COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 1

ON CERTAIN FOUR-DIMENSIONAL ALMOST KÄHLER MANIFOLDS

BY

WŁODZIMIERZ JELONEK (Kraków)

Abstract. We study four-dimensional almost Kähler manifolds (M, g, J) which admit an opposite almost Kähler structure.

0. Introduction. V. Apostolov and T. Draghici [A-D] studied almost Kähler 4-manifolds satisfying Gray's second curvature condition (G_2). They proved that such manifolds admit an opposite Kähler structure $\bar{J}$ such that $(M, g, \bar{J})$ has $\bar{J}$-invariant Ricci tensor and symmetric $*$-Ricci tensor. T. Oguro, K. Sekigawa and Y. Yamada [O-S-Y] proved that every strictly almost Kähler Einstein and weakly $*$-Einstein 4-manifold admit two complementary foliations and have to be Ricci flat. Nurowski and Przanowski [N-P] gave explicit examples of Einstein and weakly $*$-Einstein strictly almost Kähler 4-manifolds. The examples admit two opposite almost Kähler structures, one of which is Kähler.

A. Gray [G] introduced the notion of $\mathcal{A}$-manifolds. Davidov and Muškarov [D-M] gave examples of six-dimensional almost Kähler non-Kähler structures on the twistor bundle over self-dual Einstein manifolds with negative scalar curvature. Davidov, Grantcharov and Muškarov [D-G-M] proved that these examples are $\mathcal{A}$-manifolds. These examples were generalized by Alexandrov, Grantcharov and Ivanov [A-G-I] and later by the author [J-3]. All these examples are proper $\mathcal{A}$-manifolds and have Hermitian Ricci tensor. On the other hand, A. Gray (see [G], [S-V]) proved that every Kähler $\mathcal{A}$-manifold has parallel Ricci tensor. If we consider almost Kähler manifolds then a similar result does not hold in general even if we assume that the Ricci tensor is Hermitian.

The aim of this note is to give a simple proof that every four-dimensional almost Kähler $\mathcal{A}$-manifold with Hermitian Ricci tensor has parallel Ricci tensor and to characterize 4-manifolds (M, g) admitting two opposite almost

2000 *Mathematics Subject Classification*: 53C15, 53B20.

Key words and phrases: almost Kähler manifold, almost Kähler structure.

The Editorial Committee apologizes to the author and readers for the unusually long delay in the publication of this paper.

Kähler structures. In particular we prove that every strictly almost Kähler 4-manifold (M, g, J) with J-invariant Ricci tensor and symmetric $*$-Ricci tensor admits an opposite almost Kähler structure $\bar{J}$. We also prove that an almost Kähler surface whose Ricci tensor is Hermitian and has constant eigenvalues admits an opposite almost Kähler structure. In particular every Kähler surface whose Ricci tensor has constant eigenvalues admits an opposite almost Kähler structure. Our results are connected with the question of Blair and Ianuş (see [B-I], [D-1]): "Is it true that every four-dimensional almost Kähler compact manifold with Hermitian Ricci tensor is Kähler?" and with the Goldberg conjecture (see [S-2]).

The present work was inspired by the example by Nurowski and Przanowski [N-P] of a strictly almost Kähler Einstein 4-manifold admitting an opposite Kähler structure and by the paper [O-S-Y] by Oguro, Sekigawa and Yamada. After writing the paper the author has learned that similar results were obtained by Apostolov, Armstrong and Draghici in [A-A-D].

1. Preliminaries. Let (M, g) be a smooth, connected and oriented Riemannian manifold. For a tensor $T(X_1, \ldots, X_k)$ we define a tensor $\nabla T(X_0, X_1, \ldots, X_k)$ by $\nabla T(X_0, X_1, \ldots, X_k) = \nabla_{X_0} T(X_1, \ldots, X_k)$. By a *Killing tensor* on M we mean an endomorphism $S \in \operatorname{End}(TM)$ satisfying the following conditions:

(a) $g(SX, Y) = g(X, SY)$ for all $X, Y \in TM$,
(b) $g(\nabla S(X, X), X) = 0$ for all $X \in TM$.

We also write $S \in \mathcal{A}$ if S is a Killing tensor. We call S a *proper* Killing tensor if $\nabla S \neq 0$. A Riemannian manifold is called an $\mathcal{A}$-*manifold* (after Gray [G]) if its Ricci tensor is a Killing tensor. An $\mathcal{A}$-manifold is called *proper* if its Ricci tensor is a proper Killing tensor.

Let (M, g, J) be an almost Hermitian manifold. We say that (M, g, J) is an *almost Kähler manifold* if its Kähler form $\Omega(X, Y) = g(JX, Y)$ is closed ($d\Omega = 0$). In the following we shall consider four-dimensional almost Kähler manifolds (M, g, J). Such manifolds are always oriented and we choose an orientation in such a way that Ω is a self-dual form (i.e. $\Omega \in \bigwedge^+ M$). The vector bundle of self-dual forms admits a decomposition

$$\bigwedge\nolimits^+ M = \mathbb{R}\Omega \oplus LM \tag{1.1}$$

where LM denotes the bundle of real J-skew-invariant 2-forms (i.e. $LM = \{\Phi \in \bigwedge M : \Phi(JX, JY) = -\Phi(X, Y)\}$). The bundle LM is a complex line bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X, Y) = -\Phi(JX, Y)$. For a four-dimensional almost Kähler manifold the covariant derivative of the Kähler form Ω is locally expressed by

$$\nabla\Omega = \alpha \otimes \Phi - \mathcal{J}\alpha \otimes \mathcal{J}\Phi \tag{1.2}$$

where $\mathcal{J}\alpha(X) = -\alpha(JX)$. The Ricci tensor ϱ of an almost Hermitian manifold (M,g,J) is said to be *Hermitian* (or *J-invariant*) if $\varrho(X,Y) = \varrho(JX,JY)$ for all $X,Y \in \mathfrak{X}(M)$. In what follows we shall consider Killing tensors with two eigenvalues λ,μ. We denote by D_λ, D_μ the corresponding eigendistributions $D_\lambda = \ker(S-\lambda\,\mathrm{Id})$ and $D_\mu = \ker(S-\mu\,\mathrm{Id})$.

Let D be an oriented p-dimensional distribution in (M,g) and let $\{E_1,\dots,E_p\}$ be an oriented orthonormal basis of D. Then the *characteristic form* of D is the p-form ω defined by $\omega(X_1,\dots,X_p) = \det(g(E_i,X_j))$.

A distribution D is called *involutive* or a *foliation* if $[X,Y] \in \Gamma(D)$ for all local sections $X,Y \in \Gamma(D)$. A foliation D is called *minimal* if every leaf of D is a minimal submanifold of (M,g), i.e. the trace of its second fundamental form (the mean curvature) vanishes. Analogously if D is a p-dimensional distribution then its second fundamental form α (not symmetric in general) is given by the formula

$$\alpha(X,Y) = \nabla_X Y - \pi(\nabla_X Y) \quad \text{for any } X,Y \in \Gamma(D).$$

A distribution D is called *minimal* if $\mathrm{tr}_{g'}\,\alpha = 0$ where $g' = g_{|D}$. In the following we shall assume that all almost complex structures we consider are orthogonal with respect to g, i.e. $g(X,Y) = g(JX,JY)$ for all $X,Y \in \mathfrak{X}(M)$.

An almost Kähler 4-manifold (M,g,J) is said to have an *opposite almost Kähler structure* if it admits an orthogonal almost Kähler structure $\bar{J}$ with anti-self dual Kähler form $\bar{\Omega}$. For any almost Hermitian 4-manifold the following formula holds:

$$\tfrac{1}{2}(\varrho(X,Y)+\varrho(JX,JY))-\tfrac{1}{2}(\varrho^*(X,Y)+\varrho^*(Y,X)) = \tfrac{1}{4}(\tau-\tau^*)g(X,Y) \tag{1.3}$$

where ϱ^* is the $*$-*Ricci tensor* defined by

$$\varrho^*(X,Y) = \tfrac{1}{2}\,\mathrm{tr}\{Z \mapsto R(X,JY)JZ\} \tag{1.4}$$

where $R(X,Y)Z = ([\nabla_X,\nabla_Y] - \nabla_{[X,Y]})Z$ and $\tau^* = \mathrm{tr}_g\,\varrho^*$.

The first Chern class of (M,g,J) is represented by the form γ defined by

$$8\pi\gamma = -\phi + 2\psi \tag{1.5}$$

where

$$\phi(X,Y) = \mathrm{tr}(Z \mapsto J\nabla_X J \circ \nabla_Y JZ), \tag{1.6}$$

$$\psi(X,Y) = \mathrm{tr}(Z \mapsto R(X,Y)JZ). \tag{1.7}$$

Note that $\psi(X,Y) = -2\varrho^*(X,JY)$. We denote by $\mathcal{D}$ the *nullity distribution* of (M,g,J) defined by $\mathcal{D} = \{X \in TM : \nabla_X J = 0\}$. For an almost Kähler manifold it follows from (1.2) that $\mathcal{D}$ is J-invariant and $\dim\mathcal{D} = 2$ in $M_0 = \{x \in M : \nabla J_x \neq 0\}$. We shall call the nullity distribution *integrable* if $\mathcal{D}_{|M_0}$ is integrable.

The curvature tensor R of a four-dimensional manifold (M,g) determines an endomorphism $\mathcal{R}$ of the bundle $\bigwedge M$ defined by $g(\mathcal{R}(X\wedge Y), Z\wedge W) = \mathcal{R}(X\wedge Y, Z\wedge W) = -R(X,Y,Z,W) = -g(R(X,Y)Z,W)$. Note that $\varrho^* =$

$\mathcal{J}\mathcal{R}(\Omega)$ and $\tau^* = 2\mathcal{R}(\Omega,\Omega)$. Set $\mathcal{R}_{\bigwedge^+ M} = p_{\bigwedge^+ M} \circ \mathcal{R}_{|\bigwedge^+ M}$. Then $\operatorname{tr}\mathcal{R}_{\bigwedge^+ M} = \tau/4$.

2. Almost Kähler four-dimensional manifolds with Hermitian Ricci tensor. We start by recalling some properties of $\mathcal{A}$-manifolds and Killing tensors (see [J-1], [J-2]):

THEOREM 1. *Let S be a Killing tensor on (M,g) with exactly two eigenvalues λ,μ and a constant trace. Then λ and μ are constant on M. The distributions D_λ, D_μ are both integrable if and only if $\nabla S = 0$.*

COROLLARY 2. *Let (M,g) be an $\mathcal{A}$-manifold whose Ricci tensor ϱ has exactly two eigenvalues λ,μ. Then λ and μ are constant. The Ricci tensor ϱ is parallel if and only if both eigendistributions of ϱ are integrable.*

THEOREM 3. *Let S be a self-adjoint $(1,1)$-tensor $(g(SX,Y) = g(SY,X))$ with two constant eigenvalues λ,μ. Then S is a Killing tensor field if and only if*

$$\nabla S(X,X) = 0 \tag{2.1}$$

for all $X \in D_\lambda$ and all $X \in D_\mu$, or equivalently, if $\nabla_X X \in \Gamma(D_a)$ for all local sections $X \in \Gamma(D_a)$ where $a \in \{\lambda,\mu\}$.

We shall prove (cf. also [P-S]):

PROPOSITION 1. *Let (M,g) be a four-dimensional $\mathcal{A}$-manifold. Assume that (M,g,J) is an almost Kähler manifold with Hermitian Ricci tensor. Then either (M,g) is an Einstein space, or (M,g,J) is a Kähler manifold. If (M,g) is not Einstein and is complete then its covering space is a product of two Riemannian surfaces of constant curvature.*

Proof. Write $\varrho(X,Y) = g(SX,Y)$ where S is the Ricci endomorphism of (M,g). Since $\varrho(JX,JY) = \varrho(X,Y)$ we have $S\circ J = J\circ S$. Hence S has at most two eigenfunctions, and since (M,g) is an $\mathcal{A}$-manifold, this means that either (M,g) is Einstein or S has exactly two constant eigenvalues λ,μ, both of multiplicity 2. Let D_λ, D_μ be the corresponding eigendistributions. They are both J-invariant. Let $\{E_1,E_2\}$ be an orthonormal local basis in D_λ such that $JE_1 = E_2$, and $\{E_3,E_4\}$ be an orthonormal local basis in D_μ such that $JE_3 = E_4$. Since S is a Killing tensor we have $\nabla_X X \in \Gamma(D_a)$ for all $X \in \Gamma(D_a)$ where $a \in \{\lambda,\mu\}$. Thus there exist smooth functions $\alpha,\beta,\gamma,\sigma$ such that

$$\nabla_{E_1}E_1 = \alpha E_2, \quad \nabla_{E_2}E_2 = \beta E_1, \quad \nabla_{E_3}E_3 = \gamma E_4, \quad \nabla_{E_4}E_4 = \sigma E_3.$$

From the relations $JE_1 = E_2, JE_2 = -E_1, JE_3 = E_4, JE_4 = -E_3$ we get

$$\nabla J(E_1,E_1) + J(\nabla_{E_1}E_1) = \nabla_{E_1}E_2, \tag{2.2a}$$

$$\nabla J(E_2,E_2) + J(\nabla_{E_2}E_2) = -\nabla_{E_2}E_1, \tag{2.2b}$$

$$\nabla J(E_3, E_3) + J(\nabla_{E_3} E_3) = \nabla_{E_3} E_4, \tag{2.2c}$$

$$\nabla J(E_4, E_4) + J(\nabla_{E_4} E_4) = -\nabla_{E_4} E_3. \tag{2.2d}$$

Note that a four-dimensional almost Hermitian manifold (M, g, J) is almost Kähler if and only if its Kähler form is coclosed ($\delta\Omega = 0$) or equivalently if $\operatorname{tr}_g \nabla J = 0$. In dimension four an almost Hermitian manifold is almost Kähler if and only if it is semi-Kähler. We have

$$\nabla J(JX, JY) = -\nabla J(X, Y). \tag{2.3}$$

Note that from (2.3) we get

$$\nabla J(E_1, E_1) + \nabla J(E_2, E_2) = 0. \tag{2.4}$$

Consequently, summing up (2.2a) and (2.2b) we obtain $[E_1, E_2] = -\alpha E_1 + \beta E_2$. Analogously $[E_3, E_4] = -\gamma E_3 + \sigma E_4$. Thus the distributions D_λ, D_μ are both integrable. From Theorem 1 it follows that the Ricci tensor ϱ is parallel ($\nabla\varrho = 0$) and D_λ, D_μ are both parallel. Thus (M, g) is locally a product of two Riemannian surfaces and J is one of the standard Kähler structures on such a product. If (M, g) is complete and simply connected then from the de Rham theorem it follows that (M, g) is a product of two (simply connected) complete Riemannian surfaces of constant curvature. Thus $M = S(\lambda) \times S(\mu), H(\lambda) \times H(\mu), S(\lambda) \times \widetilde{T}, S(\lambda) \times H(\mu)$, or $\widetilde{T} \times H(\mu)$ where $S(\lambda) = \mathbb{CP}^1$ is the 2-sphere of constant sectional curvature $\lambda > 0$, $\widetilde{T} = \mathbb{R}^2$ (here $\mu = 0$) and $H(\mu)$ is the two-dimensional hyperbolic space of constant sectional curvature $\mu < 0$. Hence the Riemannian covering of any complete non-Einstein 4-manifold (M, g) satisfying the above conditions is one of these products. ■

From the proof of Proposition 1 we have

Corollary. *Let (M, g, J) be a four-dimensional almost Kähler $\mathcal{A}$-manifold with Hermitian Ricci tensor. Then (M, g) has parallel Ricci tensor.*

Remark. Note that there are examples of Einstein non-Kähler almost Kähler 4-manifolds (see [N-P]). Hence the two cases in the statement of our proposition are different and really occur. Note also that there are examples of almost Kähler four-dimensional $\mathcal{A}$-manifolds with non-Hermitian Ricci tensor (thus non-Kähler; Thurston's example is an almost Kähler $\mathcal{A}$-manifold with non-parallel Ricci tensor). Thus the hypothesis of Hermitian Ricci tensor in Proposition 1 is necessary. Oguro and Sekigawa [O-S-1] gave an example of a strictly almost Kähler 4-manifold with parallel and non-Hermitian Ricci tensor. L. Vanhecke informed the author that Proposition 1 is also an easy consequence of [P-S] and [J-1]. The proof we have given in our particular case is much simpler.

Let us recall the following well known fact.

Proposition 2. *A Riemannian* 4*-manifold* (M,g) *admits two* (*orthogonal*) *opposite almost Kähler structures if and only if it admits two orthogonal, two-dimensional oriented involutive and minimal foliations* D_1, D_2 *such that* $TM = D_1 \oplus D_2$. *If* ω_1, ω_2 *are characteristic forms of the foliations* D_1, D_2 *then* $\Omega = \omega_1 + \omega_2$ *and* $\overline{\Omega} = \omega_1 - \omega_2$ *give rise to two opposite almost Kähler structures* J *and* $\overline{J}$. *An almost Kähler manifold* (M,g,J) *admits an opposite almost Kähler structure if and only if it admits two orthogonal* J*-invariant two-dimensional foliations*, *or equivalently*, *if it admits two* J*-invariant two-dimensional orthogonal minimal distributions. If* (M,g) *is complete and admits two opposite Kähler structures then its covering space is a product of two Riemannian surfaces* (*complex curves*).

Recall that an almost Hermitian manifold (M,g,J) is said to satisfy *condition* (G_3) of A. Gray if

$$(G_3) \qquad R(JX,JY,JZ,JW) = R(X,Y,Z,W)$$

for all $X,Y,Z,W \in \mathfrak{X}(M)$. Note that for every manifold satisfying (G_3) we have $\mathcal{R}(LM) \subset \bigwedge^+ M$, its Ricci tensor ϱ is J-invariant and its $*$-Ricci tensor is symmetric. Indeed, since $R(j(X \wedge Y), j(Z \wedge W)) = R(X \wedge Y, Z \wedge W)$ where $j(X \wedge Y) = JX \wedge JY$, we have $\mathcal{R}(\ker(j - \mathrm{id}), \ker(j + \mathrm{id})) = 0$. Since $\ker(j - \mathrm{id}) = \bigwedge^- M \oplus \mathbb{R}\Omega$ and $\ker(j + \mathrm{id}) = LM$ we get $g(\mathcal{R}(LM), \bigwedge^- M \oplus \mathbb{R}\Omega) = 0$. Consequently, $\mathcal{R}(LM) \subset LM \subset \bigwedge^+ M$. In fact, the condition $\mathcal{R}(LM) \subset \bigwedge^+ M$ holds if and only if the Ricci tensor ϱ of (M,g) is J-invariant (see [D-2, p. 5 (i)]), and an almost Hermitian 4-manifold (M,g,J) with J-invariant Ricci tensor and symmetric $*$-Ricci tensor satisfies (G_3). In [O-S-Y] it is proved that every Einstein and weakly $*$-Einstein strictly almost Kähler manifold has both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ integrable. We shall show that this also holds in a more general situation.

Proposition 3. *Let* (M,g,J) *be an almost Kähler* 4*-manifold with Hermitian Ricci tensor. Assume that* (M,g,J) *has symmetric* $*$*-Ricci tensor and that* $|\nabla J| \neq 0$ *on* M. *Then both distributions* $\mathcal{D}, \mathcal{D}^\perp$ *are minimal foliations and* (M,g) *admits an opposite almost Kähler structure* $\overline{J}$. *Also* $\mathcal{D}^\perp \subset \ker d(\tau^* - \tau)$ *and the function* $|\nabla J|$ *is constant if and only if it is constant on the leaves of the nullity foliation* $\mathcal{D}$.

Proof. We start with a lemma:

Lemma A. *Let* (M,g) *be a Riemannian* 4*-manifold and let* D_1, D_2 *be two two-dimensional orthogonal distributions. Let* E_1, E_2 *and* E_3, E_4 *be any local oriented orthonormal bases of* D_1 *and* D_2 *respectively and let* $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ *be the dual co-basis. If there exists a positive function* f *such that* $d(f\theta_3 \wedge \theta_4) = 0$ *then* D_1 *is integrable.*

We shall prove Lemma A later. From (1.3) it follows that $\varrho^*(X,JY) = \varrho(X,JY) + \frac{1}{4}(\tau - \tau^*)\Omega$. Note (see [S-1]) that the form ϕ is equal to $\phi = \frac{1}{2}|\nabla J|^2\theta_1 \wedge \theta_2$ where θ_1, θ_2 is a co-frame dual to any orthonormal oriented basis $\{E_1, E_2\}$ of $\mathcal{D}^\perp$. Since $\psi(X,Y) = 2\varrho^*(X,JY)$ and $\tau - \tau^* = -\frac{1}{2}|\nabla J|^2$, we have

$$8\pi\gamma = -\frac{1}{2}|\nabla J|^2\theta_1 \wedge \theta_2 - 4\varrho^*(X,JY) = \frac{1}{2}|\nabla J|^2\theta_3 \wedge \theta_4 - 4\varrho(X,JY)$$

where $\{E_3, E_4\}$ is an oriented basis of $\mathcal{D}$, and $\{\theta_3, \theta_4\}$ its dual co-basis. Since the Ricci form $\varrho(X,JY)$ is closed (see Prop. 4, p. 165 of [D-1] and its proof) it follows from $d\gamma = 0$ that $d(\frac{1}{2}|\nabla J|^2\theta_3 \wedge \theta_4) = 0$. From Lemma A and Proposition 2 we infer that $\mathcal{D}^\perp$ is a minimal foliation.

Next we prove

Lemma B. *Let (M,g,J) be an almost Kähler four-dimensional manifold whose curvature tensor $\mathcal{R}$ satisfies the condition $\mathcal{R}(LM) \subset \bigwedge^+ M$. Then the Kähler form Ω of (M,g,J) is an eigenform of the positive Weyl tensor W^+, i.e. $W^+\Omega = \lambda\Omega$ for $\lambda \in C^\infty(M)$ (or equivalently (M,g,J) has symmetric $*$-Ricci tensor) if and only if the nullity distribution $\mathcal{D}$ is integrable.*

Proof. Note that it is enough to prove the lemma for (M_0,g,J). Thus we can assume that $\mathcal{D}$ is a two-dimensional J-invariant distribution. Let $\{E_3,E_4\}$ be a local orthonormal basis in $\mathcal{D}$ such that $E_4 = JE_3$. Hence

$$\nabla_{E_3} J = 0, \tag{2.5a}$$

$$\nabla_{E_4} J = 0. \tag{2.5b}$$

Consequently,

$$\nabla^2_{E_4E_3} J + \nabla_{\nabla_{E_4}E_3} J = 0, \tag{2.6a}$$

$$\nabla^2_{E_3E_4} J + \nabla_{\nabla_{E_3}E_4} J = 0. \tag{2.6b}$$

Thus $\nabla^2_{E_3E_4}J - \nabla^2_{E_4E_3}J + \nabla_{[E_3,E_4]}J = 0$. Hence

$$R(E_3,E_4).J = -\nabla_{[E_3,E_4]}J. \tag{2.7}$$

Choose a local orthonormal basis (for the details see [O-S-2]) $\{E_1,E_2\}$ of $\mathcal{D}^\perp$ such that $JE_1 = E_2$ and

$$\nabla\Omega = \alpha(\theta_1 \otimes \Phi - \theta_2 \otimes \Psi) \tag{Ω}$$

where $\Phi = \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4,\ \Psi = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3$ and α equals $-\frac{1}{2\sqrt{2}}|\nabla J|$. From (2.7) we obtain

$$R(E_3,E_4,JX,Y) + R(E_3,E_4,X,JY) = -\nabla_{[E_3,E_4]}\Omega(X,Y). \tag{2.8}$$

Consequently,

$$\mathcal{R}(E_3 \wedge E_4, E_2 \wedge E_3 + E_1 \wedge E_4) = \mathcal{R}(E_3 \wedge E_4, \Psi) = \alpha\theta_1([E_3,E_4]), \tag{2.9.a}$$

$$\mathcal{R}(E_3 \wedge E_4, E_1 \wedge E_3 - E_2 \wedge E_4) = \mathcal{R}(E_3 \wedge E_4, \Phi) = \alpha\theta_2([E_3,E_4]). \tag{2.9.b}$$

Write $a = \mathcal{R}(E_3 \wedge E_4, \Psi)$, $b = \mathcal{R}(E_3 \wedge E_4, \Phi)$, $c = \mathcal{R}(E_1 \wedge E_2, \Psi)$, and $d = \mathcal{R}(E_1 \wedge E_2, \Phi)$. Note that the form $\overline{\Omega} = E_1 \wedge E_2 - E_3 \wedge E_4$ is anti-self-dual ($\overline{\Omega} \in \bigwedge^- M$). Thus $c - a = 0 = d - b$. We also have $\mathcal{R}(\Omega, \Phi) = b + d$ and $\mathcal{R}(\Omega, \Psi) = a + c$. Consequently,

$$(2.10)\quad \mathcal{R}(\Omega, \Phi) = 2b = 2\alpha\theta_2([E_3, E_4]), \quad \mathcal{R}(\Omega, \Psi) = 2a = 2\alpha\theta_1([E_3, E_4]).$$

It is clear that Ω is an eigenform of W^+ if and only if $\mathcal{R}(\Omega, \Phi) = 0 = \mathcal{R}(\Omega, \Psi)$. The last two equations are equivalent to the symmetry of the $*$-Ricci tensor (they also mean that the component W_2^+ of the positive Weyl tensor vanishes). ∎

Consequently, both $\mathcal{D}$ and $\mathcal{D}^\perp$ are minimal foliations. The form $\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$ is the Kähler form of the almost Kähler structure J. The form $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$ gives the opposite almost Kähler structure $\bar{J}$. Since both $\mathcal{D}$ and $\mathcal{D}^\perp$ are foliations it follows that $d(\theta_3 \wedge \theta_4) = 0$ and consequently $d(|\nabla J|^2) \wedge \theta_3 \wedge \theta_4 = 0$. Since $df = \sum E_i f \theta_i$ we have $E_1|\nabla J|^2 = E_2|\nabla J|^2 = 0$. Since $\mathcal{D}^\perp \subset \ker d|\nabla J|^2$ it follows that $|\nabla J|$ is constant if and only if $\mathcal{D} \subset \ker d(|\nabla J|^2)$, which means that $|\nabla J|$ is constant on the leaves of the foliation $\mathcal{D}$. ∎

Proof of Lemma A. We have

$$(2.11)\qquad df \wedge \theta_3 \wedge \theta_4 + f d(\theta_3 \wedge \theta_4) = 0.$$

Write $df = \sum a_i \theta_i$. Thus

$$(2.12)\qquad a_1\theta_1 \wedge \theta_3 \wedge \theta_4 + a_2\theta_2 \wedge \theta_3 \wedge \theta_4 = -f(d\theta_3 \wedge \theta_4 - \theta_3 \wedge d\theta_4).$$

From (2.12) we infer that the differential ideal generated by θ_3, θ_4 is closed and consequently the distribution D_1 is integrable. ∎

We say that an almost Hermitian manifold (M, g, J) satisfies the *second condition* (G_2) of A. Gray if its curvature tensor R satisfies

$$(G_2)\quad R(X, Y, Z, W) - R(JX, JY, Z, W) = R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. It is known that an almost Kähler manifold (M, g, J) satisfies (G_2) if and only if its Ricci tensor is J-invariant, the $*$-Ricci tensor is symmetric and the component W_3^+ of the positive Weyl tensor vanishes (i.e. $\mathcal{R}_{LM} = a\,\mathrm{id}_{LM}$ where $\mathcal{R}_{LM} = p_{LM} \circ \mathcal{R}_{|LM}$ and p_{LM} is the orthogonal projection $p_{LM} : \bigwedge M \to LM$). It is well known that any almost Kähler manifold satisfying (G_2) also satisfies (G_3). On the other hand, we have the following as an application of our previous results. The implication "if (G_2) then $|\nabla J|$ is constant" is proved in [A-D].

PROPOSITION 4. *Let (M, g, J) be an almost Kähler manifold with J-invariant Ricci tensor and symmetric $*$-Ricci tensor. Then (M, g, J) satisfies condition (G_2) if and only if $|\nabla J|$ is constant on M.*

Proof. From the assumptions we have $W_2^+ = 0$. We shall show that the condition $W_3^+ = 0$ is equivalent to $|\nabla J|$ being constant. We can assume that $|\nabla J| \neq 0$ on M. From Proposition 3 it follows that both distributions $\mathcal{D}, \mathcal{D}^\perp$ are minimal foliations. Let $\{E_1, E_2, E_3, E_4\}$ be a local orthonormal frame such that (Ω) holds. Then

$$(2.12)\qquad g(\nabla_{E_1} JX, Y) = \alpha\Phi(X,Y), \quad g(\nabla_{E_2} JX, Y) = -\alpha\Psi(X,Y), \quad \nabla_{E_3} J = 0, \quad \nabla_{E_4} J = 0.$$

Consequently,

$$(2.13a)\qquad g(R(E_1,E_3).JX,Y) = -\nabla_{[E_1,E_3]}\Omega - E_3\alpha\Phi - \alpha p(E_3)\Psi,$$
$$(2.13b)\qquad g(R(E_1,E_4).JX,Y) = -\nabla_{[E_1,E_4]}\Omega - E_4\alpha\Phi - \alpha p(E_4)\Psi,$$
$$(2.13c)\qquad g(R(E_2,E_3).JX,Y) = -\nabla_{[E_2,E_3]}\Omega + E_3\alpha\Psi - \alpha p(E_3)\Phi,$$
$$(2.13d)\qquad g(R(E_4,E_2).JX,Y) = -\nabla_{[E_4,E_2]}\Omega - E_4\alpha\Psi + \alpha p(E_4)\Phi,$$

where the local 1-form p is defined by $p(X) = \frac{1}{2} g(\nabla_X \Phi, \Psi)$. Since $\mathcal{R}(LM) \subset \bigwedge^+ M$ it is clear that

$$(2.14a)\qquad g(R(E_1,E_3).JX,Y) = g(R(E_4,E_2).JX,Y),$$
$$(2.14b)\qquad g(R(E_3,E_2).JX,Y) = g(R(E_4,E_1).JX,Y).$$

Consequently, from (2.13) and (2.14) we get

$$(2.15a)\qquad \mathcal{R}(\Phi,\Psi) = -2g(R(E_1,E_3).JE_1,E_3) = 2(E_3\alpha + \alpha\theta_1([E_1,E_3])),$$
$$(2.15b)\qquad \mathcal{R}(\Phi,\Psi) = -2g(R(E_2,E_3).JE_3,E_2) = 2(E_3\alpha + \alpha\theta_2([E_2,E_3])),$$
$$(2.15c)\qquad \mathcal{R}(\Phi,\Phi) = -2g(R(E_4,E_2).JE_3,E_2) = -2(E_4\alpha - \alpha\theta_2([E_4,E_2])),$$
$$(2.15d)\qquad \mathcal{R}(\Psi,\Psi) = -2g(R(E_1,E_4).JE_1,E_3) = -2(-E_4\alpha - \alpha\theta_1([E_1,E_4])).$$

Since $\mathcal{D}^\perp$ is a minimal foliation we have $\theta_1([E_1,E_3]) + \theta_2([E_2,E_3]) = 0$ and $\theta_1([E_1, E_4]) - \theta_2([E_4,E_2]) = 0$. Thus from (2.15) we get $\mathcal{R}(\Phi,\Psi) = 2E_3\alpha$ and $\mathcal{R}(\Phi,\Phi) - \mathcal{R}(\Psi,\Psi) = -4E_4\alpha$. Since from Proposition 3 we have $E_1\alpha = E_2\alpha = 0$ it follows that $|\nabla J|$ is constant if and only if $\mathcal{R}(\Phi,\Psi) = 0$ and $\mathcal{R}(\Phi,\Phi) = \mathcal{R}(\Psi,\Psi)$. The last two equalities are equivalent to the vanishing of the component W_3^+ of the positive Weyl tensor W^+. ■

PROPOSITION 5. *Let (M,g,J) be an almost Kähler manifold with Hermitian Ricci tensor and symmetric $*$-Ricci tensor. Assume that $|\nabla J| \neq 0$ on M. Then the opposite almost Hermitian structure $\bar{J}$ determined by the minimal foliations $\mathcal{D}, \mathcal{D}^\perp$ is almost Kähler. The distribution $\mathcal{D}^\perp$ is contained in the nullity distribution of $\bar{J}$.*

Proof. The first part of the proposition is an immediate consequence of Propositions 2 and 3. We show that $\nabla_X \bar{\Omega} = 0$ for any $X \in \mathcal{D}^\perp$. Choose a local orthonormal frame $\{E_1, \ldots, E_4\}$ such that (Ω) holds. Note that (we write $\nabla_X\theta_i = \omega_i^j(X)\theta_j$, $\bar{\Phi} = \theta_1\wedge\theta_3 + \theta_2\wedge\theta_4$, $\bar{\Psi} = \theta_1\wedge\theta_4 - \theta_2\wedge\theta_3$)

$$\nabla(\theta_1\wedge\theta_2) = \tfrac{1}{2}\{\Phi(\omega_1^4+\omega_2^3) + \Psi(\omega_3^1+\omega_2^4) + \bar{\Phi}(-\omega_1^4+\omega_2^3) + \bar{\Psi}(-\omega_3^1+\omega_2^4)\}.$$

Analogously

$$\nabla(\theta_3\wedge\theta_4) = \tfrac{1}{2}\{\Phi(\omega_1^4+\omega_2^3) + \Psi(\omega_3^1+\omega_2^4) - \bar{\Phi}(-\omega_1^4+\omega_2^3) - \bar{\Psi}(-\omega_3^1+\omega_2^4)\}.$$

Note that $\nabla\Omega = a\otimes\Phi + b\otimes\Psi$ and $\nabla\bar{\Omega} = a'\otimes\bar{\Phi} + b'\otimes\bar{\Psi}$ where with our assumptions $a = \alpha\theta_1$ and $b = -\alpha\theta_2$. On the other hand, $a = \omega_1^4+\omega_2^3$, $b = \omega_3^1+\omega_2^4$ and

$$\alpha\theta_1 = \omega_1^4+\omega_2^3, \qquad -\alpha\theta_2 = \omega_3^1+\omega_2^4, \tag{2.16a}$$

$$a' = -\omega_1^4+\omega_2^3, \qquad b' = -\omega_3^1+\omega_2^4. \tag{2.16b}$$

It is clear that $\mathcal{D}^\perp$ is in the nullity distribution of $\bar{J}$ if $a'(E_1) = a'(E_2) = 0$. Write $\Gamma_{jk}^i = \omega_j^i(E_k)$. Then $a'(E_1) = \Gamma_{21}^3 - \Gamma_{11}^4$ and $a'(E_2) = \Gamma_{22}^3 - \Gamma_{12}^4$. Note that from (2.15a,b) we have

$$\Gamma_{11}^3 = \Gamma_{22}^3 = 0 \tag{2.17}$$

and since $\operatorname{tr}\mathcal{R}_{\bigwedge^+ M} = \tau/4$ we have $\mathcal{R}(\Phi,\Phi) + \mathcal{R}(\Psi,\Psi) = \tau/2 - \mathcal{R}(\Omega,\Omega) = (\tau-\tau^*)/2 = -2\alpha^2$, so that from (2.15c,d) we obtain

$$\Gamma_{11}^4 = -\Gamma_{22}^4 = \alpha/2. \tag{2.18}$$

From (2.16a) we have

$$\Gamma_{11}^4 + \Gamma_{21}^3 = \alpha, \qquad \Gamma_{22}^3 + \Gamma_{12}^4 = 0. \tag{2.19}$$

We infer from (2.18), (2.19) that $\Gamma_{11}^4 = \Gamma_{21}^3 = \alpha/2$ and $\Gamma_{22}^3 = \Gamma_{12}^4 = 0$. Consequently, $a'(E_1) = a'(E_2) = 0$ and $\mathcal{D}^\perp$ is contained in the nullity distribution of $\bar{J}$. It follows that in the set $M_0' = \{x : |\nabla\bar{J}| \neq 0\}$ the nullity distribution of $\bar{J}$ is $\mathcal{D}^\perp$. From (2.7) we also get

$$R(E_3,E_4).J = 0, \qquad R(E_1,E_2).\bar{J} = 0. \ \blacksquare \tag{2.20}$$

PROPOSITION 6. *Let (M,g,J) be a four-dimensional almost Kähler manifold. Assume that (M,g,J) has Hermitian Ricci tensor with constant eigenvalues. Then either (M,g) is Einstein, or (M,g,J) admits an opposite almost Kähler structure $\bar{J}$ such that $(M,g,\bar{J})$ has Hermitian Ricci tensor. On the other hand, a 4-manifold with constant scalar curvature which admits two opposite almost Kähler structures with Hermitian Ricci tensor is either Einstein or its Ricci tensor has two constant eigenvalues.*

Proof. We can assume that (M,g) has Ricci tensor with exactly two constant eigenvalues λ,μ since in the other case it is Einstein. Let $\{E_1,E_2\}$ be an orthonormal local basis in D_λ such that $JE_1 = E_2$ and let $\{E_3,E_4\}$ be an orthonormal local basis in D_μ such that $JE_3 = E_4$; let $\{\theta_1,\theta_2,\theta_3,\theta_4\}$ be a dual coframe. Recall that every four-dimensional almost Kähler manifold with Hermitian Ricci tensor has closed Ricci form $\alpha(X,Y) := \varrho(JX,Y)$ (see

Prop. 4, p. 165 of [D-1] and its proof). Since $\alpha = \lambda\theta_1 \wedge \theta_2 + \mu\theta_3 \wedge \theta_4$ we obtain

$$\lambda d(\theta_1 \wedge \theta_2) + \mu d(\theta_3 \wedge \theta_4) = 0. \tag{2.21}$$

On the other hand, $d(\theta_1 \wedge \theta_2) + d(\theta_3 \wedge \theta_4) = 0$. Thus we infer from (2.21) that $d(\theta_1 \wedge \theta_2) = 0 = d(\theta_3 \wedge \theta_4)$, i.e. the characteristic forms of the distributions D_λ, D_μ are both closed. The tensor ϱ is clearly invariant with respect to the almost Kähler structure given by the form $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$.

Now assume that (M, g, J) is an almost Kähler manifold with Hermitian Ricci tensor and constant scalar curvature which admits an opposite almost Kähler structure $\bar{J}$ such that ϱ is also $\bar{J}$-invariant. Let $U = \{x \in M : \varrho_x$ has two eigenvalues$\}$. Then U is an open set. Let $\lambda, \mu \in C^\infty(U)$ be eigenfunctions of ϱ in U. Choose a local orthonormal frame $\{E_1, E_2, E_3, E_4\}$ just as above. It is clear that J and $\bar{J}$ are given in U respectively by $\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$ and $\overline{\Omega} = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4$. Thus both forms $\theta_1 \wedge \theta_2$ and $\theta_3 \wedge \theta_4$ are closed. Since $\alpha = \lambda\theta_1 \wedge \theta_2 + \mu\theta_3 \wedge \theta_4$ we obtain

$$d\lambda \wedge \theta_1 \wedge \theta_2 + d\mu \wedge \theta_3 \wedge \theta_4 = 0. \tag{2.24}$$

Note that $d\lambda = \sum_{i=1}^4 a_i\theta_i$ and $d\mu = \sum_{i=1}^4 b_i\theta_i$ where $a_i = E_i\lambda$ and $b_i = E_i\mu$. From (2.12) we infer that

$$a_3\theta_3 \wedge \theta_1 \wedge \theta_2 + a_4\theta_4 \wedge \theta_1 \wedge \theta_2 + b_1\theta_1 \wedge \theta_3 \wedge \theta_4 + b_2\theta_2 \wedge \theta_3 \wedge \theta_4 = 0. \tag{2.25}$$

Thus $a_3 = a_4 = b_1 = b_2 = 0$. It follows that $\nabla\lambda \in \Gamma(D_\lambda)$ and $\nabla\mu \in \Gamma(D_\mu)$. Since $\nabla\lambda + \nabla\mu = 0$ it follows that λ and μ are constant in U. Hence $U = M$. ∎

COROLLARY. *Assume that (M, g, J) is a Kähler 4-manifold whose Ricci tensor ϱ has two constant eigenvalues. Then (M, g, J) admits an opposite almost Kähler structure $\bar{J}$, and ϱ is $\bar{J}$-invariant. The structure $\bar{J}$ is Kähler if and only if (M, g) is locally a product of two Riemannian surfaces of constant curvatures.*

Acknowledgments. The author is grateful to V. Apostolov, T. C. Draghici and K. Sekigawa for sending him the preprints of their papers. The author thanks T. Draghici for pointing out a mistake in the first version of the paper.

The work was supported by KBN grant 2 P0 3A 01615.

REFERENCES

[A-G-I] B. Alexandrov, G. Grantcharov and S. Ivanov, *Curvature properties of twistor spaces of quaternionic Kähler manifolds*, J. Geom. 62 (1998), 1–12.

[A-A-D] V. Apostolov, J. Armstrong and T. Draghici, *Local ridigity of certain classes of almost Kähler surfaces*, preprint math. DG/9911196.

[A-D] V. Apostolov and T. Draghici, *Almost Kähler 4-manifolds with J-invariant Ricci tensor and special Weyl tensor*, Quart. J. Math. Oxford 51 (2000), 275–294.

[B-I] D. E. Blair and S. Ianuş, *Critical associated metrics on symplectic manifolds*, in: Contemp. Math. 51, Amer. Math. Soc., 1986, 23–29.

[D-G-M] J. Davidov, G. Grantcharov and O. Muškarov, *Almost complex structures on twistor spaces*, in: Almost Complex Structures (Sofia, 1992), World Sci., 1994, 113–149.

[D-M] J. Davidov and O. Muškarov, *Twistor spaces with Hermitian Ricci tensor*, Proc. Amer. Math. Soc. 109 (1990), 1115–1120.

[De] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compos. Math. 49 (1983), 405–433.

[D-1] T. C. Draghici, *On some 4-dimensional almost Kähler manifolds*, Kodai Math. J. 18 (1995), 156–168.

[D-2] —, *Almost Kähler 4-manifolds with J-invariant Ricci tensor*, Houston J. Math. 25 (1999), 133–145.

[G] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata 7 (1978), 259–280.

[J-1] W. Jelonek, *On $\mathcal{A}$-tensors in Riemannian geometry*, preprint 551, Inst. Math., Polish Acad. Sci., 1995.

[J-2] —, *Self-duality and $\mathcal{A}$-manifolds*, J. London Math. Soc. 58 (1998), 697–708,

[J-3] —, *Quaternion Kähler geometry and almost Kähler $\mathcal{A}$-manifolds*, Ann. Polon. Math. 75 (2000), 111–124.

[N-P] P. Nurowski and M. Przanowski, *A four-dimensional example of a Ricci flat metric admitting almost-Kähler non-Kähler structure*, Classical Quantum Gravity 16 (1999), L9–L13.

[O-S-1] T. Oguro and K. Sekigawa, *Almost Kähler structures on the Riemannian product of a 3-dimensional hyperbolic space and a real line*, Tsukuba J. Math. 20 (1996), 151–161.

[O-S-2] —, —, *Four-dimensional almost Kähler Einstein and $*$-Einstein manifolds*, Geom. Dedicata 69 (1998), 91–112.

[O-S-Y] T. Oguro, K. Sekigawa and A. Yamada, *Four-dimensional almost Kähler Einstein and weakly $*$-Einstein manifolds*, Yokohama Math. J. 47 (1999), 75–92.

[P-S] F. Podestà and A. Spiro, *Four-dimensional Einstein-like manifolds and curvature homogeneity*, Geom. Dedicata 54 (1995), 225–243.

[S-1] K. Sekigawa, *On some 4-dimensional compact Einstein almost Kähler manifolds*, Math. Ann. 271 (1985), 333–337.

[S-2] —, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan 39 (1987), 677–684.

[S-V] K. Sekigawa and L. Vanhecke, *Symplectic geodesic symmetries on Kähler manifolds*, Quart. J. Math. Oxford (2) 37 (1986), 95–103.

Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: wjelon@pk.edu.pl

Received 15 March 1999;
revised 18 January 2007 (3712)