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SOME q-ANALOGS OF CONGRUENCES FOR CENTRAL BINOMIAL SUMS

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Abstract. We establish q-analogs for four congruences involving central binomial coefficients. The q-identities necessary for this purpose are shown via the q-WZ method.

1. Introduction. Recently, a number of papers have appeared concerning congruences for central binomial sums (see the references). Here we would like to draw the reader's attention to one aspect of the matter which has been partly neglected so far: q-analogs. In [5, 11], the authors identified a first group of such congruences which have a q-counterpart. Among them we mention: for any prime $p > 2$,

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \qquad \sum_{k=0}^{p-1} q^k \begin{bmatrix}2k\\ k\end{bmatrix}_q \equiv \left(\frac{p}{3}\right) q^{\lfloor \frac{p^2}{3}\rfloor},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k} \equiv \left(\frac{p}{5}\right) \pmod{p}, \qquad \sum_{k=0}^{p-1}(-1)^k q^{-\binom{k+1}{2}} \begin{bmatrix}2k\\ k\end{bmatrix}_q \equiv \left(\frac{p}{5}\right) q^{-\lfloor \frac{p^4}{5}\rfloor},$$

$$\sum_{k=0}^{p-1}\frac{1}{2^k}\binom{2k}{k} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}, \qquad \sum_{k=1}^{p-1} \frac{q^k}{(-q;q)_k} \begin{bmatrix}2k\\ k\end{bmatrix}_q \equiv (-1)^{\frac{p-1}{2}} q^{\lfloor \frac{p^2}{4}\rfloor},$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol and all q-congruences are modulo $[p]_q$. It has been conjectured in [5] that the first q-congruence holds modulo $[p]_q^2$, and we claim that the same can be said for the third one. However, in this short note, we are not going to refine these q-congruences. Instead, we will present a few more examples of this phenomenon. More precisely, we show that the congruences

$$\sum_{k=1}^{p-1}\frac{(1/2)^k}{k}\binom{2k}{k} \equiv -\frac{3}{2}\sum_{k=1}^{p-1}\frac{(-1/2)^k}{k}\binom{2k}{k}^{-1} \equiv Q_p(2) \pmod{p}, \tag{1}$$

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$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k}\equiv 0 \pmod{p^2}, \tag{2}$$

$$\frac{5}{2}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv -\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^3}, \tag{3}$$

where $p>5$ is a prime and $Q_p(2)=(2^{p-1}-1)/p$ is the usual Fermat quotient, have as q-analogs respectively

$$\sum_{k=1}^{p-1}\frac{q^k}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^k(1+q^k+q^{2k})}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q^{-1} \equiv Q_p(2;q) \pmod{[p]_q}, \tag{4}$$

$$\sum_{k=1}^{p-1}\frac{(1+q^k+q^{2k})\,q^{-\binom{k}{2}}}{(1+q^k)^2[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv\frac{[p]_q(p^2-1)(1-q)^2}{24} \pmod{[p]_q^2}, \tag{5}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k(1+3q^k+q^{2k})\,q^{-\binom{k}{2}}}{(1+q^k)[k]_q^2}\begin{bmatrix}2k\\k\end{bmatrix}_q \equiv -\sum_{k=1}^{p-1}\frac{q^k}{[k]_q^2}-\frac{[p]_q^2(p^4-1)(1-q)^4}{240} \pmod{[p]_q^3}, \tag{6}$$

where $Q_p(2;q)=((-q;q)_{p-1}-1)/[p]_q$. Proofs of (1)–(3) can be found in [12, Theorem 3.1] ((2) appeared first in [10]).

We are optimistically hopeful that there are plenty of interesting q-analogs to discover. For example, recently in [8], the authors proved that for $0<q<1$,

$$\sum_{k=1}^{\infty}\frac{(1+2q^k)\,q^{k^2}}{[k]_q^2}\begin{bmatrix}2k\\k\end{bmatrix}_q^{-1}=\sum_{k=1}^{\infty}\frac{q^k}{[k]_q^2}.$$

By letting $q\to 1$, it gives a well known series identity which happens to have a congruence version: in [12] we showed that for any prime $p>3$,

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\binom{2k}{k}^{-1}\equiv -\frac{1}{6}\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^3}.$$

Is there a q-analog for the above congruence?

2. Notations and preliminary results. The first two results of this section yield a family of q-analogs of the classical congruence for the harmonic sums: for any prime $p>d+2$ where d is a positive integer,

$$H_{p-1}(d):=\sum_{k=1}^{p-1}\frac{1}{k^d}\equiv\begin{cases}0 \pmod{p^2} & \text{if } d \text{ is odd},\\ 0 \pmod{p} & \text{if } d \text{ is even}.\end{cases}$$

This family depends on two integer parameters a, b and it concerns the sum

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

Several special cases have already been discussed by numerous authors (see [3, 4, 7, 9]). In particular, K. Dilcher [4] found a determinant expression in the case when $a = 1$ and $b \in \{0, 1\}$. We point out that, in this paper, two rational functions in q are congruent modulo $[p]_q^r$ for $r \geq 1$ if the numerator of their difference is congruent to 0 modulo $[p]_q^r$ in the polynomial ring $\mathbb{Z}[q]$ and the denominator is relatively prime to $[p]_q$.

THEOREM 2.1. *For any prime $p > 2$, if a, b, d are integers such that $a, d > 0$, $b \geq 0$ and $\gcd(a, p) = 1$ then*

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^d} \equiv \frac{(1-q)^d}{p^d} \cdot \left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{r_0 + sp}{d} - \sum_{s=0}^{d} (-1)^s \binom{d}{s} \binom{sp}{2d} \right) \pmod{[p]_q} \tag{7}$$

where $r_0 \equiv -b/a \pmod{p}$ such that $r_0 \in \{0, 1, \ldots, p-1\}$ and

$$c_s = \sum_{k=0}^{s} (-1)^{s-k} \binom{r_0 + kp + d - 1}{d-1} \binom{d}{s-k}.$$

Proof. Let q be a p-root of unity such that $q \neq 1$. Since

$$\sum_{k=1}^{p-1} q^{k(aj+b)} = -1 + \begin{cases} p & \text{if } p \mid (aj+b), \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{ak}z)^d} &= \sum_{k=1}^{p-1} q^{bk} \sum_{j\geq 0} \binom{j+d-1}{d-1} q^{akj} z^j \\
&= \sum_{j\geq 0} \binom{j+d-1}{d-1} z^j \sum_{k=1}^{p-1} q^{k(aj+b)} \\
&= p \sum_{s\geq 0} \binom{r_0 + sp + d - 1}{d-1} z^{r_0+sp} - \frac{1}{(1-z)^d} \\
&= \frac{p \sum_{s=0}^{d-1} c_s z^{r_0+sp}}{(1-z^p)^d} - \frac{1}{(1-z)^d}.
\end{aligned}$$

Let $z = 1 + w$. Then

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{ak})^d} = \lim_{w\to 0} \frac{p\sum_{s=0}^{d-1} c_s(1+w)^{r_0+sp} - \left(\frac{1-(1+w)^p}{-w}\right)^d}{(1-(1+w)^p)^d}$$

$$= \lim_{w\to 0} \frac{p\sum_{s=0}^{d-1} c_s\binom{r_0+sp}{d} w^d - (-1)^d \sum_{s=0}^{d}(-1)^s\binom{d}{s}\binom{sp}{2d} w^d + o(w^d)}{(-pw+o(w))^d}$$

$$= \frac{1}{p^d}\left((-1)^d p\sum_{s=0}^{d-1} c_s\binom{r_0+sp}{d} - \sum_{s=0}^{d}(-1)^s\binom{d}{s}\binom{sp}{2d}\right). \blacksquare$$

The following special cases are worth mentioning. By letting $d = 1, 2, 3$ in (7), we obtain these q-congruences which hold modulo $[p]_q$:

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q} \equiv (1-q)\left(\frac{p-1}{2} - r_0\right), \tag{8}$$

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^2} \equiv (1-q)^2\left(-\frac{(p-1)(p-5)}{12} + \frac{r_0(p-2-r_0)}{2}\right), \tag{9}$$

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{[ak]_q^3} \equiv -\frac{(1-q)^3}{24}\big(3(p-1(p-3) + 2r_0(p^2-9p+12-3r_0(p-3)+2r_0^2\big). \tag{10}$$

THEOREM 2.2. *Let $b, \bar{b}, a, d$ be non-negative integers such that $ad = b + \bar{b} > 0$. Then for any prime $p > 2$ such that* $\gcd(a, p) = 1$,

$$\sum_{k=1}^{p-1} \frac{(-1)^{d-1}q^{bk} + q^{\bar{b}k}}{[ak]_q^d} \equiv b(1-q)[p]_q \sum_{k=1}^{p-1} \frac{q^{\bar{b}k}}{[ak]_q^d} - ad[p]_q \sum_{k=1}^{p-1} \frac{q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}, \tag{11}$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^k((-1)^d q^{bk} + q^{\bar{b}k})}{[ak]_q^d} \equiv b(1-q)[p]_q \sum_{k=1}^{p-1} \frac{(-1)^k q^{\bar{b}k}}{[ak]_q^d} - ad[p]_q \sum_{k=1}^{p-1} \frac{(-1)^k q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}. \tag{12}$$

Moreover, let $b_1, \bar{b}_1, b_2, \bar{b}_2$ be non-negative integers. If $d_1 = b_1 + \bar{b}_1 > 0$ and $d_2 = b_2 + \bar{b}_2 > 0$ then

$$\sum_{1\le j<k\le p-1} \frac{q^{b_1 j+b_2 k}+(-1)^{d_1+d_2} q^{\bar{b}_1 j+\bar{b}_2 k}}{[j]_q^{d_1}[k]_q^{d_2}} \equiv \sum_{k=1}^{p-1} \frac{q^{b_1 j}}{[j]_q^{d_1}} \cdot \sum_{k=1}^{p-1} \frac{q^{b_2 k}}{[k]_q^{d_2}} - \sum_{k=1}^{p-1} \frac{q^{(b_1+b_2)k}}{[k]_q^{d_1+d_2}} \pmod{[p]_q}. \tag{13}$$

Proof. As regards (11) and (12), it suffices to note that

$$\begin{aligned} \frac{(-1)^d q^{b(p-k)}}{[a(p-k)]_q^d} &= \frac{(-1)^d q^{b(p-k)+adk}}{([ap]_q-[ak]_q)^d} = \frac{q^{bp+\bar{b}k}}{[ak]_q^d(1-[ap]_q/[ak]_q)^d} \\ &\equiv \frac{q^{\bar{b}k}(1-b(1-q)[p]_q)}{[ak]_q^d}\left(1+ad\frac{[p]_q}{[ak]_q}\right) \\ &\equiv \frac{q^{\bar{b}k}}{[ak]_q^d} - \frac{b(1-q)[p]_q q^{\bar{b}k}}{[ak]_q^d} + \frac{ad[p]_q q^{\bar{b}k}}{[ak]_q^{d+1}} \pmod{[p]_q^2}. \end{aligned} \tag{14}$$

Moreover

$$\begin{aligned} &\sum_{k=1}^{p-1} \frac{q^{b_1 j}}{[j]_q^{d_1}} \cdot \sum_{k=1}^{p-1} \frac{q^{b_2 k}}{[k]_q^{d_2}} - \sum_{k=1}^{p-1} \frac{q^{(b_1+b_2)k}}{[k]_q^{d_1+d_2}} \\ &\qquad = \sum_{1\le j<k\le p-1} \frac{q^{b_1 j+b_2 k}}{[j]_q^{d_1}[k]_q^{d_2}} + \sum_{1\le k<j\le p-1} \frac{q^{b_1 j+b_2 k}}{[j]_q^{d_1}[k]_q^{d_2}} \end{aligned}$$

and by (14) we get

$$\begin{aligned} \sum_{1\le k<j\le p-1} \frac{q^{b_1 j+b_2 k}}{[j]_q^{d_1}[k]_q^{d_2}} &= \sum_{1\le j<k\le p-1} \frac{q^{b_1(p-j)+b_2(p-k)}}{[p-j]_q^{d_1}[p-k]_q^{d_2}} \\ &\equiv \sum_{1\le j<k\le p-1} \frac{(-1)^{d_1+d_2} q^{\bar{b}_1 j+\bar{b}_2 k}}{[j]_q^{d_1}[k]_q^{d_2}} \pmod{[p]_q}. \end{aligned}$$

Hence the proof of (13) is complete. ∎

By letting $a=1$, $d=1$ and $b=0$ in (11), and by using (7), we easily find [9, Theorem 1]: for any prime $p>3$:

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{[k]_q} &= \frac{1}{2}\sum_{k=1}^{p-1} \frac{1-q^k}{[k]_q}\ \frac{1}{2}\sum_{k=1}^{p-1} \frac{1+q^k}{[k]_q} \\ &\equiv \frac{(1-q)(p-1)}{2} + \frac{(p^2-1)(1-q)^2[p]_q}{24} \pmod{[p]_q^2}. \end{aligned}$$

In a similar way, for $a=1$, $d=3$ and $b=1$, (11) yields

$$\sum_{k=1}^{p-1}\frac{q^k+q^{2k}}{[k]_q^3}\equiv(1-q)[p]_q\sum_{k=1}^{p-1}\frac{q^{2k}}{[k]_q^3}-3[p]_q\sum_{k=1}^{p-1}\frac{q^{2k}}{[k]_q^4}\tag{15}$$
$$\equiv-\frac{[p]_q(1-q)^4(p^4-1)}{240}\pmod{[p]_q^2}.$$

In order to show the q-congruences stated in the introduction, we need *suitable* q-identities. Such identities are not easy to find, but once they are guessed correctly, hopefully they can be proved via the q-WZ method (see for example [6, 13]).

For $n\geq k\geq 0$, a pair $(F(n,k),G(n,k))$ is called *q-WZ pair* if

$$\frac{F(n+1,k)}{F(n,k)},\qquad\frac{F(n,k+1)}{F(n,k)},\qquad\frac{G(n+1,k)}{G(n,k)},\qquad\frac{G(n,k+1)}{G(n,k)}$$

are all rational functions of q^n and q^k, and

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k).$$

Let

$$S(n)=\sum_{k=0}^{N-1}F(n,k).$$

Then

$$\begin{aligned}S(n+1)-S(n)&=F(n+1,n)+\sum_{k=0}^{n-1}(F(n+1,k)-F(n,k))\\&=F(n+1,n)+\sum_{k=0}^{n-1}(G(n,k+1)-G(n,k))\\&=F(n+1,n)+G(n,n)-G(n,0)\end{aligned}$$

and, by summing over n from 0 to $N-1$, we get the identity

$$\sum_{k=0}^{N-1}F(N,k)=\sum_{n=0}^{N-1}(F(n+1,n)+G(n,n))-\sum_{n=0}^{N-1}G(n,0)\tag{16}$$

which can be considered as the finite form of [6, Theorem 7.3].

The q-identities we are interested in involve the *Gaussian q-binomial coefficients*

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}(q;q)_n(q;q)_k^{-1}(q;q)_{n-k}^{-1} & \text{if } 0\leq k\leq n,\\ 0 & \text{otherwise,}\end{cases}$$

where $(a;q)_n=\prod_{j=0}^{n-1}(1-aq^j)$ (note that $\begin{bmatrix}n\\k\end{bmatrix}_q$ is a polynomial in q). The next lemma allows us to reduce a special class of q-binomial coefficients modulo a power of $[p]_q$.

Lemma 2.3. *Let p be a prime and let a be a positive integer. For $k = 1, \dots, p-1$ we have*

$$\begin{bmatrix} ap-1 \\ k \end{bmatrix}_q \equiv (-1)^k q^{-\binom{k+1}{2}} \left(1 - a[p]_q \sum_{j=1}^{k} \frac{1}{[j]_q}\right) \pmod{[p]_q^2}, \tag{17}$$

$$\begin{bmatrix} p-1+k \\ k \end{bmatrix}_q \equiv \frac{[p]_q}{[k]_q} \left(1 + [p]_q \sum_{j=1}^{k-1} \frac{q^j}{[j]_q}\right) \pmod{[p]_q^3}, \tag{18}$$

$$\begin{aligned} \begin{bmatrix} p-1+k \\ k \end{bmatrix}_q \begin{bmatrix} p-1 \\ k \end{bmatrix}_q^{-1} &\equiv \frac{(-1)^k q^{\binom{k+1}{2}} [p]_q}{[k]_q} \\ &\quad \cdot \left(1 + \frac{[p]_q}{[k]_q} + [p]_q \sum_{j=1}^{k-1} \frac{1+q^j}{[j]_q}\right) \pmod{[p]_q^3}. \end{aligned} \tag{19}$$

Proof. Since $[ap]_q \equiv a[p]_q \pmod{[p]_q^2}$, it follows that

$$\begin{aligned} \begin{bmatrix} ap-1 \\ k \end{bmatrix}_q &= (-1)^k q^{-\binom{k+1}{2}} \prod_{j=1}^{k} \left(1 - \frac{[ap]_q}{[j]_q}\right) \\ &\equiv (-1)^k q^{-\binom{k+1}{2}} \left(1 - a[p]_q \sum_{j=1}^{k} \frac{1}{[j]_q}\right) \pmod{[p]_q^2}. \end{aligned}$$

Moreover

$$\begin{bmatrix} p-1+k \\ k \end{bmatrix}_q = \frac{[p]_q}{[k]_q} \prod_{j=1}^{k-1} \left(1 + \frac{q^j [p]_q}{[j]_q}\right) \equiv \frac{[p]_q}{[k]_q} \left(1 + [p]_q \sum_{j=1}^{k-1} \frac{q^j}{[j]_q}\right) \pmod{[p]_q^3}.$$

Congruences (17) and (18) easily yield (19). ■

It should be noted that when p is an odd prime, by using (18) for $k = p-1$, we recover the q-congruence [3, (3.2)]:

$$\begin{aligned} \begin{bmatrix} ap-1 \\ p-1 \end{bmatrix}_q &\equiv q^{-\binom{p}{2}} \left(1 - a[p]_q \sum_{j=1}^{p-1} \frac{1}{[j]_q}\right) \\ &\equiv q^{-\binom{p}{2}} \left(1 - \frac{a[p]_q (p-1)(1-q)}{2}\right) \equiv q^{(a-1)\binom{p}{2}} \pmod{[p]_q^2}. \end{aligned} \tag{20}$$

3. Proof of (4). By [2, (5.17)] (see [5, (4.1)] for a generalization), if n is odd then

$$\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{\binom{n-k}{2}}}{(-q;q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q = 0.$$

Hence for $n=p$ we have

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}q^{\binom{p-k}{2}-\binom{p}{2}}}{(-q;q)_k[k]_q}\begin{bmatrix}p-1\\k-1\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q$$
$$=\frac{1}{(-q;q)_{p-1}[p]_q}\left((-q;q)_{p-1}-q^{-\binom{p}{2}}\begin{bmatrix}2p-1\\p-1\end{bmatrix}_q\right).$$

By (17) and (20) we get

$$\sum_{k=1}^{p-1}\frac{q^k}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv\sum_{k=1}^{p-1}\frac{q^{-pk+k}}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q$$
$$\equiv\frac{(-q;q)_{p-1}-1}{(-q;q)_{p-1}[p]_q}\equiv Q_p(2;q)\pmod{[p]_q},$$

and the first congruence is proved.

As regards the second one, we take

$$F(n,k)=\frac{(-1)^k}{(-q;q)_n[k+1]_q}\begin{bmatrix}n+k+1\\k+1\end{bmatrix}_q^{-1}\quad\text{and}\quad G(n,k)=\frac{q^{n+1}F(n,k)}{1+q^{n+1}}.$$

This pair can be found in [1, Subsection 2.1] in connection with the irrationality of the q-series

$$\mathrm{Ln}_q(2):=\sum_{k=1}^{\infty}\frac{(-1)^k}{q^n-1}$$

for $|q|\notin\{0,1\}$. Hence by (16) we obtain the identity

$$\sum_{k=1}^{n}\frac{(-1)^k(1+q^k+q^{2k})}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q^{-1}\tag{21}$$
$$=\frac{1}{(-q;q)_n}\sum_{k=1}^{n}\frac{(-1)^k}{[k]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q^{-1}-\sum_{k=1}^{n}\frac{q^k}{(-q;q)_k[k]_q}.$$

Let $n=p-1$. Now by (17) and [7, (1.5)],

$$\frac{1}{(-q;q)_{p-1}}\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q^{-1}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{[p]_q}\left(1-[p]_q\sum_{j=1}^{k-1}\frac{q^j}{[j]_q}\right)$$
$$\equiv-\sum_{k=1}^{p-1}(-1)^k\sum_{j=1}^{k-1}\frac{q^j}{[j]_q}\equiv-\sum_{j=1}^{p-1}\frac{q^{2j-1}}{[2j-1]_q}$$
$$\equiv\frac{(p-1)(1-q)}{2}-\sum_{k=1}^{p-1}\frac{1}{[j]_q}+\sum_{k=1}^{(p-1)/2}\frac{1}{[2j]_q}$$
$$\equiv-Q_p(2;q)\pmod{[p]_q}.$$

By the q-binomial theorem,

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\prod_{j=0}^{k-1}(x-q^j)=x^n$$

and for $n=p$, $x=-1$ together with (17), we get

$$\sum_{k=1}^{p-1}\frac{q^{-\binom{k}{2}}(-q;q)_{k-1}}{[k]_q}\equiv\sum_{k=1}^{p-1}\frac{(-1)^{k-1}(-q;q)_{k-1}}{[k]_q}\begin{bmatrix}p-1\\k-1\end{bmatrix}_q=-Q_p(2;q).$$

Hence (see the *dual* congruence [7, (5.4)])

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{q^k}{(-q;q)_k[k]_q}&\equiv\sum_{k=1}^{p-1}\frac{q^{p-k}}{(-q;q)_{p-k}[p-k]_q}\\&\equiv\sum_{k=1}^{p-1}\frac{q^{-\binom{k}{2}}(-q;q)_{k-1}}{[k]_q}\equiv-Q_p(2;q)\pmod{[p]_q}\end{aligned}$$

where we used

$$[p-k]_q=-q^{-k}[k]_q\quad\text{and}\quad(-q;q)_{p-k}^{-1}\equiv q^{-\binom{k}{2}}(-q;q)_{k-1}\pmod{[p]_q}.$$

Therefore, by identity (21),

$$\sum_{k=1}^{p-1}\frac{(-1)^k(1+q^k+q^{2k})}{(-q;q)_k[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q^{-1}\equiv-2Q_p(2;q)\pmod{[p]_q}$$

and we are done.

4. Proof of (5). Let

$$\begin{aligned}F(n,k)&=\frac{q^{-\binom{k+1}{2}}}{[k+1]_q}\begin{bmatrix}n+k+2\\n+1\end{bmatrix}_q\begin{bmatrix}n+1\\k+1\end{bmatrix}_q^{-1},\\G(n,k)&=-\frac{q^{n+2}(1-q^{k+1})(1-q^{n+1-k})F(n,k)}{(1+q^{n+2})(1-q^{n+2})^2}.\end{aligned}$$

Then (16) gives the identity

$$\sum_{k=1}^{n}\frac{(1+q^k+q^{2k})\,q^{-\binom{k}{2}}}{(1+q^k)^2[k]_q}\begin{bmatrix}2k\\k\end{bmatrix}_q=\sum_{k=1}^{n}\frac{q^{-\binom{k}{2}}}{[k]_q}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q^{-1}-\sum_{k=1}^{n}\frac{q^k}{[2k]_q}.\tag{22}$$

Let $n=p-1$. Then by (19) and (12), we obtain

$$\sum_{k=1}^{p-1}\frac{q^{-\binom{k}{2}}}{[k]_q}\begin{bmatrix}p-1+k\\k\end{bmatrix}_q\begin{bmatrix}p-1\\k\end{bmatrix}_q^{-1}\equiv[p]_q\sum_{k=1}^{p-1}\frac{(-1)^kq^k}{[k]_q^2}\equiv0\pmod{[p]_q^2}.$$

Moreover, (11) and (9) imply that

$$\sum_{k=1}^{p-1}\frac{q^k}{[2k]_q}\equiv -[p]_q\sum_{k=1}^{p-1}\frac{q^k}{[2k]_q^2}\equiv -\frac{[p]_q(p^2-1)(1-q)^2}{24} \pmod{[p]_q^2},$$

and (5) follows easily from (22).

Note that by letting $q\to 1$ in (22) we obtain the identity

$$\frac{3}{4}\sum_{k=1}^{n}\frac{1}{k}\binom{2k}{k}=\sum_{k=1}^{n}\frac{1}{k}\binom{n+k}{k}\binom{n}{k}^{-1}-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k},$$

which can be exploited to prove an improvement of [12, Theorem 4.2]:

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k}\equiv -\frac{8}{3}H_{p-1}(1)+2p^4B_{p-5} \pmod{p^5}$$

for any prime $p>3$.

5. Proof of (6). By taking

$$F(n,k)=\frac{(-1)^k q^{-\binom{k+1}{2}}(1+q^{k+1})}{[k+1]_q^2}\begin{bmatrix}n+k+2\\ n+1\end{bmatrix}_q\begin{bmatrix}n+1\\ k+1\end{bmatrix}_q^{-1},$$

$$G(n,k)=\frac{q^{n+2}(1-q^{k+1})^2(1-q^{n+1-k})F(n,k)}{(1+q^{k+1})(1-q^{n+2})^3},$$

(16) yields the identity

$$\sum_{k=1}^{n}\frac{(-1)^k(1+3q^k+q^{2k})q^{-\binom{k}{2}}}{(1+q^k)[k]_q^2}\begin{bmatrix}2k\\ k\end{bmatrix}_q =\sum_{k=1}^{n}\frac{(-1)^k(1+q^k)q^{-\binom{k}{2}}}{[k]_q^2}\begin{bmatrix}n+k\\ k\end{bmatrix}_q\begin{bmatrix}n\\ k\end{bmatrix}_q^{-1}-\sum_{k=1}^{n}\frac{q^k}{[k]_q^2}. \tag{23}$$

Let $n=p-1$. Now by (19),

$$\sum_{k=1}^{p-1}\frac{(-1)^k(1+q^k)q^{-\binom{k}{2}}}{[k]_q^2}\begin{bmatrix}p-1+k\\ k\end{bmatrix}_q\begin{bmatrix}p-1\\ k\end{bmatrix}_q^{-1}$$

$$\equiv [p]_q\sum_{k=1}^{p-1}\frac{q^k+q^{2k}}{[k]_q^3}+[p]_q^2\sum_{k=1}^{p-1}\frac{q^k+q^{2k}}{[k]_q^4}$$

$$+[p]_q^2\sum_{1\le j<k\le p-1}\frac{(1+q^j)(q^k+q^{2k})}{[j]_q[k]_q^3} \pmod{[p]_q^3}.$$

Then the q-congruence (6) follows from (23) by using (15) and

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{q^k+q^{2k}}{[k]_q^4} &\equiv -\sum_{1\le j<k\le p-1}\frac{(1+q^j)(q^k+q^{2k})}{[j]_q[k]_q^3}\\ &\equiv \frac{(1-q)^4(p^2-1)(p^2-4)}{360} \pmod{[p]_q},\end{aligned}$$

which is a straightforward application of (7) and (13).

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