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ORDER CONVOLUTION AND VECTOR-VALUED MULTIPLIERS

BY

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Abstract. Let $I = (0,\infty)$ with the usual topology. For $x, y \in I$, we define $xy = \max(x,y)$. Then I becomes a locally compact commutative topological semigroup. The Banach space $L^1(I)$ of all Lebesgue integrable functions on I becomes a commutative semisimple Banach algebra with order convolution as multiplication. A bounded linear operator T on $L^1(I)$ is called a multiplier of $L^1(I)$ if $T(f\star g) = f\star Tg$ for all $f,g\in L^1(I)$. The space of multipliers of $L^1(I)$ was determined by Johnson and Lahr. Let X be a Banach space and $L^1(I,X)$ be the Banach space of all X-valued Bochner integrable functions on I. We show that $L^1(I,X)$ becomes an $L^1(I)$-Banach module. Suppose X and Y are Banach spaces. A bounded linear operator T from $L^1(I,X)$ to $L^1(I,Y)$ is called a multiplier if $T(f\star g) = f\star Tg$ for all $f\in L^1(I)$ and $g\in L^1(I,X)$. In this paper, we characterize the multipliers from $L^1(I,X)$ to $L^1(I,Y)$.

1. Introduction. Let $I = (0,\infty)$ with the usual topology. For $x, y \in I$, we define $xy = \max(x,y)$. Then I becomes a locally compact commutative topological semigroup. Let $L^1(I)$ denote the Banach space of all Lebesgue integrable functions on I. It becomes a commutative semisimple Banach algebra if multiplication is defined to be the order convolution introduced by Lardy [3]. More specifically, if $f,g\in L^1(I)$, then

$$f\star g(s) = f(s)\int_0^s g(t)\,dt + g(s)\int_0^s f(t)\,dt.$$

A bounded linear operator T on $L^1(I)$ is called a *multiplier of* $L^1(I)$ if $T(f\star g) = f\star Tg$ for all $f,g\in L^1(I)$. Johnson and Lahr [2] characterized the multipliers of $L^1(I)$. In fact, they considered any interval in place of I, with possibly infinite end points, and which may include one or the other of the end points. Slightly earlier, Larsen [4] had characterized the multipliers of $L^1([0,1])$ with order convolution.

Let X be a Banach space. Let $L^1(I,X)$ be the Banach space of X-valued measurable functions f such that $\int_0^\infty \|f(t)\|\,dt<\infty$. If $f\in L^1(I)$ and

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$g \in L^1(I, X)$, we define

$$f \star g(s) = f(s) \int_0^s g(t)\,dt + g(s) \int_0^s f(t)\,dt.$$

It turns out that $f \star g \in L^1(I, X)$ and $L^1(I, X)$ becomes an $L^1(I)$-Banach module.

Let X and Y be Banach spaces. A bounded linear operator T from $L^1(I, X)$ to $L^1(I, Y)$ is called a *multiplier from* $L^1(I, X)$ *to* $L^1(I, Y)$ if $T(f \star g) = f \star Tg$ for all $f \in L^1(I)$ and $g \in L^1(I, X)$. In this paper, we characterize the multipliers from $L^1(I, X)$ to $L^1(I, Y)$.

2. Preliminaries. Let $I = (0, \infty)$ as before. Let $M(I)$ denote the Banach algebra of all bounded regular Borel measures on I. It can be identified with the Banach space dual of $C_0(I)$, the Banach space of all continuous functions on I vanishing at infinity. The convolution of μ and ν belonging to $M(I)$ is defined by

$$\int_I f(z)\,d\mu \star \nu(z) = \int_I \int_I f(xy)\,d\mu(x)\,d\nu(y).$$

The Banach space $L^1(I)$ consisting of all measures in $M(I)$ which are absolutely continuous with respect to Lebesgue measure on I becomes a commutative semisimple Banach algebra with the product inherited from $M(I)$. If $f, g \in L^1(I)$, this product turns out to be

$$f \star g(s) = f(s) \int_0^s g(t)\,dt + g(s) \int_0^s f(t)\,dt \qquad \text{a.e.}$$

Lardy [3] studied the algebra $L^1(I)$. Its maximal ideal space $\widehat{I}$ can be identified with the interval $(0, \infty]$ and the Gelfand transform $\widehat{f}$ of $f \in L^1(I)$ is defined by

$$\widehat{f}(s) = \int_0^s f(t)\,dt \qquad (0 < s \le \infty),$$

that is, $\widehat{f}$ is the indefinite integral of f on $(0, \infty]$. The algebra $L^1(I)$ is without identity, but it does have approximate identities. One such approximate identity is the sequence $\{u_n\}$ defined by

$$u_n(x) = \begin{cases} n & \text{if } 0 < x \le 1/n, \\ 0 & \text{if } 1/n < x < \infty, \end{cases} \qquad n = 1, 2, \ldots.$$

If T is a multiplier of $L^1(I)$ there exists a bounded continuous function ϕ on $(0, \infty]$ such that $(Tf)^\wedge = \phi \widehat{f}$ and $\|\phi\|_\infty \le \|T\|$, where $\|\phi\|_\infty = \sup_{t \in I} |\phi(t)|$. Conversely, if ϕ is a bounded continuous function on $(0, \infty]$ such that for each $f \in L^1(I)$ there exists a $g \in L^1(I)$ such that $\widehat{g} = \phi \widehat{f}$ then we define $Tf = g$ and T becomes a multiplier of $L^1(I)$ such that $(Tf)^\wedge = \phi\widehat{f}$ for all $f \in L^1(I)$.

Larsen gave a characterization of the multipliers of $L^1[0,1]$ under order convolution. Let $AC[0,1]$ denote the subalgebra of $C[0,1]$ consisting of all absolutely continuous functions on $[0,1]$. Here, $C[0,1]$ is the space of continuous functions on $[0,1]$. Suppose $\phi \in AC[0,1]$; then for all $f \in L^1[0,1]$ we have $\phi\widehat{f} \in AC[0,1]$ and $(\phi\widehat{f})(0) = 0$. Hence there exists a $g \in L^1[0,1]$ such that $\widehat{g} = \phi\widehat{f}$, in fact $g = (\phi\widehat{f})'$ almost everywhere. If we define $Tf = g$, then we get a multiplier of $L^1[0,1]$ such that $(Tf)^\wedge = \phi\widehat{f}$. Conversely, Larsen [4] proved that if T is a multiplier of $L^1[0,1]$ then there exists $\phi \in AC[0,1]$ such that $(Tf)^\wedge = \phi\widehat{f}$ for all $f \in L^1[0,1]$.

In [2] Johnson and Lahr described the multipliers of $L^1(J)$, where J is any interval, with possibly infinite end points, and which may contain one or the other of the end points. The following theorem is an immediate consequence of the multiplier results in [2] and [4].

THEOREM 2.1. *Let T be a multiplier of $L^1(I)$. Then there exists an absolutely continuous function ϕ on $(0,\infty]$ which is of bounded variation such that $(Tf)^\wedge(s) = \phi(s)\widehat{f}(s)$ for all $s \in (0,\infty]$ and $f \in L^1(I)$. Conversely, if ϕ is an absolutely continuous function on $(0,\infty]$ which is of bounded variation then there exists a multiplier T of $L^1(I)$ such that $(Tf)^\wedge(s) = \phi(s)\widehat{f}(s)$ for all $s \in (0,\infty]$ and $f \in L^1(I)$.*

REMARK. If ϕ is as in Theorem 2.1 then ϕ is differentiable almost everywhere and $\phi' \in L^1(I)$. Further, $\lim_{t\to 0+}\phi(t)$ exists. Let $\phi(0) = \lim_{t\to 0+}\phi(t)$. Then $Tf = \phi(0)f + (\phi\widehat{f})'$.

3. Main results. In this section we prove our main result which characterizes the multipliers from $L^1(I,X)$ to $L^1(I,Y)$. We begin with a proposition about $L^1(I,X)$.

PROPOSITION 3.1. *Let $\{u_n\}$ be the approximate identity of $L^1(I)$ defined earlier. Suppose $f \in L^1(I,X)$. Then*

$$\|u_n \star f - f\|_1 \to 0 \qquad \text{as } n \to \infty.$$

Proof. Let $\varepsilon > 0$. Choose $t > 0$ such that

$$\int_0^t \|f(s)\|\,ds < \varepsilon/3.$$

If $s > 1/n$, then

$$u_n \star f(s) = u_n(s)\int_0^s f(r)\,dr + f(s)\int_0^s u_n(r)\,dr = f(s),$$

since $u_n(s) = 0$ and $\int_0^s u_n(r)\,dr = 1$. Therefore,

$$\int_0^\infty \|u_n \star f(s) - f(s)\|\,ds = \int_0^{1/n} \|u_n \star f(s) - f(s)\|\,ds$$
$$= \int_0^{1/n} \Big\| u_n(s)\int_0^s f(r)\,dr + f(s)\int_0^s u_n(r)\,dr - f(s)\Big\|\,ds.$$

Choose n_0 such that $t > 1/n_0$. Then, for all $n \geq n_0$,

$$\|u_n \star f - f\|_1 \leq \int_0^{1/n} \Big[u_n(s)\int_0^s \|f(r)\|\,dr + 2\|f(s)\|\Big]\,ds.$$

Since $t > 1/n$ and $s \leq 1/n$, we have

$$\|u_n \star f - f\|_1 \leq \frac{\varepsilon}{3}\int_0^{1/n} u_n(s)\,ds + \frac{2\varepsilon}{3} = \varepsilon.$$

Definition. Let $f \in L^1(I,X)$. For each $s \in (0,\infty]$, define

$$\widehat{f}(s) = \int_0^s f(t)\,dt.$$

The function $\widehat{f}$ is called the *Gelfand transform* of f. Clearly $\widehat{f}$ is absolutely continuous. Also $(\widehat{f})'(s) = f(s)$ almost everywhere.

Note that $\widehat{f}(s) \to 0$ as $s \to 0$. Further, if $\widehat{f}(s) = 0$ for all $s \in (0,\infty]$ then $f(s) = 0$ almost everywhere.

Proposition 3.2. *Let X, Y be Banach spaces and $B(X,Y)$ be the Banach space of bounded linear maps of X into Y. Suppose T is a multiplier from $L^1(I,X)$ into $L^1(I,Y)$. Then there exists a $B(X,Y)$-valued bounded strongly continuous function ϕ on $(0,\infty]$ such that $(Tf)^\wedge(s) = \phi(s)\widehat{f}(s)$ for all $s \in (0,\infty]$ and $f \in L^1(I,X)$, where $\phi(s)\widehat{f}(s)$ is the value of $\phi(s)$ at $\widehat{f}(s)$.*

Proof. Let $f, g \in L^1(I)$ and $x \in X$. Then the function fx defined by $(fx)(s) = f(s)x$ belongs to $L^1(I,X)$. It is easy to see that $(f \star g)x = (fx) \star g = f \star gx$. Since T is a multiplier, we have

$$T((f \star g)x) = f \star T(gx) = T(fx) \star g.$$

Therefore, $\widehat{f}(s)(T(gx))^\wedge(s) = (T(fx))^\wedge(s)\widehat{g}(s)$. We define

$$\phi(s)x = \frac{(T(fx))^\wedge(s)}{\widehat{f}(s)}, \quad \text{provided} \quad \widehat{f}(s) \neq 0.$$

We see that the definition of $\phi(s)$ does not depend on the choice of f and $\phi(s)$ is a linear map from X into Y. We now show that $\phi(s) \in B(X,Y)$ and

$\|\phi(s)\| \le \|T\|$. Let $s \in (0,\infty]$. Choose n such that $s \ge 1/n$. Then $\widehat{u}_n(s) = 1$ and we have

$$\|\phi(s)x\| = \|(T(u_n x))^\wedge(s)\| \le \|T(u_n(x))\|_1 \le \|T\|\,\|x\|,$$

as $\|u_n\|_1 = 1$. Therefore $\phi(s) \in B(X,Y)$ and $\|\phi(s)\| \le \|T\|$. It follows from the definition of $\phi(s)$ that it is continuous in the strong operator topology and $(T(fx))^\wedge(s) = \phi(s)(fx)^\wedge(s)$ for $f \in L^1(I)$ and $x \in X$. Since $\{\sum_{i=1}^n f_i x_i : f_i \in L^1(I),\ x_i \in X\}$ is dense in $L^1(I,X)$, we conclude that

$$(Tf)^\wedge(s) = \phi(s)\widehat{f}(s) \quad \text{for all } f \in L^1(I,X).$$

Proposition 3.3. *Let $U : X \to Y$ be a bounded linear map. Then the map $\widetilde{U} : L^1(I,X) \to L^1(I,Y)$ defined by $\widetilde{U}f = U \circ f$ is a multiplier from $L^1(I,X)$ to $L^1(I,Y)$ and*

$$\|\widetilde{U}f\| \le \|U\|\,\|f\|_1.$$

Proof. Let $f \in L^1(I)$ and $g \in L^1(I,X)$. Then

$$f \star g(s) = f(s)\int_0^s g(t)\,dt + g(s)\int_0^s f(t)\,dt.$$

Hence

$$\begin{aligned}\widetilde{U}(f\star g)(s) &= U((f\star g)(s))\\ &= f(s)\int_0^s U(g(t))\,dt + U(g(s))\int_0^s f(t)\,dt = f \star \widetilde{U}g(s).\end{aligned}$$

Also

$$(\widetilde{U}f)^\wedge(s) = \int_0^s U(f(t))\,dt = U(\widehat{f}(s)).$$

Thus the multiplier function ϕ corresponding to $\widetilde{U}$ is $\phi(s) = U$ for all $s \in (0,\infty]$.

The following definitions are taken from Hille and Phillips [1].

Definition. Let $\phi(s)$ be an operator-valued function defined on $(0,\infty)$. We say that ϕ is of *strong bounded variation* on $(0,\infty)$ if for each $x \in X$ the function $s \mapsto \phi(s)x$ is of strong bounded variation, that is,

$$\sup \sum_{i=1}^n \|\phi(t_i)x - \phi(t_{i-1})x\| < \infty,$$

where all possible finite sets $\{t_0, t_1, \ldots, t_n\} \subset (0,\infty)$ such that $t_0 < t_1 < \cdots < t_n$ are allowed.

ϕ is called *strongly absolutely continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\{(s_i,t_i)\}$ is a finite sequence of disjoint open intervals for which $\sum(t_i - s_i) < \delta$, we have

$$\sum_{i=1}^{n} \|\phi(t_i)x - \phi(s_i)x\| < \varepsilon.$$

For all results and notions regarding operator-valued functions on $(0,\infty)$, we refer to [1].

PROPOSITION 3.4. *Let T be a multiplier from $L^1(I,X)$ to $L^1(I,Y)$. Let ϕ be the corresponding multiplier operator-valued function. Then $\phi(s)$ is of strong bounded variation on $(0,\infty)$.*

Proof. Let $\{t_0, t_1, \ldots, t_n\} \subset (0,\infty)$ be such that $t_0 < t_1 < \cdots < t_n$. Let $a = t_0$ and $b = t_n$. Then $\{t_0, t_1, \ldots, t_n\}$ is a partition of $[a,b]$. Choose an integer m such that $1/m \le a$. Then $\widehat{u}_m(s) = 1$ for all $s \ge 1/m$. Let $x \in X$. For $s \ge 1/m$, we have $\phi(s)x = (T(u_m x))^\wedge(s)$. Therefore,

$$\sum_{i=1}^{n} \|\phi(t_i)x - \phi(t_{i-1})x\| = \sum_{i=1}^{n} \|(T(u_m x))^\wedge(t_i) - (T(u_m x))^\wedge(t_{i-1})\|$$

$$= \sum_{i=1}^{n} \Big\| \int_{t_{i-1}}^{t_i} T(u_m x)(r)\,dr \Big\| \le \int_a^b \|T(u_m x)(r)\,dr\| \le \|T(u_m x)\|_1 \le \|T\|\,\|x\|.$$

This completes the proof.

PROPOSITION 3.5. *Let T and ϕ be as in Proposition 3.4. Then the map $s \mapsto \phi(s)x$ is weakly absolutely continuous.*

Proof. We have

$$(Tf)^\wedge(s) = \phi(s)\widehat{f}(s) \quad \text{for all } s \in (0,\infty] \text{ and } f \in L^1(I,X).$$

Let $x \in X$ and $y^* \in Y^*$. Define $S : L^1(I) \to L^1(I)$ by

$$Sf(s) = \langle T(fx)(s), y^* \rangle \quad \text{for } f \in L^1(I).$$

It can be easily seen that

$$S(f \star g) = f \star Sg \quad \text{for } f, g \in L^1(I).$$

Hence S is a multiplier of $L^1(I)$. By Theorem 2.1, there exists an absolutely continuous function h on $(0,\infty]$ such that $(Sf)^\wedge(s) = h(s)\widehat{f}(s)$ for all $f \in L^1(I)$. It also follows from the definition of S that $(Sf)^\wedge(s) = \langle \widehat{f}(s)\phi(s)x, y^* \rangle$. Choosing $f \in L^1(I)$ such that $\widehat{f}(s) \ne 0$, we see that $\langle \phi(s)x, y^* \rangle = h(s)$. This shows that the map $s \mapsto \phi(s)(x)$ is weakly absolutely continuous.

PROPOSITION 3.6. *Let T and ϕ be as in Proposition 3.4. Then for each $x \in X$, the function $s \mapsto \phi(s)x$ is strongly differentiable almost everywhere.*

Proof. Let n be any positive integer. Then for all $s > 1/n$,

$$\phi(s)x = (T(u_n x))^\wedge(s) = \int_0^s T(u_n x)(t)\,dt.$$

It follows from Corollary 2, p. 88 of [1] that the function $s \mapsto \phi(s)x$ is strongly differentiable almost everywhere on $(1/n, \infty)$, and hence on $(0, \infty)$, since n is arbitrary. The derivative of this function is denoted by $\phi'(s)x$. It is easy to see that $\phi'(s) \in B(X, Y)$ and $\|\phi'(s)\| \le \|T\|$ for almost all $s \in (0, \infty)$.

COROLLARY 3.7. *Let T and ϕ be as in Proposition* 3.4. *Then for all $x \in X$, $s \mapsto \phi(s)x$ is strongly absolutely continuous and the function $s \mapsto \phi'(s)x$ is Bochner integrable such that*

$$\int_0^\infty \|\phi'(s)x\|\,ds \le \|T\|\,\|x\|.$$

Proof. It follows from Propositions 3.4 to 3.6 and Theorem 3.8.6 of [1] that the function $s \mapsto \phi(s)x$ is strongly absolutely continuous and $s \mapsto \phi'(s)x$ is Bochner integrable. From the proof of Proposition 3.6, we see that $\phi'(s)x = T(u_n x)(s)$ almost everywhere on $(1/n, \infty)$. Therefore

$$\int_{1/n}^\infty \|\phi'(s)x\|\,ds \le \int_{1/n}^\infty \|T(u_n x)(s)\|\,ds \le \|T\|\,\|x\|.$$

Since n is arbitrary,

$$\int_0^\infty \|\phi'(s)x\|\,ds \le \|T\|\,\|x\|.$$

PROPOSITION 3.8. *Let T and ϕ be as in Proposition* 3.4. *For $f \in L^1(I, X)$, let $M_{\phi'}(f) = \phi' f$. Then $M_{\phi'}$ is a bounded linear map from $L^1(I, X)$ into $L^1(I, Y)$.*

Proof. Let $f \in L^1(I, X)$. Then $(Tf)^\wedge(s) = \phi(s)\widehat{f}(s)$. Therefore,

$$\phi(s)\widehat{f}(s) = \int_0^s (Tf)(t)\,dt.$$

Differentiating, we have

$$\phi(s)f(s) + \phi'(s)\widehat{f}(s) = Tf(s) \qquad \text{a.e.}$$

Hence $M_{\phi'}(f) = Tf - \phi f$. Since ϕ is bounded,

$$\int_0^\infty \|\phi(s)f(s)\|\,ds \le \|\phi\|_\infty \int_0^\infty \|f(s)\|\,ds.$$

Therefore $\phi f \in L^1(I,Y)$. It follows that $M_{\phi'}(f) \in L^1(I,Y)$. We also have

$$\|M_{\phi'}(f)\|_1 \le (\|T\| + \|\phi\|_\infty)\|f\|_1.$$

We are now in a position to prove our main result.

THEOREM 3.9. *Let T be a multiplier from $L^1(I,X)$ to $L^1(I,Y)$ and ϕ be the corresponding multiplier function from $(0,\infty)$ to $B(X,Y)$. Then, for all $x \in X$,*

(i) *$s \mapsto \phi(s)x$ is of strong bounded variation on $(0,\infty)$,*
(ii) *$s \mapsto \phi(s)x$ is strongly differentiable almost everywhere,*
(iii) *$s \mapsto \phi(s)x$ is strongly absolutely continuous,*
(iv) *if $M_{\phi'}(f) = \phi'\widehat{f}$ then $M_{\phi'} : L^1(I,X) \to L^1(I,Y)$ is a bounded linear map,*
(v) *$(Tf)^\wedge(s) = \phi(s)(\widehat{f}(s))$ for all $s \in (0,\infty)$ and $f \in L^1(I,X)$.*

Conversely, if ϕ is a bounded $B(X,Y)$-valued function on $(0,\infty)$ satisfying (i) *to* (iv) *then there exists a multiplier T from $L^1(I,X)$ to $L^1(I,Y)$ satisfying* (v).

Proof. Let T be a multiplier from $L^1(I,X)$ to $L^1(I,Y)$ and ϕ be the corresponding multiplier function. Then (i) and (ii) follow from Proposition 3.4 and 3.6 respectively. Further, (iii) and (iv) follow from Corollary 3.7 and Proposition 3.8 respectively. Moreover, (v) is a consequence of the relationship between T and ϕ.

Conversely, suppose ϕ is a bounded $B(X,Y)$-valued function on $(0,\infty)$ which satisfies all conditions (i) to (iv). We define

$$T : L^1(I,X) \to L^1(I,Y)$$

by

$$Tf(s) = \phi(s)f(s) + \phi'(s)\widehat{f}(s) \qquad \text{a.e. for } f \in L^1(I,X).$$

Since ϕ is bounded and continuous in the strong operator topology, the function $\phi(s)f(s)$ is strongly measurable and

$$\int_0^\infty \|\phi(s)f(s)\|\,ds \le \|\phi\|_\infty \|f\|_1.$$

Therefore, $\|Tf\|_1 \le [\|\phi\|_\infty + \|M_{\phi'}\|]\|f\|_1$ and we conclude that T is a bounded linear map from $L^1(I,X)$ to $L^1(I,Y)$.

We can easily see that the derivative of $\phi\widehat{f}$ equals $\phi(s)f(s) + \phi'(s)\widehat{f}(s) = Tf(s)$ almost everywhere. Hence, $(Tf)^\wedge(s) = \phi(s)\widehat{f}(s)$ for all $s \in (0,\infty]$. This completes the proof of the theorem.

NOTE. If the bounded operator-valued function ϕ satisfies (i)–(iii) and

$$\int_0^\infty \|\phi'(s)\|\, ds < \infty,$$

then (iv) automatically holds. But this condition is much stronger than (iv). Finally, we remark that we could have taken I to be any finite or infinite subinterval of the real line and obtained analogous multiplier results in the more general setting.

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