COLLOQUIUM MATHEMATICUM

VOL. 133 2013 NO. 1

ON THE CAUCHY PROBLEM FOR CONVOLUTION EQUATIONS

BY

JAN KISYŃSKI (Lublin)

Abstract. We consider one-parameter (C_0)-semigroups of operators in the space $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ with infinitesimal generator of the form $(G\,*)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$ where G is an $M_{m\times m}$-valued rapidly decreasing distribution on $\mathbb{R}^n$. It is proved that the Petrovskiĭ condition for forward evolution ensures not only the existence and uniqueness of the above semigroup but also its nice behaviour after restriction to whichever of the function spaces $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)$, $p\in[1,\infty]$, $(\mathcal{O}_a)(\mathbb{R}^n;\mathbb{C}^m)$, $a\in\,]0,\infty[$, or the spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n;\mathbb{C}^m)$, $q\in\,]1,\infty]$, of bounded distributions.

1. Preliminaries. We shall use the spaces

- $\mathcal{S}(\mathbb{R}^n)$ of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$,
- $\mathcal{S}'(\mathbb{R}^n)$ of slowly increasing (or tempered) distributions on $\mathbb{R}^n$,
- $\mathcal{O}_M(\mathbb{R}^n)$ of infinitely differentiable slowly increasing functions on $\mathbb{R}^n$,
- $\mathcal{O}'_C(\mathbb{R}^n)$ of rapidly decreasing distributions on $\mathbb{R}^n$.

A function f belongs to $\mathcal{O}_M(\mathbb{R}^n)$ if and only if $f\in C^\infty(\mathbb{R}^n)$ and for every multiindex $\alpha\in\mathbb{N}_0^n$ there are $K=K_{f,\alpha}\in[0,\infty[$ and $k=k_{f,\alpha}\in\mathbb{N}$ such that

$$|\partial^\alpha f(\xi)|\le K(1+|\xi|)^k \quad \text{for every } \xi\in\mathbb{R}^n.$$

According to L. Schwartz [S1, Sec. VII.5] a distribution T on $\mathbb{R}^n$ belongs to $\mathcal{O}'_C(\mathbb{R}^n)$ if and only if for every $k>0$ the distribution $(1+|x|^2)^{k/2}T$ is bounded (i.e. it extends to a continuous linear functional on $\mathcal{D}_{L^1}(\mathbb{R}^n)=\{\varphi\in C^\infty(\mathbb{R}^n):\partial^\alpha\varphi\in L^1(\mathbb{R}^n)$ for every $\alpha\in\mathbb{N}_0^n\}$). The distribution space $\mathcal{O}'_C(\mathbb{R}^n)$ is a convolution algebra.

The Fourier transformation maps $\mathcal{O}'_C(\mathbb{R}^n)$ in one-to-one manner onto $\mathcal{O}_M(\mathbb{R}^n)$, so that for every distribution $U\in\mathcal{O}'_C(\mathbb{R}^n)$ its Fourier transform is a function $\mathcal{F}U=\hat{U}\in\mathcal{O}_M(\mathbb{R}^n)$.

By $M_{m\times m}$ we shall denote the set of $m\times m$ matrices with complex entries. We shall use $\mathbb{C}^m$-valued and $M_{m\times m}$-valued functions and distributions.

The results of [S2] and [K2] imply the following

2010 *Mathematics Subject Classification*: Primary 35E15, 47D06, 46F99, 42B99.

Key words and phrases: one-parameter operator semigroup, one-parameter convolution semigroup, Cauchy problem, rapidly decreasing distribution, slowly increasing function.

DOI: 10.4064/cm133-1-8

THEOREM 1. *If* $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, *then we have* $(G\,*)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)} \in L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ *and the following five conditions are equivalent:*

(1.1) *there exists a unique one-parameter operator semigroup* $(T_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ *of class* (C_0) *whose infinitesimal generator is equal to* $(G\,*)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$,

(1.2) G *is equal to the generating distribution of an infinitely differentiable convolution semigroup* $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$,

(1.3) $0 \vee \max \operatorname{Re} \sigma(\hat{G}(\xi)) = O(\log|\xi|)$ *as* $\xi \in \mathbb{R}^n$ *and* $|\xi| \to \infty$ *where* σ *denotes the spectrum of a matrix belonging to* $M_{m\times m}$,

(1.4) *there are* $K \in [0,\infty[$ *and* $k \in \mathbb{N}$ *such that*

$$\rho(\exp \hat{G}(\xi)) \leq K(1+|\xi|)^k \quad \textit{for every } \xi \in \mathbb{R}^n$$

where ρ *denotes the spectral radius of a matrix belonging to* $M_{m\times m}$,

(1.5) *there are* $K \in [0,\infty[$ *and* $k \in \mathbb{N}$ *such that*

$$\|\exp \hat{G}(\xi)\|_{L(\mathbb{C}^m;\mathbb{C}^m)} \leq K(1+|\xi|)^k \quad \textit{for every } \xi \in \mathbb{R}^n.$$

Moreover, if the above equivalent conditions are satisfied, then the operator semigroup $(T_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ *occurring in* (1.1) *has the form*

$$T_t\Phi = S_t * \Phi, \quad t \in [0,\infty[, \quad \Phi \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m),$$

where $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ *is the convolution semigroup occurring in* (1.2).

If $\operatorname{supp} G = \{0\}$, i.e. $G = \mathcal{G}(\partial_1,\ldots,\partial_n)\otimes\delta$ where δ is the Dirac distribution on $\mathbb{R}^n$ and $\mathcal{G}(\partial_1,\ldots,\partial_n)$ is an $m\times m$ matrix whose entries are scalar PDOs with constant coefficients, then the equivalences (1.1)⇔(1.2)⇔(1.5) are consequences of L. Schwartz's Theorem III from [S2]. The equivalence (1.3)⇔(1.4) is an immediate consequence of [E-N, Sec. I.3, Lemma 3.19]. The implication (1.5)⇒(1.4) is trivial. The non-trivial implication (1.4)⇒(1.5) is a consequence of the G. E. Shilov inequality discussed in [K2] whose proof is based on elaborated results of the theory of functions of matrices. The conditions (1.3) and (1.4) do not occur in [S2]. The conditions (1.3) and (1.5) appeared first in the case of $G = \mathcal{G}(\partial_1,\ldots,\partial_n)\otimes\delta$ in the paper of I. G. Petrovskiĭ [P].

If $G = \mathcal{G}(\partial_1,\ldots,\partial_n)\otimes\delta$, then $\hat{G}(\xi) = \mathcal{G}(i\xi_1,\ldots,i\xi_n)$ and condition (1.3) takes the form

$$(1.3)_1 \qquad \max\{0\vee\operatorname{Re}\lambda : (\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n, \det(\lambda\mathbb{1}_{m\times m} - \mathcal{G}(i\xi)) = 0\} = O(\log|\xi|) \quad \text{as } |\xi|\to\infty.$$

Since $\det(\lambda \mathbb{1}_{m\times m} - \mathcal{G}(i\xi))$ is a polynomial in $(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n$, $(1.3)_1$ is equivalent to the condition

$$(1.3)_2 \qquad \sup\{\operatorname{Re}\lambda : (\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n, \det(\lambda\mathbb{1}_{m\times m} - \mathcal{G}(i\xi)) = 0\} < \infty,$$

which can also be written in the form

$$(1.3)_3 \qquad \sup_{\xi\in\mathbb{R}^n} \operatorname{Re}\sigma(\hat{G}(\xi)) = \sup_{\xi\in\mathbb{R}^n} \operatorname{Re}\sigma(\mathcal{G}(i\xi)) < \infty.$$

The equivalence of $(1.3)_1$ and $(1.3)_2$, conjectured by I. G. Petrovskiĭ [P, footnote on p. 24], was proved by L. Gårding [G, pp. 11–14], and reproved by L. Hörmander [H1, proof of Lemma 3.9], [H2, Appendix], [H3, Appendix] by means of the Tarski–Seidenberg projection theorem for semi-algebraic sets.

2. The result. Consider the following locally convex vector spaces:

(2.1) the function spaces $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m) = \{u \in C^\infty(\mathbb{R}^n;\mathbb{C}^m) : \partial^\alpha u \in L^p(\mathbb{R}^n;\mathbb{C}^m)$ for every $\alpha\in\mathbb{N}_0^n\}$, $p\in[1,\infty]$, and $(\mathcal{O}_a)(\mathbb{R}^n;\mathbb{C}^m) = \{u\in C^\infty(\mathbb{R}^n;\mathbb{C}^m) : \sup_{x\in\mathbb{R}^n}(1+|x|)^{-a}\|\partial^\alpha u(x)\|_{\mathbb{C}^m} < \infty$ for every $\alpha\in\mathbb{N}_0^n\}$, $a\in\,]0,\infty[$,

(2.2) the distribution spaces $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{O}'_C(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}'_{L^q}(\mathbb{R}^n;\mathbb{C}^m)$, $q\in\,]1,\infty]$, where $\mathcal{D}'_{L^q}(\mathbb{R}^n;\mathbb{C}^m)$ is equal to the m-th cartesian power of $(\mathcal{D}_{L^p}(\mathbb{R}^n))'$, $p = q/(q-1)$, and the distribution spaces $(\mathcal{O}'_a)(\mathbb{R}^n;\mathbb{C}^m)$, $a\in\,]0,\infty[$, all equipped with the strong dual topology.

The space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n) = (\mathcal{D}_{L^1}(\mathbb{R}^n))'$ is denoted by $\mathcal{B}'(\mathbb{R}^n)$ and its elements are called *bounded distributions*. Whenever $q\in\,]1,\infty[$, then $\mathcal{D}'_{L^q}(\mathbb{R}^n)\subset\mathcal{B}'(\mathbb{R}^n)$. The dual space of $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ is not a space of distributions. The space $(\mathcal{O}_a)(\mathbb{R}^n;\mathbb{C}^m)$ can be defined for every $a\in\mathbb{R}$; it is a Fréchet space with topology determined by the system of seminorms

$$p_\alpha(u) = \sup_{x\in\mathbb{R}^n}(1+|x|)^{-a}\|\partial^\alpha u(x)\|_{\mathbb{C}^m}, \quad u\in(\mathcal{O}_a)(\mathbb{R}^n;\mathbb{C}^m), \quad \alpha\in\mathbb{N}_0^n. \text{ (}^1\text{)}$$

The aim of the present paper is to prove the following result announced in [K2, pp. 50–51]:

THEOREM 2. *Suppose that* $G\in\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ *satisfies* (1.3), *and* E *is whichever of the l.c.v.s.* (2.1) *or* (2.2). *Let* $(T_t)_{t\ge0}\subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ *be the one-parameter semigroup of operators occurring in* (1.1), *and* $(S_t)_{t\ge0}\subset\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ *the one-parameter convolution semigroup in* (1.2). *Then*

(1) This topology induces the convergence of countable sequences which is stronger than the convergence adopted in [B, Sec. 6.2]. Therefore our Lemma 4.4 cannot be deduced from [B, Sec. 6.4] in spite of the common idea.

(2.3) $(T_t|_E)_{t\geq 0} = ((S_t *)|_E)_{t\geq 0} \subset L(E;E)$ *is a one-parameter operator semigroup of class* (C_0) *with infinitesimal generator* $(G*)|_E$,

(2.4) E *is sequentially complete,*

(2.5) *whenever* $u_0 \in E$ *and* $f \in C^k([0,\infty[;E)$ *where* $k \in \mathbb{N}$ *or* $k = \infty$, *then the Cauchy problem*

$$\begin{cases} \dfrac{du(t)}{dt} = G * u(t) + f(t) & \text{for } t \in [0,\infty[, \\ u(0) = u_0 \end{cases}$$

has a solution $u \in C^k([0,\infty[;E)$ *which is unique in the class* $C^1([0,\infty[;\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$. *Moreover, this solution can be expressed by the formula*

$$u(t) = S_t * u_0 + \int_0^t S_\tau * f(t-\tau)\,d\tau, \qquad t \in [0,\infty[,$$

where the integrand is a continuous E-valued function of τ and the integral is understood in the Riemann sense. If, in addition,

$$s(G) := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re}\sigma(\hat{G}(\xi))$$

is finite, then

(2.6) $\inf\{\omega \in \mathbb{R} :$ *the semigroup* $(e^{-\omega t}T_t|_E)_{t\geq 0} \subset L(E;E)$ *is equicontinuous*$\}$ $\leq s(G)$, *with equality if* $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ *or* $E = \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$.

Remark 1. The appearance of various l.c.v.s. E in Theorem 2 should be compared with [S1, Sec. VII.5, remarks after Theorem XI].

Remark 2. If P is a real polynomial on $\mathbb{R}^n$ of degree ≥ 2, then $\exp(iP) \in [\mathcal{O}_M \setminus \bigcup_{a>0}(\mathcal{O}_a)](\mathbb{R}^n)$. If $a \in]0,\infty[$, $k \in \mathbb{R}^n$ and

$$\varphi_{a,k}(x) = (1+|x|^2)^{a/2}\exp(ik\cdot x) \qquad \text{for } x \in \mathbb{R}^n,$$

then $\varphi_{a,k} \in (\mathcal{O}_a)(\mathbb{R}^n)$. If, in addition, $k_\nu \neq 0$ for $\nu = 1,\ldots,n$, then

$$\lim_{|x|\to\infty} |x|^{-a}|\partial^\alpha\varphi_{a,k}(x)| = 1 \qquad \text{for every } \alpha \in \mathbb{N}_0^n.$$

Remark 3. If $E = \mathcal{D}_{L^\infty}(\mathbb{R}^n;\mathbb{C}^m)$ and $G = \mathcal{G}(\partial_1,\ldots\partial_n)\otimes\delta$, then the equivalences (1.3)⇔(1.5)⇔(2.3) and the assertion (2.5) of Theorem 2 coincide with the results of I. G. Petrovskiĭ formulated in Chapter I of [P] in terms of classical analysis.

Remark 4. Theorem 2 is not true for $E = \mathcal{O}_M(\mathbb{R}^n)$ (although it is true for $E' = \mathcal{O}'_C(\mathbb{R}^n)$). To see this, let $G = i\Delta\delta$ (on $\mathbb{R}^n$). Then $\operatorname{supp} G = \{0\}$, so that $G \in \mathcal{O}'_C(\mathbb{R}^n)$, and $\hat{G}(\xi) \equiv -i|\xi|^2$, so that $s(G) = s(-G) = 0$. The i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n)$ with generating distribution $i\Delta\delta$ extends to a one-parameter convolution group $(S_t)_{t\in\mathbb{R}} \subset \mathcal{O}'_C(\mathbb{R}^n)$ such that for every $t \in \mathbb{R}\setminus\{0\}$ the distribution $S_t \in \mathcal{O}'_C(\mathbb{R}^n)$ is equal to the function

$$S_t(x) = \left(\frac{1 - i\,\mathrm{sgn}\,t}{2\sqrt{2\pi|t|}}\right)^m \exp\left(\frac{i|x|^2}{4t}\right), \qquad x \in \mathbb{R}^n,$$

belonging to $\mathcal{O}_M(\mathbb{R}^n)$. See [Go, Sec. 1.8.13]. Consider the Cauchy problem for the Schrödinger equation

$$\text{(2.7)} \qquad \begin{cases} \dfrac{\partial}{\partial t}u(t) = i\Delta\delta * u(t) = i\Delta u(t) & \text{for } t \in [0,\infty[, \\ u(0) = S_{-t_0}, \end{cases}$$

where $t_0 \in\,]0,\infty[$. The choice of the initial distribution $u(0) = S_{-t_0}$, $t_0 \in\,]0,\infty[$, resembles some formulas from [R, Sec. 3.4, Problem 2] concerning dispersion phenomena. In the class $C^1([0,\infty[;\mathcal{S}'(\mathbb{R}^n))$ the Cauchy problem (2.7) has a unique solution. Since this solution has the form $u(t) = S_t * S_{-t_0}$, one has $u(0) = S_{-t_0} \in \mathcal{O}_M(\mathbb{R}^n)$ and $u(t_0) = S_{t_0} * S_{-t_0} = \delta \in \mathcal{S}'(\mathbb{R}^n)\setminus\mathcal{O}_M(\mathbb{R}^n)$. Consequently, the Cauchy problem (2.7) with initial distribution being the function $S_{-t_0} \in \mathcal{O}_M(\mathbb{R}^n)$ has no solution in the class $C^1([0,\infty[;\mathcal{O}_M(\mathbb{R}^n))$.

Remark 5. From (2.6) it follows that if $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ and $s(G) < \infty$, then the semigroups $(T_t|_E)_{t\geq 0} \subset L(E;E)$ for E of the form (2.1) or (2.2) are uniformly exponential. More exactly, if $\omega \in\,]s(G),\infty[$, then, for each such E, the (C_0)-semigroup of operators $(e^{-\omega t}T_t|_E)_{t\geq 0} \subset L(E;E)$ is equicontinuous in the topology of E. According to the theory of equicontinuous one-parameter operator semigroups of class (C_0) in a sequentially complete l.c.v.s. (see [Y, Chapter IX]), the operator $(G\,*)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)} \in L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m), \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ has the $L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m),\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$-valued resolvent defined in the half-plane $\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > s(G)\}$, and this resolvent is equal to the Laplace transform of the semigroup $(T_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m),\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$. Moreover, if E is whichever of the l.c.v.s. (2.1) or (2.2), then the restrictions to $L(E;E)$ of the values of this resolvent constitute an $L(E;E)$-valued resolvent defined in the half-plane $\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > s(G)\}$. The $L(E;E)$-valued resolvent obtained in this way is equal to the Laplace transform of the semigroup $(T_t|_E)_{t\geq 0} \subset L(E;E)$, and is equal to the resolvent of the operator $(G\,*)|_E \in L(E;E)$.

Moreover, still under the assumptions that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ and $s(G) < \infty$, all the above resolvents have the form

$$\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > s(G)\} \ni \lambda \mapsto (R_\lambda\,*)|_E \in L(E,E)$$

where for each $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda > s(G)$ the distribution $R_\lambda \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is defined by the formula

$$R_\lambda = \int_0^\infty e^{-\lambda}S_t\,dt$$

with improper Riemann integral in the sequentially complete l.c.v.s. $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m}) \subset L_b(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$. The mapping $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > s(G)\} \ni \lambda \mapsto R_\lambda \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is holomorphic, and is an $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$-valued convolutional pseudoresolvent, i.e. it satisfies the equality

$$R_\lambda - R_\mu = (\mu-\lambda) R_\lambda * R_\mu \quad \text{for every } \lambda,\mu\in\mathbb{C} \text{ such that } \operatorname{Re}\lambda, \operatorname{Re}\mu > s(G).$$

For every $k\in\mathbb{N}$ and $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda > s(G)$ one has

$$(\lambda \mathbb{1}_{m\times m}\otimes\delta - G) * R_\lambda = R_\lambda * (\lambda \mathbb{1}_{m\times m}\otimes\delta - G) = \mathbb{1}_{m\times m}\otimes\delta$$

and

$$(R_\lambda)^{*,k} = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial\lambda}\right)^{k-1} R_\lambda.$$

3. Strongly $(G\,*)$-invariant locally convex vector spaces. Assume that $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is an infinitely differentiable convolution semigroup, and let $G\in\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ be its generating distribution.

By an $(S.\,*)$-*invariant l.c.v.s.* we mean a sequentially complete l.c.v.s. E continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; M_{m\times m})$ which satisfies the three conditions:

(3.1) $S_t * E \subset E$ for every $t\in[0,\infty[$,
(3.2) the mapping $[0,\infty[\, \times E \ni (t,u) \mapsto S_t * u \in E$ is continuous,
(3.3) $((S_t\,*)|_E)_{t\ge 0} \subset L(E;E)$ is a one-parameter (C_0)-semigroup with infinitesimal generator G_E such that $\operatorname{Dom}(G_E) = \{u\in E : G*u\in E\}$, $G_E u = G*u$ for $u\in\operatorname{Dom}(G_E)$.

We say that a l.c.v.s. E is *strongly* $(G\,*)$-*invariant* if $G*E\subset E$ and E is $(S.\,*)$-invariant.

In the above definitions the sequential completeness of a l.c.v.s. is important for two reasons: (i) one can use Riemann integrals of continuous E-valued functions, and (ii) one can use [E2, Theorem 7.4.4].

Every distribution $T\in\mathcal{S}'(\mathbb{R}^n; M_{m\times m})$ is represented by an $m\times m$-matrix whose entries are scalar distributions belonging to $\mathcal{S}'(\mathbb{R}^n)$. Then $\check{T}^\dagger \in \mathcal{S}'(\mathbb{R}^n; M_{m\times m})$ is defined as the distribution represented by the matrix transpose of the matrix representing T, the distribution-entries being reflected. Whenever $G\in\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies the condition (1.3), then so does $\check{G}^\dagger$.

THEOREM 3. *Suppose that a distribution $G\in\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies the condition* (1.3), *and that $(S_t)_{t\ge 0}\subset\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the i.d.c.s. whose generating distribution is G. Under these assumptions the following two assertions hold true:*

(A) *If E is a sequentially complete l.c.v.s. continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; M_{m\times m})$ such that $(S_t\,*)E\subset E$ for every $t\in[0,\infty[$ and the mapping $[0,\infty[\,\times E\ni(t,u)\mapsto S_t*u\in E$ is separately continuous,* *then* $(S_t|_E)_{t\geq 0} \subset L(E;E)$ *is a one-parameter operator semigroup of class* (C_0) *whose infinitesimal generator is the operator* G_E *such that* $D(G_E) = \{u \in E : G * u \in E\}$, *and* $G_E u = G * u$ *whenever* $u \in D(G_E)$.

(B) *Suppose that* F *is a barrelled l.c.v.s. continuously imbedded in* $\mathcal{S}'(\mathbb{R}^n)$ *such that* $\mathcal{S}(\mathbb{R}^n)$ *is continuously and densely imbedded in* F. *Let* F'_s *be the dual space of* F *equipped with the strong dual topology. If* $E := \times_{\mu=1}^m F$ *is strongly* $(\check{G}^\dagger *)$*-invariant, then* $E' := \times_{\mu=1}^m F'_s$ *is strongly* $(G\,*)$*-invariant. If the semigroup* $((\check{S}_t^\dagger *)|_E)_{t\geq 0} \subset L(E;E)$ *is equicontinuous, then the semigroup* $((S_t\,*)|_{E'})_{t\geq 0} \subset L(E';E')$ *is equicontinuous.*

Proof of (A). It is obvious that $((S_t\,*)|_E)_{t\geq 0} \subset L(E;E)$ is a (C_0)-semigroup. Denote by A_E its infinitesimal generator. We have to prove that $A_E = G_E$. To this end notice that whenever $T \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ is fixed, the mapping $\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m}) \ni U \mapsto U * T \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ is continuous, and hence the mapping $[0,\infty[\, \ni h \mapsto S_h * T \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ is infinitely differentiable. Moreover,

$$\frac{d}{dh}(S_h * T) = \left(\frac{d}{dh}S_h\right) * T = (S_h * G) * T \quad \text{for every } h \in [0,\infty[.$$

Since $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ is complete, it follows that whenever $h \in [0,\infty[$ and $T \in \mathcal{S}'(\mathbb{R}^n;M_{m\times m})$, then

$$S_h * T - T = \int_0^h S_\tau * G * T\,d\tau$$

where the Riemann integral on the right-hand side is convergent in the topology of $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$. This last equality implies that $A_E = G_E$. Indeed, if $T = u \in D(G_E)$, then the integrand on the right-hand side is a continuous E-valued function of τ, so that, by sequential completeness of E, the Riemann integral makes sense. It follows that if $u \in D(G_E)$ and $h \downarrow 0$, then

$$E\text{-}\lim_{h\downarrow 0} \frac{1}{h}(S_h * u - u) = E\text{-}\lim_{h\downarrow 0} \frac{1}{h}\int_0^h S_\tau * G_E u\,d\tau = G_E u,$$

proving that $G_E \subset A_E$. On the other hand, if $u \in D(A_E)$ and $h \downarrow 0$, then

$$\begin{aligned} A_E u &= E\text{-}\lim \frac{1}{h}(S_h * u - u) = \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)\text{-}\lim \frac{1}{h}(S_h * u - u) \\ &= \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)\text{-}\lim \frac{1}{h}\int_0^h S_\tau * G * u\,d\tau = G * u \end{aligned}$$

where the Riemann integral is understood in the sense of $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$. It follows that if $u \in D(A_E)$, then $A_E u = G * u$, and so $G * u \in E$ and $A_E u = G_E u$, proving that $A_E \subset G_E$.

Proof of (B). Since $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is continuously and densely imbedded in F, it follows that every element of F' has a unique extension to a distribution belonging to $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, and $E' = \times_{\mu=1}^m F'_s$ is continuously imbedded in $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$. Furthermore, the barrelledness of F implies the sequential completeness of F'. Indeed, suppose that T_ν, $\nu = 1, 2, \ldots$, is a Cauchy sequence in F'. Then, by the Banach–Steinhaus theorem, there is $T \in F'$ such that $\lim_{\nu\to\infty}\langle T_\nu, u\rangle = \langle T, u\rangle$ for every $u \in F$. Let B be a bounded subset of F. Since T_ν, $\nu = 1, 2, \ldots$, is a Cauchy sequence uniformly on B, there is a sequence $\nu_1 < \nu_2 < \cdots$ such that

$$\sup_{u\in B} |\langle T_{\nu_k}, u\rangle - \langle T_{\nu_{k+1}}, u\rangle| \le 1/2^k.$$

Since

$$\langle T_{\nu_k}, u\rangle - \langle T, u\rangle = \sum_{l=k}^{\infty}(\langle T_{\nu_l}, u\rangle - \langle T_{\nu_{l+1}}, u\rangle)$$

it follows that

$$\sup_{u\in B} |\langle T_{\nu_k}, u\rangle - \langle T, u\rangle| \le \sum_{l=k}^{\infty} \sup_{u\in B} |\langle T_{\nu_l}, u\rangle - \langle T_{\nu_{l+1}}, u\rangle| \le 1/2^{k-1}.$$

From this and the fact that

$$\lim_{\mu,\nu\to\infty} \sup_{u\in B} |\langle T_\nu, u\rangle - \langle T_\mu, u\rangle| = 0$$

it follows that

$$\lim_{k\to\infty} \sup_{u\in B} |\langle T_{\nu_k}, u\rangle - \langle T, u\rangle| = 0.$$

Thanks to the already proved assertion (A), in order to complete the proof of (B) we have to show that if E is strongly $(\check{G}^\dagger *)$-invariant, then:

(3.4) $(G*)E' \subset E'$,
(3.5) $(S_t*)E' \subset E'$ for every $t \in [0,\infty[$,
(3.6) the mapping $[0,\infty[\times E' \ni (t,u) \mapsto S_t * u \in E'$ is continuous (so that $((S_t*)|_{E'})_{t\ge0} \subset L(E';E')$ is an operator semigroup of class (C_0)),
(3.7) if the semigroup $((\check{S}_t^\dagger *)|_E)_{t\ge0} \subset L(E;E)$ is equicontinuous, then so is $((S_t*)|_{E'})_{t\ge0} \subset L(E';E')$,
(3.8) for every $u \in E'$ the mapping $[0,\infty[\ni t \mapsto S_t * u \in E'$ is continuously differentiable (so that $D(G_{E'}) = E'$ and hence $G_{E'} = (G*)|_{E'}$, by (A)).

For the proof of (3.4) observe that whenever $T \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ and $u \in S(\mathbb{R}^n;\mathbb{C}^m)$, then

$$\langle G*T, u\rangle = \langle T, \check{G}^\dagger * u\rangle.$$

If $T \in E'$, then the right-hand side of this equality can be uniquely extended to a linear functional of u continuous on E. This implies that $G*T \in E'$. An analogous argument proves (3.5).

In order to prove (3.6) we have to show that whenever $t_0 \in [0,\infty[$ and $T_0 \in E'$, then for every neighbourhood U of $S_{t_0} * T_0$ in E' there are a neighbourhood $V \subset [0,\infty[$ of t_0 and a neighbourhood W of T_0 in E' such that whenever $t \in V$ and $T \in W$, then $S_t * T \in U$. It is sufficient to prove this for

$$U = \Big\{X \in E' : \sup_{u\in B} |\langle X - S_{t_0} * T_0, u\rangle| < \varepsilon\Big\} \tag{3.9}$$

where $\varepsilon \in\,]0,\infty[$ and B is a bounded subset of E. It will appear that if U has the form (3.9), then one can take

$$W = \Big\{T \in E' : \sup_{v\in C} |\langle T - T_0, v\rangle| < \varepsilon/2\Big\} \tag{3.10}$$

where

$$C = \{\check{S}_t^\dagger * u : t \in [0, t_0 + 1],\ u \in B\}.$$

The boundedness of C follows from [E2, Theorem 7.4.4] because whenever $u \in E$ is arbitrarily fixed, then $\{\check{S}_t^\dagger * u : t \in [0, t_0 + 1]\}$ is a bounded subset of E.

The construction of V is more complicated. Notice first that for every $t \in [0,\infty[$, $T \in E'$ and $u \in E$ one has

$$\begin{aligned}\langle S_t * T - S_{t_0} * T_0, u\rangle &= \langle (S_t - S_{t_0}) * T_0, u\rangle + \langle S_t * (T - T_0), u\rangle \\ &= \langle T_0, (\check{S}_t^\dagger - \check{S}_{t_0}^\dagger) * u\rangle + \langle T - T_0, \check{S}_t^\dagger * u\rangle.\end{aligned}$$

It follows that if U and W have the form (3.9) and (3.10), then $S_t * T \in U$ if only $T \in W$ and

$$\sup_{u\in B} |\langle T_0, (\check{S}_t^\dagger - \check{S}_{t_0}^\dagger) * u\rangle| < \varepsilon/2 \quad \text{for every } t \in V. \tag{3.11}$$

In order to construct V which satisfies (3.11) and has the form $V = \{t \in [0, t_0 + 1[\, : |t - t_0| < \delta\}$, observe that for every $t \in [0, t_0 + 1]$ and $u \in E$ one has

$$(\check{S}_t^\dagger - \check{S}_{t_0}^\dagger) * u = \int_{t_0}^{t} \check{S}_\tau^\dagger * \check{G}^\dagger * u \, d\tau,$$

so that whenever B is a bounded subset of E, then

$$\sup_{u\in B} |\langle T_0, (\check{S}_t^\dagger - \check{S}_{t_0}^\dagger) * u\rangle| \le |t - t_0| \sup_{v\in D} |\langle T_0, v\rangle|$$

where, by [E2, Theorem 7.4.4],

$$D = \{\check{S}_\tau^\dagger * \check{G}^\dagger * u : \tau \in [0, t_0 + 1],\ u \in B\}$$

is a bounded subset of E. It follows that if $\delta = \frac{1}{2}\varepsilon(1 + \sup_{v\in D} |\langle T_0, v\rangle|)^{-1}$, then (3.11) holds. This completes the proof of (3.6).

In order to prove (3.7), take a neighbourhood U of zero in E' of the form

$$U = \Big\{X \in E' : \sup_{u\in B} |\langle X, u\rangle| < \varepsilon\Big\}$$

where $\varepsilon \in \,]0,\infty[$ and B is a bounded subset of E. Put

$$W = \Big\{T \in E' : \sup_{v\in C} |\langle T, v\rangle| < \varepsilon\Big\} \quad \text{where} \quad C = \{\check{S}_t^\dagger * u : t \in [0,\infty[,\ u \in B\}.$$

By [E2, Theorem 7.4.4], C is a bounded subset of E, because the equicontinuity of the semigroup $((\check{S}_t^\dagger *)|_E)_{t\ge 0} \subset L(E;E)$ implies that whenever $u \in E$, then $\{\check{S}_t^\dagger * u : t \in [0,\infty[\}$ is a bounded subset of E. Hence W is a neighbourhood of zero in E'. Since $\langle S_t * T, u\rangle = \langle T, \check{S}_t^\dagger * u\rangle$, it follows that whenever $T \in W$, then $S_t * T \in U$ for every $t \in [0,\infty[$.

Finally, notice that (3.8) means that whenever $u \in E'$, $t_0 \in \,]0,\infty[$ and a bounded set $B \subset E$ are fixed, then the mappings

$$[0,t_0] \ni t \mapsto \langle S_t * u, \varphi\rangle \in \mathbb{C}, \qquad \varphi \in B,$$

are continuously differentiable uniformly with respect to φ ranging over B. To prove this, it is sufficient to observe that

$$\begin{aligned}\sup_{\varphi\in B,\, t\in[0,t_0]} \Big|\frac{d^2}{dt^2}\langle S_t * u, \varphi\rangle\Big| &= \sup_{\varphi\in B,\, t\in[0,t_0]} \Big|\frac{d^2}{dt^2}\langle u, \check{S}_t^\dagger * \varphi\rangle\Big| \\ &= \sup_{\varphi\in B,\, t\in[0,t_0]} |\langle u, \check{S}_t^\dagger * \check{G}^\dagger * \check{G}^\dagger * \varphi\rangle| \\ &= \sup_{\psi\in C} |\langle u, \psi\rangle| < \infty\end{aligned}$$

where

$$C = \{\check{S}_t^\dagger * \check{G}^\dagger * \check{G}^\dagger * \varphi : t \in [0,t_0],\ \varphi \in B\}$$

is a bounded subset of E. To prove the boundedness of C it is sufficient to recall that E is strongly $(\check{G}^\dagger *)$-invariant, and to apply [E2, Theorem 7.4.4].

4. The strong $(G*)$-invariance and property (2.6) of the function spaces (2.1). By [E1, Sec. devoted to factors of class (S,S)] or [K1, Theorem 2.1], one has

(4.1) $$\mathcal{O}'_C(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n) : (T*)|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))\}.$$

Recall that by [K2, p. 53]

(4.2) the topology in $\mathcal{O}'_C(\mathbb{R}^n)$ is induced from $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ via the mapping

$$\mathcal{O}'_C(\mathbb{R}^n) \ni T \mapsto (T*)|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n)).$$

From (4.1) and (4.2), by the Banach–Steinhaus theorem, it follows that if $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is an i.d.c.s., then for every $t_0 \in \,]0,\infty[$ the set of operators $\{(S_t\,*)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)} : t \in [0,t_0]\} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. This implies that for $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ the conditions (3.1) and (3.2) are satisfied.

Furthermore, if $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ is an i.d.c.s., then for every $\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ the mapping $[0,\infty[\ \ni t \mapsto S_t * \varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is infinitely differentiable, so that the domain of the infinitesimal generator A of the operator semigroup $((S_t\,*)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)})_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equal to the whole $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$.

By Theorem 3(A), it follows that $A = (G\,*)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$ where $G \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ is the generating distribution of the i.d.c.s. Thus for $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ the condition (3.3) is satisfied with $G_E = (G\,*)|_E$. Altogether, this means that $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is strongly $(G\,*)$-invariant. A direct consequence of [K2, Theorem 2.2] is that (2.6) holds for $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$.

In order to establish analogous facts for $E = \mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)$, $p \in [1,\infty]$, and $E = (\mathcal{O}_a)(\mathbb{R}^n;\mathbb{C}^m)$, $a \in\ ]0,\infty[$, we shall use the following four lemmas. For every $k \in\ ]0,\infty[$ let

$$B_k(\mathbb{R}^n;M_{m\times m}) = \Big\{f \in C(\mathbb{R}^n;M_{m\times m}) : \sup_{x\in\mathbb{R}^n} |x|^k \|f(x)\|_{M_{m\times m}} < \infty\Big\}.$$

Equipped with the norm $\|f\|_{B_k} = \sup_{x\in\mathbb{R}^n}(1+|x|)^k\|f(x)\|_{M_{m\times m}}$, the space $B_k(\mathbb{R}^n;M_{m\times m})$ is a Banach space.

LEMMA 4.1. *Suppose that the distribution* $G \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ *satisfies* (1.3), *and let* $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ *be the i.d.c.s. with generating distribution* G. *Then for every* $t_0 \in\ ]0,\infty[$ *and* $k \in \mathbb{N}$ *there are* $j_0 \in \mathbb{N}$ *and a continuously differentiable mapping* $[0,t_0] \ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n;M_{m\times m})$ *such that whenever* $t \in [0,t_0]$, *then*

$$f_t \in B_{2k}(\mathbb{R}^n;M_{m\times m}) \cap \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m}) \tag{4.3}$$

and

$$S_t = (1-\Delta)^{j_0} f_t. \tag{4.4}$$

If in addition $s(G) := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re}\sigma(\hat{G}(\xi)) < \infty$, *then for every* $k \in \mathbb{N}$ *there are* $j_0 \in \mathbb{N}$ *and a continuously differentiable mapping* $[0,\infty[\ \ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n;M_{m\times m})$ *such that* (4.3) *and* (4.4) *hold for every* $t \in [0,\infty[$, *and moreover*

$$\sup_{t\in[0,\infty[} e^{-\omega t}\|f_t\|_{B_{2k}} < \infty \quad \textit{for every } \omega \in\ ]s(G),\infty[. \tag{4.5}$$

It is instructive to compare Lemma 4.1 with [S1, Sec. VII.5, Theorem IX, 1^0] which implies that an $M_{m\times m}$-valued distribution T on $\mathbb{R}^n$ belongs to $\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ (i.e. is rapidly decreasing) if and only if for every $k \in\ ]0,\infty[$ there is $m_k \in \mathbb{N}$ and a finite collection $\{f_{k,\alpha} : \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq m_k\} \subset B_k(\mathbb{R}^n;M_{m\times m})$ such that $T = \sum_{|\alpha|\leq m_k} \partial^\alpha f_{k,\alpha}$. The argument of the proof of Lemma 4.1 (applied to a single $T \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$) yields a new proof of Theorem IX, 1^0 of [S1]. This new proof depends on the Fourier transformation but is independent of the knowledge about fundamental solutions of iterated laplacians.

Lemma 4.2. *Suppose that $a > 0$ and $k > a + n$. Then*

$$B_k(\mathbb{R}^n; M_{m\times m}) * (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m) \subset (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$$

where $*$ denotes convolution of functions. Moreover, the mapping

$$B_k(\mathbb{R}^n; M_{m\times m}) \times (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m) \ni (f, u) \mapsto f * u \in (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$$

is continuous.

Lemma 4.3. *Suppose that $0 < h < k < \infty$, and let $(\varphi_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ be a non-negative sequence such that $\operatorname{supp}\varphi_\nu \subset \{x \in \mathbb{R}^n : |x| \le 1/\nu\}$ and $\int_{\mathbb{R}^n} \varphi_\nu(x)\,dx = 1$ for every $\nu \in \mathbb{N}$. Let B be a bounded subset of $B_k(\mathbb{R}^n)$ consisting of equicontinuous functions. Then*

$$\lim_{\nu\to\infty} \sup_{f\in B} \|f * \varphi_\nu - f\|_{B_h} = 0.$$

Lemma 4.4. *Whenever $a > 0$, $k > a + n$, $f \in B_k(\mathbb{R}^n) \cap \mathcal{O}'_C(\mathbb{R}^n)$ and $u \in (\mathcal{O}_a)(\mathbb{R}^n)$, the distribution-theoretical convolution (defined via the duality between distributions and sample functions) of $f \in \mathcal{O}'_C(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ coincides with the convolution of the functions $f \in B_k(\mathbb{R}^n)$ and $u \in (\mathcal{O}_a)(\mathbb{R}^n)$.*

Before proving the above lemmas, let us show how they imply that whenever $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies (1.3) and either $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$ or $E = (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$, $a \in]0, \infty[$, then E is a strongly $(G*)$-invariant l.c.v.s. such that (2.6) holds if $s(G) < \infty$. To this end fix a and $k \in \mathbb{N}$ such that $a > 0$ and $2k > a + n$. Then, by Lemmas 4.1 and 4.2, for every $t_0 \in]0, \infty[$ there are $j_0 \in \mathbb{N}$ and a continuously differentiable mapping $[0, t_0] \ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ such that (4.3) and (4.4) hold, so that

$$S_t * u = ((1 - \Delta)^{j_0} f_t) * u = f_t * ((1 - \Delta)^{j_0} u) \quad \text{for every } t \in [0, t_0] \text{ and } u \in (\mathcal{O}_a)(\mathbb{R}^n; M_{m\times m}). \tag{4.6}$$

In (4.6) the symbols $*$ denote the distribution-theoretical convolution of an element $(1-\Delta)^{j_0} f_t$ or f_t of $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ with an element u or $(1-\Delta)^{j_0} u$ of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. By Lemma 4.4, for every fixed $t \in [0, t_0]$ the last term in (4.6) is equal to the elementary convolution of the $M_{m\times m}$-valued function $f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ with the $\mathbb{C}^m$-valued function $(1 - \Delta)^{j_0} u \in (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$. If $E = (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$, then, by Lemma 4.2, the mapping

$$[0, t_0] \times E \ni (t, u) \mapsto S_t * u = f_t * ((1 - \Delta)^{j_0} u) \in E \tag{4.7}$$

is continuous, and for every fixed $u \in E$ the mapping

$$[0, t_0] \ni t \mapsto S_t * u = f_t * ((1 - \Delta)^{j_0} u) \in E \tag{4.8}$$

is continuously differentiable. The statements (4.7) and (4.8) remain valid for $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$ because if $2k > n$, then $B_{2k}(\mathbb{R}^n; M_{m\times m})$ is continuously imbedded in $L^1(\mathbb{R}^n; M_{m\times m})$, and the mapping $L^1(\mathbb{R}^n; M_{m\times m}) \times \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m) \ni (f, u) \mapsto f * u \in \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$ is continuous by the Young inequality. From (4.7) it follows that if either $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$, or $E = (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$, $a \in \,]0, \infty[$, then the conditions (3.1) and (3.2) are satisfied. From (4.8) it follows that $G_E = (G\,*)|_E$, so that $G * E \subset E$. Hence if either $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$, or $E = (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$, $a \in \,]0, \infty[$, then E is a strongly $(G\,*)$-invariant l.c.v.s.

Suppose now that $s(G) < \infty$ and either $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$, or $E = (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$, $a \in \,]0, \infty[$. The topology in E is determined by the system of seminorms p_α, $\alpha \in \mathbb{N}_0^n$, where either $p_\alpha(u) = \|\partial^\alpha u\|_{L^p(\mathbb{R}^n;\mathbb{C}^m)}$ or $p_\alpha(u) = \sup_{x\in\mathbb{R}^n}(1 + |x|)^{-a}\|\partial^\alpha u(x)\|_{\mathbb{C}^m}$. If $s(G) < \infty$, then (4.6) holds for every $t \in [0, \infty[$ and $u \in E$, so that whenever $\omega > s(G)$, then, by (4.5), Lemma 4.2, continuity of the imbedding $B_{2k}(\mathbb{R}^n; M_{m\times m}) \subset L^1(\mathbb{R}^n; M_{m\times m})$ and the Young inequality,

$$\begin{aligned} p_\alpha(e^{-\omega t} S_t * u) &= p_\alpha(e^{-\omega t} f_t * ((1 - \Delta)^{j_0} u)) = e^{-\omega t} p_\alpha(f_t * ((1 - \Delta)^{j_0} u)) \\ &\le C_\omega p_\alpha((1 - \Delta)^{j_0} u) \end{aligned}$$

for every $t \in [0, \infty[$ and $u \in E$, with some constant $C_\omega \in \,]0, \infty[$. This proves that for every $\omega > s(G)$ the operator semigroup $((e^{-\omega t} S_t\,*)|_E)_{t\ge0} \subset L(E; E)$ is equicontinuous, so that (2.6) holds.

Proof of Lemma 4.1. Suppose that $(S_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is an i.d.c.s. with generating distribution G. Fix $k \in \,]0, \infty[$. Take $j_0 \in \mathbb{N}$ whose value will be determined later. For every $t \in [0, \infty[$ the formula

$$g_t(\xi) = (1 + |\xi|^2)^{-j_0} \exp(t\hat{G}(\xi)), \quad \xi \in \mathbb{R}^n, \tag{4.9}$$

defines an element g_t of $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$. Let

$$f_t = \mathcal{F}^{-1} g_t. \tag{4.10}$$

Then

$$f_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m}) \quad \text{and} \quad (1 - \Delta)^{j_0} f_t = \mathcal{F}^{-1}(\exp t\hat{G}) = S_t.$$

Lemma 4.1 follows once it is proved that

(4.11) whenever $t_0 \in \,]0, \infty[$, there is $j_0 \in \mathbb{N}$ such that $f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ for every $t \in [0, t_0]$ and the mapping $[0, t_0] \ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ is continuously differentiable,

(4.12) whenever $s(G) < \infty$, there is $j_0 \in \mathbb{N}$ such that $f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ for every $t \in [0, \infty[$, the mapping $[0, \infty[\ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ is continuously differentiable and (4.5) holds.

The condition (4.11) is satisfied if for every $t_0 \in]0,\infty[$ there is $j_0 \in \mathbb{N}$ such that

(4.13) $\{(d/dt)^l(1-\Delta)^k g_t : t \in [0,t_0],\ l=0,1,2\}$ is a bounded subset of $L^1(\mathbb{R}^n; M_{m\times m})$.

Notice that

$$\left(\frac{d}{dt}\right)^l \partial_\xi^\kappa g_t(\xi) = \sum_{|\alpha|+|\beta|+|\gamma|=|\kappa|} \frac{\kappa!}{\alpha!\beta!\gamma!}[\partial_\xi^\alpha(1+|\xi|^2)^{-j_0}][\partial_\xi^\beta \hat{G}(\xi)^l][\partial_\xi^\gamma \exp(t\hat{G}(\xi))]$$

where the first bracketed factor is a scalar function, and the second and third factors are $M_{m\times m}$-valued functions. Since

$$\partial_\xi^\alpha(1+|\xi|^2)^{-j_0} = (1+|\xi|^2)^{-j_0-\alpha}P_\alpha(\xi)$$

where P_α is a polynomial of degree $|\alpha|$, the function $\mathbb{R}^n \ni \xi \mapsto (1+|\xi|^2)^{-j_0} \in \mathbb{R}$ belongs to $(\mathcal{O}_{-2j_0})(\mathbb{R}^n)$. Moreover the function $\mathbb{R}^n \ni \xi \mapsto \hat{G}(\xi)^l \in M_{m\times m}$ belongs to $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$, and by [K2, Proposition 3.2], $\{\exp(t\hat{G}) : t \in [0,t_0]\}$ is a set of uniformly slowly increasing C^∞-functions on $\mathbb{R}^n$. It follows that whenever $j_0 \in \mathbb{N}$ is sufficiently large, then $\{(d/dt)^l \partial_\xi^\kappa g_t : t \in [0,t_0],\ l = 0,1,2,\ |\kappa| \le 2k\}$ is a bounded subset of $L^1(\mathbb{R}^n; M_{m\times m})$, which implies (4.13).

The proof of (4.12) is similar to that of (4.11), but this time we make use of the fact that whenever $\omega > s(G)$ then, by [K2, Proposition 3.3], $\{e^{-\omega t}\exp(t\hat{G}) : t \in [0,\infty]\} \subset \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is a set of uniformly slowly increasing C^∞-functions on $\mathbb{R}^n$. This implies that whenever $\omega > s(G)$ and $j_0 \in \mathbb{N}$ is sufficiently large, then $\{e^{-\omega t}(d/dt)^l(1-\Delta_\xi)^k g_t : t \in [0,\infty[,\ l = 0,1,2\}$ is a bounded subset of $L^1(\mathbb{R}^n; M_{m\times m})$. Consequently, the mapping $[0,\infty[\ \ni t \mapsto f_t \in B_{2k}(\mathbb{R}^n; M_{m\times m})$ is continuously differentiable and (4.5) holds.

Proof of Lemma 4.2. Suppose that $a > 0$ and $k > a+n$. Let $f \in B_k(\mathbb{R}^n; M_{m\times m})$ and $u \in (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$. Then, for every $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} \|f(y)\partial^\alpha u(x-y)\|_{\mathbb{C}^m}\,dy &\le \|f\|_{B_k} p_\alpha(u) \int_{\mathbb{R}^n} (1+|y|)^{-k}(1+|x-y|)^a\,dy \\ &\le \|f\|_{B_k} p_\alpha(u) \int_{\mathbb{R}^n} (1+|y|)^{-k}(1+|x|+|y|)^a\,dy \\ &\le \|f\|_{B_k} p_\alpha(u)\Big(\int_{\mathbb{R}^n} (1+|y|)^{a-k}\,dy\Big)(1+|x|)^a,\end{aligned}$$

where $\int_{\mathbb{R}^n}(1+|y|)^{a-k}\,dy = C < \infty$ because $a-k < -n$. This implies that $f * u \in (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m)$ and $p_\alpha(f*u) \le C\|f\|\,p_\alpha(u)$ for every $\alpha \in \mathbb{N}_0^n$. Hence it is easy to deduce the continuity of the mapping

$$B_k(\mathbb{R}^n; M_{m\times m}) \times (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m) \ni (f,u) \mapsto f*u \in (\mathcal{O}_a)(\mathbb{R}^n; \mathbb{C}^m).$$

Proof of Lemma 4.3. Denote by ω the common modulus of continuity of functions belonging to B. Whenever $f \in B$ and $\nu \ge 2$, then for every $m > 0$

one has

$$\begin{aligned}\|f*\varphi_\nu - f\|_{B_h} &= \sup_{x\in\mathbb{R}^n}(1+|x|)^h\Big|\int\limits_{\mathbb{R}^n}(f(x-y)-f(x))\varphi_\nu(y)\,dy\Big| \\ &\le (1+m)^h\omega(1/\nu) + \sup_{|x|\ge m}(1+|x|)^h(1+|x|-1/\nu)^{-k}2\|f\|_{B_k} \\ &\le (1+m)^h\omega(1/\nu) + \sup_{|x|\ge m}(1+|x|)^h(\tfrac12+\tfrac12|x|)^{-k}2\|f\|_{B_k} \\ &\le (1+m)^h\omega(1/\nu) + (1+m)^{h-k}2^{k+1}\sup_{f\in B}\|f\|_{B_k},\end{aligned}$$

whence

$$\lim_{\nu\to\infty}\sup_{f\in B}\|f*\varphi_\nu - f\|_{B_h} = 0.$$

Proof of Lemma 4.4. In the present proof, denote by $*$ the general distribution-theoretical convolution, and by $*_0$ convolution of functions. Fix a sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}\subset\mathcal{D}(\mathbb{R}^n)$ of non-negative functions such that $\operatorname{supp}\varphi_\nu\subset\{x\in\mathbb{R}^n : |x|\le 1/\nu\}$ and $\int_{\mathbb{R}^n}\varphi_\nu(x)\,dx = 1$ for every $\nu\in\mathbb{N}$. Then, by Lemma 4.3, for every $\phi\in\mathcal{S}(\mathbb{R}^n)$ the sequence $(\varphi_\nu*\phi)_{\nu\in\mathbb{N}} = (\varphi_\nu *_0 \phi)_{\nu\in\mathbb{N}}\subset\mathcal{S}(\mathbb{R}^n)$ converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$, and this convergence is uniform with respect to ϕ ranging over any bounded subset of $\mathcal{S}(\mathbb{R}^n)$. By (4.1) it follows that whenever $f\in\mathcal{O}'_C(\mathbb{R}^n)$ and $\phi\in\mathcal{S}(\mathbb{R}^n)$, then the distributions $(f*\varphi_\nu)*\phi = f*(\varphi_\nu*\phi)$, $\nu\in\mathbb{N}$, are functions belonging to $\mathcal{S}(\mathbb{R}^n)$, and the sequence $((f*\varphi_\nu)*\phi)_{\nu\in\mathbb{N}}$ converges to $f*\phi$ in the topology of $\mathcal{S}(\mathbb{R}^n)$, uniformly with respect to ϕ ranging over any bounded subset of $\mathcal{S}(\mathbb{R}^n)$. This means that the sequence $(f*\varphi_\nu)_{\nu\in\mathbb{N}}\subset\mathcal{S}(\mathbb{R}^n)\subset\mathcal{O}'_C(\mathbb{R}^n)$ converges to $f\in\mathcal{O}'_C(\mathbb{R}^n)$ in the topology of $\mathcal{O}'_C(\mathbb{R}^n)$. (In this way we have proved that $\mathcal{S}(\mathbb{R}^n)$ is sequentially dense in $\mathcal{O}'_C(\mathbb{R}^n)$.) Since for every fixed $u\in(\mathcal{O}_a)(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$ the mapping $\mathcal{O}'_C(\mathbb{R}^n)\ni f\mapsto f*u\in\mathcal{S}'(\mathbb{R}^n)$ is continuous (in the strong dual topology of $\mathcal{S}'(\mathbb{R}^n)$), it follows that

(4.14) whenever $f\in\mathcal{O}'_C(\mathbb{R}^n)$ and $u\in(\mathcal{O}_a)(\mathbb{R}^n)$, then the sequence of distributions $((f*\varphi_\nu)*u)_{\nu\in\mathbb{N}}\subset\mathcal{S}'(\mathbb{R}^n)$ converges to the distribution $f*u$ in the strong dual topology of $\mathcal{S}'(\mathbb{R}^n)$.

Suppose now that $a>0$, $k>a+n$, $f\in\mathcal{O}'_C(\mathbb{R}^n)\cap B_k(\mathbb{R}^n)$ and $u\in(\mathcal{O}_a)(\mathbb{R}^n)$. Then, by (4.1), $f*\varphi_\nu\in\mathcal{S}(\mathbb{R}^n)\subset B_k(\mathbb{R}^n)$ for every $\nu\in\mathbb{N}$, so that, by Lemma 4.2, the distribution $(f*\varphi_\nu)*u\in\mathcal{S}'(\mathbb{R}^n)$ is equal to the function $(f*\varphi_\nu)*_0u\in(\mathcal{O}_a)(\mathbb{R}^n)$ (because the distribution u is a slowly increasing continuous function). Fix now $h\in\,]a+n,k[$. Then $B_k(\mathbb{R}^n)\subset B_h(\mathbb{R}^n)$ and whenever $f\in B_k(\mathbb{R}^n)$ (so that $f\in L^1(\mathbb{R}^n)$), then Lemma 4.3 applied to the singleton $B=\{f\}$ implies that the sequence of functions $(f*\varphi_\nu)_{\nu\in\mathbb{N}}$ converges to the function f in the norm of $B_h(\mathbb{R}^n)$. By Lemma 4.2 it follows that

(4.15) whenever $f \in B_k(\mathbb{R}^n) \cap \mathcal{O}'_C(\mathbb{R}^n)$ and $u \in (\mathcal{O}_a)(\mathbb{R}^n)$, then the distributions $(f * \varphi_\nu) * u$, $\nu \in \mathbb{N}$, are functions belonging to $(\mathcal{O}_a)(\mathbb{R}^n)$ and the sequence $((f * \varphi_\nu) * u)_{\nu \in \mathbb{N}}$ converges to the function $f *_0 u$ in $(\mathcal{O}_a)(\mathbb{R}^n)$.

From (4.14) and (4.15) it follows that the distribution $f * u$ is equal to the function $f *_0 u$.

5. The strong $(G*)$-invariance and property (2.6) of the distribution spaces (2.2). Suppose that the matrix-valued distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies the condition (1.3). Then also $\check{G}^\dagger \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, and $\check{G}^\dagger$ satisfies (1.3). Moreover, $s(\check{G}^\dagger) = s(G)$. In Section 4 it was proved that $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is strongly $(G*)$-invariant and satisfies (2.6). Similarly, $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is strongly $(\check{G}^\dagger *)$-invariant. Hence, by Theorem 3(B), $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is strongly $(G*)$-invariant. (The strong $(G*)$-invariance of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ also follows directly from condition (1.1) of Theorem 1.) Moreover, since the pair $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ is reflexive ($\mathcal{S}'(\mathbb{R}^n)$ being equipped with the strong dual topology), from Theorem 3(B) it follows that for every $\omega \in \mathbb{R}^n$ the equicontinuity of the semigroup $(e^{-\omega t}\check{T}_t^\dagger|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)})_{t\ge 0} \subset L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ is equivalent to the equicontinuity of the semigroup $(e^{-\omega t}T_t)_{t\ge 0} \subset L(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$. Therefore $\omega_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)} = \omega_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)} = s(\check{G}^\dagger) = s(G)$.

Similar reasonings, based on Theorem 3(B) and the results of Section 4, prove that each of the distribution spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n; \mathbb{C}^m) = \times_{\mu=1}^m (\mathcal{D}_{L^p}(\mathbb{R}^n))'$, $q \in]1, \infty]$, $p = q/(q-1)$, and $(\mathcal{O}'_a)(\mathbb{R}^n; \mathbb{C}^m)$, $a \in]0, \infty[$, is strongly $(G*)$-invariant and satisfies (2.6).

The strong $(G*)$-invariance and property (2.6) of the distribution space $\mathcal{O}'_C(\mathbb{R}^n; \mathbb{C}^m)$ follow directly from the properties of the convolution semigroup $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; \mathbb{C}^m)$ with generating distribution G.

6. Proof of Theorem 2. Assume that the matrix-valued distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies (1.3). By Theorem 1 (or by the strong $(G*)$-invariance of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ proved in Section 5), the assertion (1.1) is true. Moreover, $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m) = \times_{\mu=1}^m \mathcal{S}'(\mathbb{R}^n)$ where $\mathcal{S}'(\mathbb{R}^n)$ is equipped with the strong dual topology, so that $\mathcal{S}'(\mathbb{R}^n)$ is reflexive and hence barrelled. Therefore $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is barrelled, and the uniqueness in the class $C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ of a solution of the Cauchy problem (2.5) can be proved by an argument analogous to the one used in the proof of [H-P, Theorem 23.8.1]. See also [E-N, Sec. II.6, Proposition 6.4], [Go, Sec. 1.2, Remark 2.14], [Pa, Sec. 1.2, Theorem 2.6].

Thanks to the results of Sections 4 and 5, in order to complete the proof of Theorem 2, it remains to prove the existence of solutions of the Cauchy problem (2.5) and the Duhamel formula representing them. So, it remains to show that whenever E is a strongly $(G*)$-invariant l.c.v.s., $u_0 \in E$, $f \in C^k([0, \infty[; E)$ where $k \in \mathbb{N}$, and $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the i.d.c.s. with generating distribution G, then the formula

$$(6.1)\qquad u(t) = S_t * u_0 + \int_0^t S_\tau * f(t-\tau)\,d\tau, \quad t \in [0,\infty[,$$

represents a solution of the Cauchy problem (2.5) belonging to $C^k([0,\infty[;E)$. To this end we shall consider separately the cases of $f \equiv 0$ and of $u_0 = 0$.

If $f \equiv 0$ and $u_0 \in E$, then, by (2.3), the formula (6.1) takes the form $u(t) = (T_t|_E)u_0$. Hence, again by (2.3), $u(\cdot) \in C^\infty([0,\infty[;E)$ and $(d/dt)^k u(t) = (G*)^k u(t)$ for $t \in [0,\infty[$.

If $u_0 = 0$ and $f \in C^1([0,\infty[;E)$, then (6.1) takes the form $u(t) = \nu_f(t)$ where

$$(6.2)\qquad \nu_f(t) = \int_0^t S_\tau * f(t-\tau)\,d\tau$$

or equivalently

$$(6.3)\qquad \nu_f(t) = \int_0^t S_{t-\tau} * f(\tau)\,d\tau.$$

By (3.2), the mapping $[0,\infty[\times E \ni (t,u) \mapsto S_t * u \in E$ is continuous, so the integrands in (6.2) and (6.3) are E-valued functions of (τ,t) continuous in the topology of E on the set $\{(\tau,t) \in \mathbb{R}^2 : 0 \le \tau \le t < \infty\}$. Since E is sequentially complete, the Riemann integrals in (6.2) and (6.3) make sense, and so is the case for all integrands and integrals in the subsequent formulas.

We shall follow [Ph], [Kr, Sec. I.6.2], [Go, Sec. II.1.3] and [Pa, Sec. 4.2]. From (6.2) and (6.3) we infer that whenever $h \neq 0$ and $t, t+h \in [0,\infty[$, then

$$(6.4)\qquad \frac{1}{h}(v_f(t+h) - v_f(t)) = \frac{1}{h}\int_t^{t+h} S_\tau * f(t+h-\tau)\,d\tau + \int_0^t S_\tau * \frac{1}{h}(f(t+h-\tau) - f(t-\tau))\,d\tau$$

and

$$(6.5)\qquad \frac{1}{h}(S_h - \delta) * v_f(t) = \frac{1}{h}(v_f(t+h) - v_f(t)) - \frac{1}{h}\int_t^{t+h} S_{t+h-\tau} * f(\tau)\,d\tau.$$

By continuity of $[0,\infty[\times E \ni (t,u) \mapsto S_t * u \in E$, from (6.2) and (6.3) one concludes that $v_f, v_{f'} \in C([0,\infty[;E)$, and (6.4) and (6.5) imply that the derivative $\frac{d}{dt}v_f(t)$ exists in the topology of E for every $t \in [0,\infty[$, and

$$\frac{d}{dt}v_f(t) = S_t * f(0) + v_{f'}(t) = G * v_f(t) + f(t).$$

From the left equality it follows that if $f \in C^k([0,\infty[;E)$ where $k \in \mathbb{N}$, then for every $l = 1, \ldots, k$ the expression

$$v_{f^{(l)}}(t) + S_t * (f^{(l-1)}(0) + G * f^{(l-2)}(0) + \cdots + (G*)^{l-1} f(0))$$

is equal to $(d/dt)^l v_f(t)$, so that $v_f \in C^k([0,\infty[; E)$.

REFERENCES

[B] H. Bremermann, *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley, 1965.

[E1] R. E. Edwards, *On factor functions*, Pacific J. Math. 5 (1955), 367–378.

[E2] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, 1965.

[E-N] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, 2000.

[G] L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. 85 (1951), 1–62.

[Go] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, Clarendon Press, 1985.

[H-P] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc., Providence, 1957.

[H1] L. Hörmander, *On the theory of general partial differential operators*, Acta Math. 94 (1955), 161–284.

[H2] L. Hörmander, *Linear Partial Differential Operators*, Springer, 1963.

[H3] L. Hörmander, *The Analysis of Partial Differential Operators, Vol. II, Differential Operators with Constant Coefficients*, Springer, 1983.

[K1] J. Kisyński, *Equicontinuity and convergent sequences in the spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$*, Bull. Polish Acad. Sci. Math. 59 (2011), 223–235.

[K2] J. Kisyński, *One-parameter semigroups in the convolution algebra of rapidly decreasing distributions*, Colloq. Math. 128 (2012), 49–68.

[Kr] S. G. Kreĭn, *Linear Differential Equations in Banach Space*, Amer. Math. Soc., 1971; Russian original: Nauka, Moscow, 1963.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[P] I. G. Petrovskiĭ, *Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen*, Bulletin de l'Université d'Etat de Moscou 1 (1938), no. 7, 1–74.

[Ph] R. S. Phillips, *Perturbation theory for semi-groups of linear operators*, Trans. Amer. Math. Soc. 74 (1953), 199–221.

[R] J. Rauch, *Partial Differential Equations*, Springer, 1991.

[S1] L. Schwartz, *Théorie des Distributions*, nouvelle éd., Hermann, Paris, 1966.

[S2] L. Schwartz, *Les équations d'évolution liées au produit de composition*, Ann. Inst. Fourier (Grenoble) 2 (1950), 19–49.

[Y] K. Yosida, *Functional Analysis*, 6th ed., Springer, 1980.

Jan Kisyński
E-mail: jan.kisynski@gmail.com

Received 25 March 2013;
revised 17 September 2013 (5907)