COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 1

ON ALMOST KÄHLER TYPE $(2G_3)$ *4-MANIFOLDS*

BY

WŁODZIMIERZ JELONEK (Kraków)

Abstract. We study four-dimensional almost Kähler manifolds (M, g, J) which satisfy A. Gray's condition (G_3).

0. Introduction. In [J] we have proved that every strictly almost Kähler manifold (M, g, J) (i.e. with $|\nabla J| \neq 0$) satisfying condition (G_3) admits an opposite almost Kähler structure $\bar{J}$. In the present paper we prove that every compact strongly non-Kähler almost Kähler 4-manifold (M, g, J) of type $(2G_3)$ (i.e. of type (G_3) according to A. Gray's notation and such that $\bar{J}$ is also of type (G_3)) admits a global opposite Kähler structure if the scalar curvature τ satisfies the condition $\{0\} \cap \tau(M_0) = \emptyset$ where $M_0 = \{x : \nabla J_x = 0\}$. In particular we prove that there does not exist a compact strongly non-Kähler almost Kähler type $(2G_3)$ 4-manifold with negative scalar curvature. We also prove that for a compact four-dimensional almost Kähler manifold (M, g, J) the opposite almost Kähler structure $\bar{J}$ defined on the set $\operatorname{dom} \bar{J} = \{x : \nabla J_x \neq 0\}$ is Kähler. Our results are connected with the question of Blair and Ianuş (see [B-I], [D-1]): "Is it true that every four-dimensional almost Kähler compact manifold with Hermitian Ricci tensor is Kähler?" and with the Goldberg conjecture (see [S-1], [S-2]).

1. Preliminaries. Let (M, g, J) be an almost Hermitian manifold. We say that (M, g, J) is an *almost Kähler manifold* if its *Kähler form* $\Omega(X, Y) = g(JX, Y)$ is closed ($d\Omega = 0$). In what follows we shall consider four-dimensional almost Kähler manifolds (M, g, J). Such manifolds are always oriented and we choose an orientation in such a way that Ω is a self-dual form (i.e. $\Omega \in \bigwedge^+ M$). The vector bundle of self-dual forms admits a decomposition

$$\textstyle\bigwedge^+ M = \mathbb{R}\Omega \oplus LM$$

where LM denotes the bundle of real J-skew-invariant 2-forms (i.e. $LM = \{\Phi \in \bigwedge M : \Phi(JX, JY) = -\Phi(X, Y)\}$). The bundle LM is a complex line

2000 *Mathematics Subject Classification*: 53C15, 53B20.

Key words and phrases: almost Kähler manifold, almost Kähler structure.

The Editorial Committee apologizes to the author and readers for the unusually long delay in the publication of this paper.

bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X,Y) = -\Phi(JX,Y)$.

The curvature tensor R of a four-dimensional manifold (M,g) determines an endomorphism $\mathcal{R}$ of the bundle $\bigwedge M$ defined by $g(\mathcal{R}(X\wedge Y), Z\wedge W) = \mathcal{R}(X\wedge Y, Z\wedge W) = -R(X,Y,Z,W) = -g(R(X,Y)Z,W)$.

The Ricci tensor ϱ of an almost Hermitian manifold (M,g,J) is said to be *Hermitian* (or *J-invariant*) if $\varrho(X,Y) = \varrho(JX,JY)$ for all $X,Y\in\mathfrak{X}(M)$. In the following we shall assume that (M,g,J) has Hermitian Ricci tensor. This condition is equivalent to $\mathcal{R}(LM)\subset\bigwedge^+ M$.

For any local frame $\{E_1,E_2,E_3,E_4\}$ we shall write $\nabla_{E_i}E_j = \sum_k \Gamma_{ij}^k E_k$. From [O-S] and [J] we get (cf. also [O-S-Y])

LEMMA A. *Assume that (M,g,J) is an almost Kähler (G_3) 4-manifold and $|\nabla J|\neq 0$ on M. Let $\{E_1,E_2\}$ be any local orthonormal basis of $\mathcal{D}^\perp$. Then there exists a unique orthonormal basis $\{E_3,E_4\}$ of $\mathcal{D}$ such that*

$$(\Omega)\qquad \nabla\Omega = \alpha(\theta_1\otimes\Phi - \theta_2\otimes\Psi)$$

where $\Phi = \theta_1\wedge\theta_3 - \theta_2\wedge\theta_4$, $\Psi = \theta_1\wedge\theta_4 + \theta_2\wedge\theta_3$ and $\alpha = -\frac{1}{2\sqrt{2}}|\nabla J|$. Also,

$$E_4 = (2/\alpha)\alpha_{\mathcal{D}^\perp}(E_1,E_1),\qquad E_3 = (2/\alpha)\alpha_{\mathcal{D}^\perp}(E_1,E_2),$$

where $\alpha_{\mathcal{D}^\perp}$ is the second fundamental form of $\mathcal{D}^\perp$. Moreover $\Gamma_{11}^3 = \Gamma_{22}^3 = \Gamma_{12}^4 = 0$ and $\Gamma_{11}^4 = -\Gamma_{22}^4 = \Gamma_{12}^3 = \frac{1}{2}\alpha$.

Let $\{E_1,E_2,E_3,E_4\}$ be a local orthonormal frame satisfying (Ω). Then

$$(1.1)\qquad \begin{gathered} g(\nabla_{E_1}JX,Y) = \alpha\Phi(X,Y),\qquad g(\nabla_{E_2}JX,Y) = -\alpha\Psi(X,Y),\\ \nabla_{E_3}J = 0,\qquad \nabla_{E_4}J = 0.\end{gathered}$$

Recall that an almost Hermitian manifold (M,g,J) is said to satisfy *condition (G_3) of A. Gray* if

$$(G_3)\qquad R(JX,JY,JZ,JW) = R(X,Y,Z,W)$$

for all $X,Y,Z,W\in\mathfrak{X}(M)$. Note that for every manifold satisfying (G_3) we have $\mathcal{R}(LM)\subset\bigwedge^+ M$, the Ricci tensor ϱ is J-invariant and the $*$-Ricci tensor is symmetric. Indeed, since $R(j(X\wedge Y), j(Z\wedge W)) = R(X\wedge Y, Z\wedge W)$ where $j(X\wedge Y) = JX\wedge JY$, we have $\mathcal{R}(\ker(j-\mathrm{id}), \ker(j+\mathrm{id})) = 0$. Since $\ker(j-\mathrm{id}) = \bigwedge^- M\oplus\mathbb{R}\Omega$ and $\ker(j+\mathrm{id}) = LM$ we get $g(\mathcal{R}(LM), \bigwedge^- M\oplus\mathbb{R}\Omega) = 0$. Consequently, $\mathcal{R}(LM)\subset LM\subset\bigwedge^+ M$. In fact, $\mathcal{R}(LM)\subset\bigwedge^+ M$ if and only if the Ricci tensor ϱ of (M,g) is J-invariant (see [D-2, p. 5(i)]), and an almost Hermitian 4-manifold (M,g,J) with J-invariant Ricci tensor and symmetric $*$-Ricci tensor satisfies (G_3).

Let us recall that we have proved in [J] that an almost Kähler 4-manifold (M,g,J) with Hermitian Ricci tensor, symmetric $*$-Ricci tensor (i.e. (M,g,J) satisfies the condition (G_3) of A. Gray) with $\nabla J\neq 0$ on M has integrable both distributions $\mathcal{D} = \{X\in TM : \nabla_X J = 0\}$ and $\mathcal{D}^\perp$, and they determine an opposite almost Kähler structure $\bar{J}$. In the following we shall assume that $(M, g, \bar{J})$ also satisfies condition (G_3) and we shall call such manifolds $(2G_3)$ almost Kähler manifolds.

The distribution $\mathcal{D}^{\perp}$ is contained in the nullity distribution $\mathcal{D}_{\bar{J}} = \{X \in TM : \nabla_X \bar{J} = 0\}$ of the almost Kähler structure $\bar{J}$. For a general almost Kähler 4-manifold satisfying condition $(2G_3)$ the opposite almost Kähler structure $\bar{J}$ is defined only on the set $\operatorname{dom} \bar{J} = \{x \in M : \nabla J \neq 0\}$. The distributions $\mathcal{D}, \mathcal{D}^{\perp}$ are J-invariant foliations which are eigendistributions of the Ricci tensor ϱ of (M, g). We shall denote by μ, λ the eigenvalues corresponding to the distributions $\mathcal{D}, \mathcal{D}^{\perp}$ respectively. We shall also write $\beta = -\frac{1}{2\sqrt{2}}|\nabla \bar{J}|$.

Assume that $\beta \neq 0$ and let $\{E_1', E_2'\}$ be an orthonormal frame of $\mathcal{D}^{\perp}$ determined by $\{E_3, E_4\}$, i.e. such that

$$\nabla \bar{\Omega} = \beta(\theta_4 \otimes \bar{\Phi} - \theta_3 \otimes \bar{\Psi}).$$

Then $E_1' = E_1 \cos\phi + E_2 \sin\phi$, $E_2' = -E_1 \sin\phi + E_2 \cos\phi$ and from Lemma A it follows that

$$E_1' = \tfrac{2}{\beta}\, q(\nabla_{E_3} E_4), \qquad E_2' = \tfrac{2}{\beta}\, q(\nabla_{E_4} E_4),$$

where q is the orthogonal projection $q : TM \to \mathcal{D}^{\perp}$. Hence the connection coefficients Γ^i_{kj} with respect to the orthonormal frame $\{E_1, E_2, E_3, E_4\}$ satisfy

$$\Gamma^1_{34} = -\Gamma^2_{33} = \Gamma^2_{44} = \tfrac{1}{2}\,\beta\cos\phi, \qquad \Gamma^1_{33} = -\Gamma^1_{44} = \Gamma^2_{34} = \tfrac{1}{2}\,\beta\sin\phi.$$

Note that these relations are also valid when $\beta = 0$ (in this case $\mathcal{D}$ is a totally geodesic foliation). In [J] we have proved that $R(E_1, E_2).J = 0$ and $R(E_3, E_4).\bar{J} = 0$. From (1.1) we obtain (see [J])

$$\text{(1.2a)} \qquad g(R(E_1, E_3).JX, Y) = -\nabla_{[E_1, E_3]}\Omega - E_3\alpha\Phi - \alpha p(E_3)\Psi,$$

$$\text{(1.2b)} \qquad g(R(E_1, E_4).JX, Y) = -\nabla_{[E_1, E_4]}\Omega - E_4\alpha\Phi - \alpha p(E_4)\Psi,$$

$$\text{(1.2c)} \qquad g(R(E_2, E_3).JX, Y) = -\nabla_{[E_2, E_3]}\Omega + E_3\alpha\Psi - \alpha p(E_3)\Phi,$$

$$\text{(1.2d)} \qquad g(R(E_4, E_2).JX, Y) = -\nabla_{[E_4, E_2]}\Omega - E_4\alpha\Psi + \alpha p(E_4)\Phi,$$

$$\text{(1.2e)} \qquad g(R(E_1, E_2).JX, Y) = -\nabla_{[E_1, E_2]}\Omega - \alpha\nabla_{E_1}\Psi - \alpha\nabla_{E_2}\Phi,$$

where the local 1-form p is defined by $p(X) = \frac{1}{2} g(\nabla_X \Phi, \Psi)$. Since $\mathcal{R}(LM) \subset \bigwedge^+ M$ it is clear that

$$\text{(1.3a)} \qquad g(R(E_1, E_3).JX, Y) = g(R(E_4, E_2).JX, Y),$$

$$\text{(1.3b)} \qquad g(R(E_3, E_2).JX, Y) = g(R(E_4, E_1).JX, Y).$$

Note that $p(X) = \omega^2_1(X) + \omega^4_3(X)$. Hence

$$\text{(1.4a)} \qquad \begin{aligned} 0 = g(R(E_1, E_2).JE_1, E_3) &= \alpha(p(E_1) - \theta^1([E_1, E_2])) \\ &= \alpha(2\Gamma^2_{11} + \Gamma^4_{13}), \end{aligned}$$

$$(1.4b)\qquad 0 = g(R(E_1,E_2).JE_1,E_4) = \alpha(-p(E_2)+\theta^2([E_1,E_2])) = \alpha(2\Gamma^1_{22}-\Gamma^4_{23}).$$

Note also that

$$(1.5a)\qquad g(R(E_1,E_4).JE_2,E_3) = \alpha(-p(E_4)+\theta^2([E_1,E_4])),$$

$$(1.5b)\qquad g(R(E_2,E_3).JE_2,E_3) = \alpha(\theta^2([E_2,E_3])+E_3\alpha).$$

We have $\Gamma^4_{12} = \Gamma^2_{33} = 0$. Consequently, (1.5a,b) imply

$$(1.6)\qquad E_3\alpha = \alpha(-p(E_4)+\theta^2([E_1,E_4])) = \alpha(2\Gamma^1_{42}+\Gamma^3_{44}).$$

Analogously we get

$$(1.7)\qquad E_4\alpha = \alpha(2\Gamma^2_{31}+\Gamma^4_{33}).$$

2. Almost Kähler $(2G_3)$ manifolds. Recall that an almost Hermitian 4-manifold (M,g,J) satisfies condition (G_3) if and only if it has J-invariant Ricci tensor and symmetric $*$-Ricci tensor. We start with

Lemma B. *Assume that (M,g,J) is a $(2G_3)$ almost Kähler 4-manifold and $|\nabla J| \neq 0$ on M. Then $\mathcal{D},\mathcal{D}^\perp$ are eigendistributions of the Ricci tensor ϱ of (M,g). The eigenvalue corresponding to the distribution $\mathcal{D}^\perp$ is $\lambda = -\frac{1}{32}|\nabla\bar{J}|^2$.*

Proof. The first part of the lemma is proved in [J]. From (1.4) we obtain, similarly to [O-S, p. 109],

$$(2.1)\qquad R_{1234} = -2R_{1212} - \tfrac{3}{8}(\tau^*-\tau).$$

We also have

$$\mathcal{R}(\Omega,\Omega) = \tau^*/2, \qquad \mathcal{R}(\bar\Omega,\bar\Omega) = \bar\tau^*/2.$$

Hence

$$(2.2a)\qquad R_{1212}+R_{3434}+2R_{1234} = -\tau^*/2,$$

$$(2.2b)\qquad R_{1212}+R_{3434}-2R_{1234} = -\bar\tau^*/2.$$

From (2.2) we have

$$(2.3)\qquad R_{1212}+R_{3434} = -\frac{\tau^*+\bar\tau^*}{4},$$

$$(2.4)\qquad R_{1234} = \frac{\bar\tau^*-\tau^*}{8}.$$

Thus from (2.1) it follows that

$$(2.5a)\qquad R_{1212} = \frac{3\tau-2\tau^*-\bar\tau^*}{16},$$

$$(2.5b)\qquad R_{3434} = \frac{-3\tau-2\tau^*-3\bar\tau^*}{16}.$$

Since $\tau^* - \tau = \frac{1}{2}|\nabla J|^2$ we can rewrite (2.4) and (2.5) in the following way:

$$R_{1212} = -\frac{2|\nabla J|^2 - |\nabla \bar{J}|^2}{32}, \tag{2.6a}$$

$$R_{1234} = \frac{-|\nabla J|^2 + |\nabla \bar{J}|^2}{16}, \tag{2.6b}$$

$$R_{3434} = -\frac{2|\nabla J|^2 + 3|\nabla \bar{J}|^2 + 16\tau}{32}. \tag{2.6c}$$

Note that

$$\lambda = \varrho(E_1, E_1) = K(E_1 \wedge E_2) + K(E_1 \wedge E_3) + K(E_1 \wedge E_4),$$
$$\lambda = \varrho(E_2, E_2) = K(E_1 \wedge E_2) + K(E_2 \wedge E_3) + K(E_2 \wedge E_4).$$

Thus

$$2\lambda = -2R_{1212} + 2\mu + 2R_{3434}.$$

Consequently, since $\tau = 2\lambda + 2\mu$,

$$\lambda - \mu = R_{3434} - R_{1212} = -\frac{|\nabla \bar{J}|^2}{16} - \lambda - \mu. \tag{2.7}$$

We infer from (2.7) that $\lambda = -\frac{1}{32}|\nabla \bar{J}|^2$. ∎

Corollary 1. *Assume that (M, g, J) is a $(2G_3)$ almost Kähler 4-manifold. If an opposite almost Kähler structure $\bar{J}$ satisfies the condition $\nabla \bar{J} \neq 0$ on a set $U \subset M$, then on U the scalar curvature τ is negative and $16\tau = -(|\nabla J|^2 + |\nabla \bar{J}|^2)$. In particular, if $\tau \geq 0$ on M then $(M, g, \bar{J})$ is Kähler.*

Proof. Since $(M, g, \bar{J})$ also satisfies condition (G_3) the calculations similar to those in the proof of Lemma B give us $\mu = -\frac{1}{32}|\nabla J|^2$. Consequently, $\tau = 2\lambda + 2\mu = -\frac{1}{16}(|\nabla J|^2 + |\nabla \bar{J}|^2)$. ∎

Proposition 1. *Let (M, g, J) be a compact $(2G_3)$ almost Kähler 4-manifold such that $|\nabla J| \neq 0$ on M. Then $(M, g, \bar{J})$ is Kähler and the Euler characteristic and signature of M vanish. The Ricci tensor ϱ of (M, g) has two eigenvalues $\lambda = 0, \mu = \frac{1}{2}\tau$. The foliation $\mathcal{D}$ is totally geodesic and every leaf of $\mathcal{D}$ is a geodesically complete submanifold of M. If (M, g) has negative total scalar curvature then $(M, \bar{J})$ is a minimal class VI surface.*

Proof. Assume that the function $b(x) = |\nabla \bar{J}_x|$ is not identically 0 on M. Then b is positive and continuous on M. Its image $F = \operatorname{im} b$ is a connected subset of $\mathbb{R}_+$. Let $c \in F$ be different from 0. The set $F_c = \{x \in M : b(x) = c\}$ is closed, hence compact. Note that if D is a leaf of the foliation $\mathcal{D}$ and $D \cap F_c \neq \emptyset$ then $D \subset F_c$ since $\mathcal{D} \subset \ker db$. Let $\kappa = \sup\{\alpha(x) : x \in F_c\}$. Then there exists a point $x_0 \in F_c$ such that $\alpha(x_0) = \kappa$. Note that since α is constant on the leaves of the foliation $\mathcal{D}^\perp$, at x_0 there is a local maximum of α. On the other hand, we have (cf. [O-S-Y, (3.24), (3.25)])

$$R_{3434} = E_3\Gamma^4_{43} - E_4\Gamma^4_{33} + (\Gamma^3_{34})^2 + (\Gamma^4_{34})^2 - 2H, \tag{2.8}$$

where $H = (\Gamma^1_{33})^2 + (\Gamma^1_{34})^2 = \frac{1}{4}\beta^2 = \frac{1}{32}|\nabla\bar{J}|^2$. Note that

$$R_{3412} = E_3\Gamma^2_{41} - E_4\Gamma^2_{31} + 2\Gamma^1_{34}\Gamma^2_{44} + 2\Gamma^1_{33}\Gamma^2_{43} + \Gamma^2_{31}\Gamma^4_{33} - \Gamma^2_{41}\Gamma^3_{44}. \tag{2.9}$$

We have $\Gamma^2_{41} = \frac{1}{2}\Gamma^3_{44} - \frac{1}{2}E_3 \ln|\alpha|$ and $\Gamma^2_{31} = -\frac{1}{2}\Gamma^4_{33} + \frac{1}{2}E_4 \ln|\alpha|$. Thus we obtain

$$R_{3412} = -\tfrac{1}{2}R_{3434} + \tfrac{1}{32}|\nabla\bar{J}|^2 - \tfrac{1}{2}\Delta \ln|\alpha|,$$

where $\Delta \ln|\alpha| = E_3^2 \ln|\alpha| + E_4^2 \ln|\alpha| - \Gamma^3_{44}E_3 \ln|\alpha| - \Gamma^4_{33}E_4 \ln|\alpha|$ is the usual Laplacian of the function $\ln|\alpha|$ on (M, g). Thus it follows from (2.6) that

$$32\Delta \ln|\alpha| = 16\tau + |\nabla\bar{J}|^2 + 6|\nabla J|^2. \tag{2.10}$$

If $\nabla\bar{J}_{x_0} \neq 0$ then $16\tau = -|\nabla J|^2 - |\nabla\bar{J}|^2$ and at x_0 we get

$$32\Delta \ln|\alpha| = 5|\nabla J|^2. \tag{2.11}$$

But since at x_0 the function $|\alpha|$ has a local maximum we get $\Delta \ln|\alpha|_{x_0} \leq 0$ and consequently $|\nabla J| = 0$, a contradiction. Thus $b = 0$ on M. It follows that the Ricci tensor ϱ of (M, g) has the form $\varrho = \frac{1}{2}\tau g_{|\mathcal{D}}$. Note that $8\pi\gamma_J(X, Y) = \frac{1}{2}|\nabla J|^2\theta^3 \wedge \theta^4 - 4\varrho(X, JY)$ and $8\pi\gamma_{\bar{J}}(X, Y) = -4\varrho(X, \bar{J}Y)$ where $\gamma_J, \gamma_{\bar{J}}$ are the 2-forms representing the first Chern classes of (M, J) and $(M, \bar{J})$ respectively and $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is the coframe dual to $\{E_1, E_2, E_3, E_4\}$. Consequently, $\gamma_J^2 = 0 = \gamma_{\bar{J}}^2$. Thus $c_1(M, J)^2 = 0 = c_1(M, \bar{J})^2$. Since $c_1(M, J)^2 = 2\chi(M) + 3\sigma(M)$ and $c_1(M, \bar{J})^2 = 2\chi(M) - 3\sigma(M)$ we obtain $\chi(M) = 0 = \sigma(M)$. It is well known (see [B-P-V]) that a compact Kähler surface of negative total scalar curvature is either a ruled surface with base of genus at least 2, or its Kodaira dimension is at least 1. Since $\sigma(M) = 0 = \chi(M)$ we conclude (see [A-D], [B-P-V]) that $(M, \bar{J})$ belongs to class VI of the Kodaira classification and it is a minimal surface. From (2.16) in [J] we find that $a' = b' = 0$ and the connection forms satisfy

$$\omega^3_2 = \omega^4_1 = \tfrac{1}{2}\alpha\theta_1, \qquad \omega^1_3 = \omega^2_4 = -\tfrac{1}{2}\alpha\theta_2.$$

Hence $\omega^4_1(E_3) = -\Gamma^1_{34} = 0$ and $\omega^4_1(E_4) = -\Gamma^1_{44} = 0$. Consequently, the second fundamental form of $\mathcal{D}$ vanishes and $\mathcal{D}$ is totally geodesic. Now it is easy to see that every leaf D of $\mathcal{D}$ must be geodesically complete. ∎

From the proof of Proposition 1 we obtain:

COROLLARY 2. *Let (M, g, J) be a (G_3) almost Kähler 4-manifold such that $|\nabla J| \neq 0$ on M. Then on M we have*

$$32\Delta \ln|\alpha| = 16\tau + |\nabla\bar{J}|^2 + 6|\nabla J|^2.$$

What is more, if $(M, g, \bar{J})$ is Kähler then the nullity foliation $\mathcal{D}$ is totally geodesic.

We say that an almost Hermitian manifold (M, g, J) satisfies the *second condition* (G_2) *of A. Gray* if its curvature tensor R satisfies the condition

$$(G_2) \quad R(X, Y, Z, W) - R(JX, JY, Z, W) = R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. It is known that an almost Kähler manifold (M, g, J) satisfies (G_2) if and only if it satisfies (G_3) and the function $|\nabla J|$ is constant on M. In [A-D] it is proved that every (G_2) manifold is a $(2G_3)$ manifold. Our next result which follows directly from Corollary 2 was first proved in [A-D] under the additional assumption that M is compact.

PROPOSITION 2. *Let (M, g, J) be a strictly almost Kähler 4-manifold satisfying condition (G_2). Then the opposite almost Kähler structure $\bar{J}$ determined by the minimal foliations $\mathcal{D}, \mathcal{D}^\perp$ is Kähler, the scalar curvature τ is constant and $\tau = -\frac{3}{8}|\nabla J|^2$. Every leaf of $\mathcal{D}$ is totally geodesic and has constant negative Gauss curvature $K = -\frac{1}{8}|\nabla J|^2$.*

Proof. Recall that (M, g, J) satisfies (G_2) if and only if $|\nabla J|$ (hence $|\alpha|$) is constant on M. From Corollary 2 we obtain

$$16\tau = -6|\nabla J|^2 - |\nabla \bar{J}|^2. \tag{2.12}$$

Assume that $|\nabla \bar{J}| \neq 0$ at a point $x_0 \in M$. Then at x_0 we have

$$16\tau = -|\nabla J|^2 - |\nabla \bar{J}|^2. \tag{2.13}$$

From (2.12) and (2.13) it follows that $|\nabla J|_{x_0} = 0$, a contradiction. Thus $|\nabla \bar{J}| = 0$ on M and from (2.12) we derive $\tau = -\frac{3}{8}|\nabla J|^2$. Since every leaf of $\mathcal{D}$ is totally geodesic it follows from (2.6) that it has constant Gauss curvature $K = \frac{\tau}{2} + \frac{1}{16}|\nabla J|^2 = -\frac{1}{8}|\nabla J|^2$. ∎

COROLLARY 3. *Let (M, g, J) be a strictly almost Kähler $(2G_3)$ 4-manifold. Assume that the scalar curvature τ of (M, g) is constant. Then $(M, g, \bar{J})$ is Kähler and either (M, g, J) is Einstein with zero scalar curvature (hence self-dual) or the Ricci tensor ϱ of (M, g) has two constant eigenvalues $\lambda = 0$, $\mu = \frac{1}{2}\tau$.*

Proof. From our assumption $|\nabla J| \neq 0$ on M. Thus the opposite almost Kähler structure $\bar{J}$ is defined on the whole of M. Let $V := \{x \in M : |\nabla \bar{J}| \neq 0\}$. Let λ, μ be the eigenvalues of the Ricci tensor ϱ corresponding respectively to the eigendistributions $\mathcal{D}^\perp, \mathcal{D}$. Then on V we have $\lambda = -\frac{1}{32}|\nabla \bar{J}|^2$, $\mu = -\frac{1}{32}|\nabla J|^2$. Since τ is constant it follows from Proposition 6 of [J] that both λ and μ are constant. Hence $(M, g, \bar{J})$ satisfies (G_2). Consequently, $\nabla J = 0$ on V, a contradiction. Thus $\nabla \bar{J} = 0$ and $\lambda = 0, \mu = \frac{1}{2}\tau$. It is also known (see [De]) that a Kähler surface with zero scalar curvature is anti-selfdual with respect to the natural orientation. ∎

REMARK. In [O-S-Y] it is proved that every Einstein and weakly Einstein strictly almost Kähler manifolds is self-dual and has zero scalar curvature. The classification of such manifolds was also given by J. Armstrong.

PROPOSITION 3. *Let* (M, g, J) *be a compact* $(2G_3)$ *almost Kähler* 4-*manifold. Then the opposite almost Kähler structure* $\bar{J}$ *defined on* $\operatorname{dom}\bar{J} = \{x : |\nabla J|_x \neq 0\}$ *is Kähler.*

Proof. Note that $\operatorname{dom}\bar{J} = \bigcup_{c>0} F_c$ where $F_c = \{x : |\alpha(x)| = c\}$. Note that the function β is well defined and continuous on any F_c with $c \in \operatorname{im}|\alpha|,\ c > 0$. Let $\kappa = \sup\{|\beta(x)| : x \in F_c\}$. Every set F_c is compact. Let a point $x_0 \in F_c$ satisfy $|\beta(x_0)| = \kappa$. Since β is constant on the leaves of the foliation $\mathcal{D}$ it follows that at x_0 the function $|\beta|$ has a local maximum. On the other hand, since $(\operatorname{dom}\bar{J}, g, \bar{J})$ also satisfies condition (G_3) we have at x_0, if $\kappa > 0$,

$$32\Delta \ln|\beta| = 16\tau + 6|\nabla\bar{J}|^2 + |\nabla J|^2. \tag{2.14}$$

If $\kappa \neq 0$ then $\nabla\bar{J}_{x_0} \neq 0$ and hence $16\tau = -|\nabla J|^2 - |\nabla\bar{J}|^2$ at x_0. Consequently, $32\Delta \ln|\beta|_{x_0} = 5|\nabla\bar{J}|^2_{x_0} > 0$, which is a contradiction. It follows that $\nabla\bar{J} = 0$ wherever $\bar{J}$ is defined. ∎

PROPOSITION 4. *Let* (M, g, J) *be a compact* $(2G_3)$ *almost Kähler* 4-*manifold such that* $|\nabla J| \neq 0$ *on* M. *Then the scalar curvature* τ *of* (M, g) *attains positive and negative values on* M. *In particular*, τ *cannot be constant or nonpositive.*

Proof. From [D-2] it follows that τ cannot be nonnegative on M. Assume that $\tau \leq 0$ on M. It is clear that the set $M_- = \{x \in M : \tau(x) < 0\}$ is nonempty and open. It follows that the Ricci tensor $\varrho = \frac{1}{2}\tau g_{|\mathcal{D}}$ is nonpositive definite on M and nonzero on M_-. Now we can follow [A-D], [A-D-K]. Since $(M, \bar{J})$ is a minimal properly elliptic surface of Kodaira dimension 1 it is clear ([B-P-V], [A-D-K]) that $(M, \bar{J})$ is an elliptic fibration over a smooth curve such that all smooth fibers are isomorphic and the singular fibers are multiples of smooth elliptic curves (note that (6) of [A-D-K] is clearly satisfied). Taking a finite covering of $(M, \bar{J})$ we can assume that $(M, \bar{J})$ is a logarithmic transform of a total space of a principal elliptic fibre bundle, which means that it admits a nonvanishing holomorphic vector field X. The Ricci tensor ϱ is nonpositive definite on M. Then a Bochner type argument (this idea comes from the first version of [A-D]) shows that the field X is parallel and on M_- its real and imaginary parts belong to $\mathcal{D}^\perp$, thus the distribution $\mathcal{D}^\perp$ is parallel on M_-. Thus $\mathcal{D}$ is also parallel on M_- and consequently $\nabla J = 0$ on M_-, which is a contradiction. ∎

As an application of Proposition 4 we get a result of Draghici and Apostolov [A-D]:

COROLLARY 4. *Every compact almost Kähler* 4*-manifold satisfying condition* (G_2) *is Kähler.*

DEFINITION. We shall say that an almost Kähler manifold (M, g, J) is *strongly non-Kähler* if the set $M' = \{x : \nabla J_x \neq 0\}$ is dense in M. This means that the set $M_0 = \{x : \nabla J_x = 0\}$ is nowhere dense.

Note that if the manifold (M, g, J) is almost Kähler non-Kähler such that the function $\phi(x) = g(\nabla J_x, \nabla J_x)$ is real-analytic on M then (M, g, J) is strongly non-Kähler.

PROPOSITION 5. *Let* (M, g, J) *be strongly non-Kähler almost Kähler compact* 4*-manifold satisfying condition* $(2G_3)$*. Assume that* $0 \in \tau(M_0)$*. Then the opposite Kähler structure* $\bar{J}$ *defined on* M' *extends to a global Kähler structure* $\bar{J}$ *defined on* M*. Moreover,* $\sigma(M) = 0 = \chi(M)$*. If* (M, g) *has a negative total scalar curvature then the Kähler surface* $(M, g, \bar{J})$ *is a minimal class VI surface.*

Proof. Since the eigenvalues of the Ricci tensor ϱ are continuous functions on M it follows that ϱ has two eigenvalues $\lambda = 0$, $\mu = \frac{1}{2}\tau$. From our assumptions it follows that the form $\omega_1(X, Y) = (2/\tau)\varrho(X, JY)$ is well defined and smooth on an open set $U \supset M_0$ (note that M_0 is compact and $\tau \neq 0$ on M_0). Note that $\omega_1^2 = 0$ and $g(\omega_1, \omega_1) = 1$ on U. Hence the form ω_1 is a smooth extension of the characteristic form $\omega_{\mathcal{D}}$ of the foliation $\mathcal{D}$. Consequently, the form $\omega_2 = \Omega - \omega_1$ is a smooth extension of the characteristic form $\omega_{\mathcal{D}^\perp}$ of the foliation $\mathcal{D}^\perp$. Thus the form $\bar{\Omega} = \omega_2 - \omega_1 = \Omega - 2\omega_1$ is a smooth extension of the Kähler form $\bar{\Omega}_1$ of $(M', g, \bar{J})$. In particular, $\nabla\bar{\Omega} = 0$ and thus $\bar{\Omega}$ gives a global Kähler structure which is an extension of $\bar{J}$. It follows that the Ricci tensor ϱ of (M, g) has the form $\varrho = \frac{1}{2}\tau g_{|\mathcal{D}}$. Consequently, $\gamma_J^2 = 0 = \gamma_{\bar{J}}^2$ where $\gamma_J, \gamma_{\bar{J}}$ are the 2-forms representing the first Chern classes of (M, J) and $(M, \bar{J})$ respectively. Thus $c_1(M, J)^2 = 0 = c_1(M, \bar{J})^2$. Since $c_1(M, J)^2 = 2\chi(M) + 3\sigma(M)$ and $c_1(M, \bar{J})^2 = 2\chi(M) - 3\sigma(M)$ we obtain $\chi(M) = 0 = \sigma(M)$. The rest of the proof is exactly the same as in the proof of Proposition 1. ∎

PROPOSITION 6. *Let* (M, g, J) *be an almost Kähler compact* 4*-manifold satisfying condition* $(2G_3)$*. Assume that the function* $\phi = g(\nabla J, \nabla J)$ *is real-analytic on* M *and the scalar curvature* τ *of* (M, g) *is negative. Then* (M, g, J) *is Kähler.*

Proof. From our assumptions it follows that if (M, g, J) is not Kähler then it is a strongly non-Kähler almost Kähler surface. Thus from Proposition 5 it follows that the opposite Kähler structure extends to a global opposite Kähler structure $\bar{J}$ such that $(M, g, \bar{J})$ is a properly elliptic surface of Kodaira dimension 1 with vanishing Euler number. Now the considerations analogous to those in the proof of Proposition 4 show that (M, g, J) has to be Kähler, which is a contradiction. ∎

In fact, we have proved

COROLLARY 5. *There does not exist a strongly non-Kähler almost Kähler compact* 4*-manifold of type* $(2G_3)$ *with negative scalar curvature. In particular*, *there does not exist a strongly non-Kähler almost Kähler compact* 4*-manifold of type* $(2G_3)$ *with constant scalar curvature.*

Proof. The proof of the frist part of the corollary is just the same as the proof of Proposition 6. Note that Draghici has proved in [D-2] that a compact almost Kähler manifold with J-invariant Ricci tensor and nonnegative scalar curvature is Kähler. Thus if the scalar curvature is constant it is enough to consider the case when it is negative. From the first part of the proof it follows that such manifolds do not exist. ∎

Acknowledgments. The author is grateful to V. Apostolov, T. C. Draghici and K. Sekigawa for sending him the preprints of their papers. The work was supported by KBN grant 2 P0 3A 01615.

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Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: wjelon@pk.edu.pl

Received 28 April 1999;
revised 18 January 2007 (3739)