for every $i, j \in \{1, \ldots, q\}$ there is $n \in \{1, \ldots, t\}$ with $U_j \subset f^n(U_i)$. Let $\mathcal{P} = \{[0,1]\}$, and $\mathcal{V}$ be any countable base of open balls of radius < 1 for $[0,1]$. Consider $V = B(z, r) \in \mathcal{V}$. We will show that the hypothesis of Theorem 3.2 is satisfied with constants $\beta = \log\lambda/\log\eta$ and $C = C_{[0,1],V} = \eta^t$. Let $x \in [0,1]$. We denote by $|L|$ the length of a subinterval $L \subset [0,1]$. Let $\delta = \min\{\eta^{-1}, |U_1|, \ldots, |U_q|\}$, and let (δ_k) be a sequence in $(0, \delta)$ converging to 0. Fix $k \in \mathbb{N}$, and choose $i_0 \in \{1, \ldots, q\}$ such that the interval $L_0 := B(x, \delta_k) \cap U_{i_0}$ has length $\geq \delta_k$.

We claim that there exist a natural number $m_k \leq \log(1/\delta_k)/\log\eta$ and $i \in \{1, \ldots, q\}$ such that $U_i \subset f^{m_k}(L_0) \subset f^{m_k}(B(x, \delta_k))$. Indeed, observe that $f(L_0)$ is an interval of length $\geq 2\eta|L_0|$ by property (i). If there is $i \in \{1, \ldots, q\}$ with $U_i \subset f(L_0)$, we stop the process. Otherwise, we note that the interval $f(L_0)$ can intersect at most two adjacent U_i's, and therefore there is $i_1 \in \{1, \ldots, q\}$ such that the interval $L_1 := f(L_0) \cap U_{i_1}$ satisfies

$$|L_1| \geq |f(L_0)|/2 \geq \eta|L_0| \geq \eta\delta_k.$$

Now we repeat the argument with L_1 in the place of L_0. If the process does not stop with $m-1$ steps, then at the mth step we will obtain an interval $L_m \subset f^m(L_0)$ with $|L_m| \geq \eta^m\delta_k$. Here we must have $\eta^m\delta_k \leq 1$ since $L_m \subset [0,1]$. This implies that the claim must be true.

Let $j \in \{1, \ldots, q\}$ be with $U_j \cap B(z, r/2) \neq \emptyset$, and $s_k \in \{1, \ldots, t\}$ be with $U_j \subset f^{s_k}(U_i)$. Then we have $U_j \cap B(z, r/2) \subset f^{m_k+s_k}(B(x, \delta_k))$. Pick $x_k \in B(x, \delta_k) \setminus \{x\}$ with $f^{m_k+s_k}(x_k) \in B(z, r/2)$. Put $n_k = m_k + s_k$ and $\varepsilon_k = |x - x_k| < \delta_k$. By assuming that (δ_k) converges to 0 sufficiently fast, we may ensure that (n_k) is an increasing sequence. By what is proved in the previous paragraph, $m_k \leq \log(1/\varepsilon_k)/\log\eta$. Also recall that $\beta = \log\lambda/\log\eta$, and $C = \eta^t$ so that $t = \log C/\log\eta$. Therefore,

$$n_k = m_k + s_k \leq m_k + t \leq \frac{\log(1/\varepsilon_k)}{\log\eta} + \frac{\log C}{\log\eta} = \frac{\log(C/\varepsilon_k)}{\log\eta} = \frac{\beta\log(C/\varepsilon_k)}{\log\lambda}.$$

Thus the hypothesis of Theorem 3.2 is satisfied, and so $[0,1] \setminus \mathrm{Trans}(f)$ is σ-polynomially porous of degree $\beta = \log\lambda/\log\eta$ in $[0,1]$. ■

Examples of maps satisfying the hypothesis of Theorem 4.5 are easy to construct, and we give one such example:

EXAMPLE 4.6. Let $a_0 = 0$, $a_1 = 2/7$, $a_2 = 5/7$, $a_3 = 1$, and $U_j = (a_{j-1}, a_j)$ for $1 \leq j \leq 3$. Let $f : [0,1] \to [0,1]$ be the continuous *piecewise linear* map defined by the conditions that $f(a_0) = 1/7$, $f(a_1) = 1$, $f(a_2) = 0$, $f(a_3) = 6/7$, and the graph of f is linear on $\overline{U_j}$ for $1 \leq j \leq 3$. We have $|f'| = 3$ on $U_1 \cup U_3$ and $|f'| = 7/3$ on U_2. Hence f is 3-Lipschitz; and $|f'(x)| \geq 2\eta$ for every $x \in \bigcup_{j=1}^{3} U_j$ if we put $\eta = 7/6$. Moreover, $U_i \subset f^2(U_j)$ for every $i, j \in \{1,2,3\}$. Hence by Theorem 4.5, the set of non-transitive points of f is σ-polynomially porous of degree $\log 3/\log(7/6)$ in $[0,1]$.

5. Non-recurrent points, non-proximal pairs, and porosity. The methods that we used to analyze the set of non-transitive points can also be used to analyze the set of non-recurrent points and the set of non-proximal pairs (i.e., distal pairs). The similarity between the theories of recurrence and proximality was exploited by the author in an earlier work [TKS] also. The material of this section may look a little repetitive, but we choose to provide most of the details for the benefit of the reader since various constants and estimates are involved. The idea is to imitate Lemma 3.1 and then deduce an analogue of Theorem 3.2. We state the results for σ-polynomially porous sets, but observe that when the degree β is equal to 1, we get the case of σ-porous sets as a special case. First we look at the set of non-recurrent points.

LEMMA 5.1. *Let (X, f) be a dynamical system, and suppose that f is λ-Lipschitz for some $\lambda > 1$. Let $0 < r \leq 1$, and $P \subset X$ be an arbitrary subset. Assume that there are constants $\beta, C \geq 1$ such that for any $x \in P$ the following is satisfied:*

There is a sequence (x_k) in $X \setminus \{x\}$ with $1 > \varepsilon_k := d(x, x_k) \to 0$ as $k \to \infty$, and there is an increasing sequence (n_k) of natural numbers such that

$$d(x_k, f^{n_k}(x_k)) < r/3 \quad \text{and} \quad n_k \leq \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \text{for every } k \in \mathbb{N}.$$

Then the set $E := \{x \in P : d(x, f^n(x)) \geq r \text{ for every } n \in \mathbb{N}\}$ is polynomially porous of type $\theta := 3^{-1}C^{-\beta}r$ and degree β in X.

Proof. Let $x \in E$, and (x_k), (n_k) be as given. For any $y_k \in B(x_k, \theta\varepsilon_k^\beta)$, we have $d(f^{n_k}(x_k), f^{n_k}(y_k)) < \lambda^{n_k}\theta\varepsilon_k^\beta < r/3$ as in the proof of Lemma 3.1. Moreover, $d(x_k, y_k) < \theta\varepsilon_k^\beta < \theta < r/3$. Hence

$$\begin{aligned} d(y_k, f^{n_k}(y_k)) &\leq d(y_k, x_k) + d(x_k, f^{n_k}(x_k)) + d(f^{n_k}(x_k), f^{n_k}(y_k)) \\ &< r/3 + r/3 + r/3 = r, \end{aligned}$$

which implies $E \cap B(x_k, \theta\varepsilon_k^\beta) = \emptyset$ for every $k \in \mathbb{N}$. ∎

Theorem 5.2. *Let* (X, f) *be a dynamical system, and suppose that* f *is* λ*-Lipschitz for some* $\lambda > 1$. *Assume that there exist a constant* $\beta \geq 1$, *a countable collection* $\mathcal{P}$ *of subsets of* X *covering* X, *and a countable base* $\mathcal{V}$ *of open balls with radius* < 1 *for the topology of* X *such that the following condition is satisfied:*

For each $P \in \mathcal{P}$ *and* $r > 0$, *there is a constant* $C = C_{P,r} \geq 1$ *so that for any* $x \in P$ *there is a sequence* (x_k) *in* $X \setminus \{x\}$ *with* $1 > \varepsilon_k := d(x, x_k) \to 0$ *as* $k \to \infty$, *and there is an increasing sequence* (n_k) *of natural numbers such that*

$$d(x_k, f^{n_k}(x_k)) < r/3 \quad \text{and} \quad n_k \leq \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \text{for every } k \in \mathbb{N}.$$

Then $X \setminus \mathrm{Recu}(f)$ *is* σ*-polynomially porous of degree* β *in* X. *In particular, if* $\beta = 1$, *then* $X \setminus \mathrm{Recu}(f)$ *is* σ*-porous.*

Proof. For $P \in \mathcal{P}$ and $k \in \mathbb{N}$, let $E_{P,k} = \{x \in P : d(x, f^n(x)) \geq 1/k$ for every $n \in \mathbb{N}\}$. Then each $E_{P,k}$ is polynomially porous of degree β by Lemma 5.1, and $X \setminus \mathrm{Recu}(f) = \bigcup_{(P,k)\in\mathcal{P}\times\mathbb{N}} E_{P,k}$. ■

Next we look at the set of non-proximal pairs (i.e., distal pairs) in X^2 for a dynamical system (X, f). If (X, d) is a complete separable metric space, we equip X^2 with the metric d' given by $d'((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$.

Lemma 5.3. *Let* (X, f) *be a dynamical system, and suppose* f *is* λ*-Lipschitz for some* $\lambda > 1$. *Let* $0 < r \leq 1$, *and* $P \subset X^2$ *be an arbitrary subset. Assume that there are constants* $\beta, C \geq 1$ *such that for any* $(x, y) \in P$ *the following is satisfied:*

There is a sequence (x_k, y_k) *in* $X^2 \setminus \{(x, y)\}$ *with* $1 > \varepsilon_k := d(x, x_k) + d(y, y_k) \to 0$ *as* $k \to \infty$, *and there is an increasing sequence* (n_k) *of natural numbers such that*

$$d(f^{n_k}(x_k), f^{n_k}(y_k)) < r/3 \quad \text{and} \quad n_k \leq \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \text{for every } k \in \mathbb{N}.$$

Then the set $E := \{(x, y) \in P : d(f^n(x), f^n(y)) \geq r$ *for every* $n \in \mathbb{N}\}$ *is polynomially porous of type* $\theta := 3^{-1}C^{-\beta}r$ *and degree* β *in* X^2.

Proof. Let $(x, y) \in E$, and $((x_k, y_k))$, (n_k) be as given. Consider $(a_k, b_k) \in B((x_k, y_k), \theta\varepsilon_k^{\beta}) \subset X^2$. We have $d(f^{n_k}(a_k), f^{n_k}(x_k)) < \lambda^{n_k}\theta\varepsilon_k^{\beta} < r/3$ as in the proof of Lemma 3.1, and similarly $d(f^{n_k}(b_k), f^{n_k}(y_k)) < r/3$. Moreover, $d(f^{n_k}(x_k), f^{n_k}(y_k)) < r/3$ by hypothesis. Hence

$$\begin{aligned} &d(f^{n_k}(a_k), f^{n_k}(b_k)) \\ &\quad \leq d(f^{n_k}(a_k), f^{n_k}(x_k)) + d(f^{n_k}(x_k), f^{n_k}(y_k)) + d(f^{n_k}(y_k), f^{n_k}(b_k)) < r, \end{aligned}$$

which implies $E \cap B((x_k, y_k), \theta\varepsilon_k^{\beta}) = \emptyset$ for every $k \in \mathbb{N}$. ■

Theorem 5.4. *Let (X, f) be a dynamical system, and suppose f is λ-Lipschitz for some $\lambda > 1$. Assume that there exist a constant $\beta \geq 1$ and a countable collection $\mathcal{P}$ of subsets of X^2 covering X^2 such that the following condition is satisfied:*

For each $P \in \mathcal{P}$ and $r > 0$, there is a constant $C = C_{P,r} \geq 1$ so that for any $(x, y) \in P$ there is a sequence $((x_k, y_k))$ in $X^2 \setminus \{(x, y)\}$ with $1 > \varepsilon_k := d(x, x_k) + d(y, y_k) \to 0$ as $k \to \infty$, and there is an increasing sequence (n_k) of natural numbers such that

$$d(f^{n_k}(x_k), f^{n_k}(y_k)) < r/3 \quad \text{and} \quad n_k \leq \frac{\beta \log(C/\varepsilon_k)}{\log \lambda} \quad \text{for every } k \in \mathbb{N}.$$

Then $X^2 \setminus \mathrm{Prox}(f)$ is σ-polynomially porous of degree β in X^2. In particular, if $\beta = 1$, then $X^2 \setminus \mathrm{Prox}(f)$ is σ-porous.

Proof. For $P \in \mathcal{P}$ and $k \in \mathbb{N}$, let $E_{P,k} = \{(x, y) \in P : d(f^n(x), f^n(y)) \geq 1/k \text{ for every } n \in \mathbb{N}\}$. Then each $E_{P,k}$ is polynomially porous of degree β in X^2 by Lemma 5.3, and $X^2 \setminus \mathrm{Prox}(f) = \bigcup_{(P,k) \in \mathcal{P} \times \mathbb{N}} E_{P,k}$. ∎

We may also transfer the conclusions of Theorems 5.2 and 5.4 to some special factor systems:

Theorem 5.5. *Let $(X, f), (Y, g)$ be dynamical systems. Suppose that:*

(i) *f is λ_1-Lipschitz and g is λ_2-Lipschitz for some $\lambda_2 \geq \lambda_1 > 1$.*
(ii) *There is a factor map $h : (X, f) \to (Y, g)$ such that $h^{-1}(y)$ is a finite set for each $y \in Y$, and h is (λ, α)-Hölder continuous for some $\lambda > 1$ and $0 < \alpha \leq 1$.*

Then we have the following:

(a) *If (X, f) satisfies the hypothesis of Theorem 5.2 with constant $\beta_1 \geq 1$, then $Y \setminus \mathrm{Recu}(g)$ is σ-polynomially porous of degree $\beta_2 := (\beta_1 \log \lambda_2)/(\alpha \log \lambda_1)$ in Y.*
(b) *If (X, f) satisfies the hypothesis of Theorem 5.4 with constant $\beta_1 \geq 1$, then $Y^2 \setminus \mathrm{Prox}(g)$ is σ-polynomially porous of degree $\beta_2 := (\beta_1 \log \lambda_2)/(\alpha \log \lambda_1)$ in Y^2.*

Proof. Just imitate the proof of Theorem 3.3. ∎

6. Directions for further research. Till now we have been discussing the notion of porosity for Lipschitz maps. Since Hölder continuity is a generalization of Lipschitz continuity, the next natural step in research can be to study whether the results given above can be extended to the case of Hölder continuous maps. Though we could not do this extension step, we provide some remarks. First we observe the following:

Proposition 6.1. *Let (X, f) be a dynamical system, and suppose f is (λ, α)-Hölder continuous for some $\lambda > 1$ and $0 < \alpha \leq 1$.*

(i) *Let $s_n = \sum_{j=0}^{n-1} \alpha^j$ for $n \in \mathbb{N}$. Then f^n is $(\lambda^{s_n}, \alpha^n)$-Hölder continuous for every $n \in \mathbb{N}$.*
(ii) *Assume further that $\alpha < 1$. Then for every $x, y \in X$ with $d(x, y) \leq 1$, we have $d(f^n(x), f^n(y)) \leq \lambda^{1/(1-\alpha)}$ for every $n \in \mathbb{N}$.*

Proof. (i) This is an easy verification by induction on n, where the initial step is covered by the definition of Hölder continuity. Assuming the nth step, we obtain

$$\begin{aligned} d(f^{n+1}(x), f^{n+1}(y)) &\leq \lambda d(f^n(x), f^n(y))^\alpha \\ &\leq \lambda(\lambda^{s_n} d(x, y)^{\alpha^n})^\alpha = \lambda^{s_{n+1}} d(x, y)^{\alpha^{n+1}} \end{aligned}$$

for every $x, y \in X$ since $1 + s_n\alpha = s_{n+1}$.

(ii) This is a direct consequence of (i) since $d(x, y)^{\alpha^n} \leq 1$ and $\sum_{j=0}^{\infty} \alpha^j = 1/(1 - \alpha)$. ∎

The above observation already implies that there are some restrictions for a Hölder continuous, non-Lipschitz map to be transitive when the metric is unbounded:

Corollary 6.2. *Let (X, f) be a dynamical system, and suppose f is (λ, α)-Hölder continuous for some $\lambda > 1$ and $0 < \alpha < 1$ (note that we have excluded the case $\alpha = 1$). Also assume that the metric d on X under consideration is unbounded. Then:*

(i) *f cannot be weakly mixing.*
(ii) *If f has a bounded orbit, then f cannot be transitive.*

Proof. (i) If f is weakly mixing, there must exist $(x, y) \in \mathrm{Trans}(f \times f)$ with $d(x, y) \leq 1$. Since d is unbounded and $(x, y) \in \mathrm{Trans}(f \times f)$ we should have $\sup_{n\in\mathbb{N}} d(f^n(x), f^n(y)) = \infty$. This contradicts Proposition 6.1(ii).

(ii) Let $y \in X$ be such that the orbit $O(f, y)$ is bounded. If f is transitive, there must exist $x \in \mathrm{Trans}(f)$ with $d(x, y) \leq 1$. Since d is unbounded and $x \in \mathrm{Trans}(f)$, we should have

$$\sup_{n\in\mathbb{N}} d(f^n(x), f^n(y)) \geq \sup_{n\in\mathbb{N}} \mathrm{dist}(f^n(x), \overline{O(f, y)} = \infty,$$

contradicting Proposition 6.1(ii). ∎

In contrast, spaces with unbounded metric can support weakly mixing Lipschitz maps. In fact, every infinite-dimensional separable Banach space admits bounded linear operators that are mixing [ANS, BER]. Moreover, Hölder continuous, non-Lipschitz maps can be mixing on compact metric spaces. We give one such example:

EXAMPLE 6.3. Let $U_1 = (0, 1/9)$, $U_2 = (1/9, 5/9)$ and $U_3 = (5/9, 1)$. Let $f : [0,1] \to [0,1]$ be the continuous map defined by the following conditions: $f(0) = f(5/9) = 0$, $f(1/9) = f(1) = 1$, $f(x) = 3\sqrt{x}$ for $x \in U_1$ and the graph of f is linear on each of U_2, U_3. For $x \in U_1$, we have $|f'(x)| \geq 9/2$ since $f'(x) = 3/(2\sqrt{x})$. For $x \in U_2 \cup U_3$, we have $|f'(x)| = 9/4$. Therefore, by imitating the proof of Theorem 4.5 it can be shown that for any non-degenerate subinterval $L \subset [0,1]$, there is $n \in \mathbb{N}$ with $f^n(L) = [0,1]$. This implies f is mixing. Since f is $(3, 1/2)$-Hölder continuous on $[0, 1/9]$ and $(9/4)$-Lipschitz on $[1/9, 1]$, it follows that f is $(3, 1/2)$-Hölder continuous on $[0,1]$. But f is not Lipschitz on $[0,1]$ since $x \mapsto \sqrt{x}$ is not Lipschitz on $[0,a]$ for any $a > 0$.

Many examples similar to the one given above can easily be constructed. Thus it is worth investigating sufficient conditions for the set of non-transitive points of a Hölder continuous map to be σ-(polynomially) porous. As noted in Proposition 6.1(i), if f is (λ, α)-Hölder continuous and $s_n = \sum_{j=0}^{n-1} \alpha^j$, then $d(f^n(x), f^n(y)) \leq \lambda^{s_n} d(x,y)^{\alpha^n}$. The presence of the power α^n in this inequality creates obstructions to imitating the estimate technique from the proof of Lemma 3.1. We invite the reader to come up with new techniques to analyze the Hölder continuous case.

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