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TOP-STABLE AND LAYER-STABLE DEGENERATIONS AND HOM-ORDER

BY

S. O. SMALØ and A. VALENTA (Trondheim)

Abstract. Using geometrical methods, Huisgen-Zimmermann showed that if M is a module with simple top, then M has no proper degeneration $M <_{\deg} N$ such that $\mathfrak{r}^t M/\mathfrak{r}^{t+1}M \simeq \mathfrak{r}^t N/\mathfrak{r}^{t+1}N$ for all t. Given a module M with square-free top and a projective cover P, she showed that $\dim_k \operatorname{Hom}(M,M) = \dim_k \operatorname{Hom}(P,M)$ if and only if M has no proper degeneration $M <_{\deg} N$ where $M/\mathfrak{r}M \simeq N/\mathfrak{r}N$. We prove here these results in a more general form, for hom-order instead of degeneration-order, and we prove them algebraically. The results of Huisgen-Zimmermann follow as consequences from our results. In particular, we find that her second result holds not just for modules with square-free top, but also for indecomposable modules in general.

1. Introduction. Let Λ be a finitely generated associative k-algebra with unit, where k is an algebraically closed field. By $\operatorname{mod}\Lambda$ we denote the category of finite-dimensional left Λ-modules.

A d-dimensional left Λ-module M is a d-dimensional vector space M together with a multiplication on M by Λ from the left. By choosing a basis in M one can identify M with the vector space k^d. If $\lambda_1, \ldots, \lambda_r$ is a generating set for Λ over k, then the multiplication on M by λ_i induces a k-endomorphism of k^d which can be represented by a $d \times d$-matrix over k. Thus, M corresponds to a unique r-tuple $m = (m_1, \ldots, m_r)$ of $d \times d$-matrices over k such that for all polynomials f in r non-commuting variables over k with the property that $f(\lambda_1, \ldots, \lambda_r) = 0$ in Λ, we have $f(m_1, \ldots, m_r) = 0$ in the ring of $d \times d$-matrices over k. Conversely, each such r-tuple m corresponds to a ring homomorphism $\phi_m : \Lambda \to \operatorname{End}_k(k^d)$ and hence gives a module structure M on k^d in the obvious way. We denote by $\operatorname{mod}_\Lambda^d(k)$ the set of all such r-tuples. It is an affine variety.

The general linear group $\mathrm{Gl}_d(k)$ acts on $\operatorname{mod}_\Lambda^d(k)$ by conjugation, $g * x = (gx_1g^{-1}, \ldots, gx_rg^{-1})$ for $g \in \mathrm{Gl}_d(k)$, $x \in \operatorname{mod}_\Lambda^d(k)$, and the orbits correspond to the isomorphism classes of d-dimensional Λ-modules (see [K]). Let M and N be two d-dimensional left Λ-modules corresponding to m and n in $\operatorname{mod}_\Lambda^d(k)$ respectively. We say that M *degenerates* to N if n belongs

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to the Zariski closure $\overline{\mathcal{O}(m)}$ of $\mathcal{O}(m)$, and we denote this by $M \leq_{\deg} N$ (see [B]).

Clearly, $\leq_{\deg}$ is a partial order on the set of isomorphism classes of d-dimensional modules.

Riedtmann showed in [R] that if there is an exact sequence of the form

$$0 \to U \to U \oplus M \to N \to 0 \tag{1}$$

for some $U \in \operatorname{mod} \Lambda$, or of the form

$$0 \to N \to M \oplus V \to V \to 0 \tag{2}$$

for some $V \in \operatorname{mod} \Lambda$, then M degenerates to N. In particular, if

$$0 \to A \to B \to C \to 0$$

is an exact sequence, then $B \leq_{\deg} A \oplus C$.

Zwara showed in [Z1] that the existence of a sequence of the form (1) is equivalent to the existence of a sequence of the form (2), and he showed in [Z2] that this is equivalent to $M \leq_{\deg} N$.

From the equivalence of the degeneration-order with the existence of Riedtmann sequences above, it is easy to see that if $M \leq_{\deg} N$ then

$$\dim_k \operatorname{Hom}_\Lambda(X, M) \leq \dim_k \operatorname{Hom}_\Lambda(X, N)$$

for each Λ-module X. We shall use the notation (X, Y) for $\operatorname{Hom}_\Lambda(X, Y)$ and $[X, Y]$ for $\dim_k \operatorname{Hom}_\Lambda(X, Y)$.

We will write $M \leq_{\mathrm{hom}} N$ if $[X, M] \leq [X, N]$ for each Λ-module X.

Auslander has proved the following results:

- If $M \leq_{\mathrm{hom}} N$ and $N \leq_{\mathrm{hom}} M$, then M and N are isomorphic. Hence $\leq_{\mathrm{hom}}$ is a partial order on the set of isomorphism classes of d-dimensional modules.
- $M \leq_{\mathrm{hom}} N$ if and only if $[M, X] \leq [N, X]$ for all X.

Bongartz proved in [B] that $M <_{\mathrm{hom}} N$ implies $[M, M] < [N, N]$. Note that the hom-order is also defined when the field k is not algebraically closed.

These two partial orders are not equivalent. Degeneration-order implies hom-order as shown above, but Carlson gave an example (see [R]) of two modules M and N such that $M \leq_{\mathrm{hom}} N$, while $M \not\leq_{\deg} N$.

2. The main results. B. Huisgen-Zimmermann defines in her preprint [H-Z2] top-stable and layer-stable degenerations. She calls a degeneration $M \leq_{\deg} N$ *top-stable* if $M/\mathfrak{r}M \simeq N/\mathfrak{r}N$, and *layer-stable* if $\mathfrak{r}^t M/\mathfrak{r}^{t+1} M \simeq \mathfrak{r}^t N/\mathfrak{r}^{t+1} N$ for all t.

EXAMPLE. Let Λ be the Kronecker algebra which is the path algebra of the quiver $1 \rightrightarrows 2$ over k. The degeneration

$$(k^2 \underset{\binom{0\ 0}{1\ 0}}{\overset{\binom{1\ 0}{0\ 1}}{\rightrightarrows}} k^2) \leq_{\deg} (k \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} k^2 \oplus k \rightrightarrows 0)$$

is both top-stable and layer-stable.

EXAMPLE. Let Λ be the path algebra of the quiver $1 \to 2 \to 3$. The degeneration

$$(k \xrightarrow{1} k \xrightarrow{1} k \oplus 0 \to k \to 0) \leq_{\deg} (k \xrightarrow{1} k \to 0 \oplus 0 \to k \xrightarrow{1} k)$$

is top-stable, but not layer-stable.

EXAMPLE. Given an almost split exact sequence $0 \to A \to B \to C \to 0$ of Λ-modules, let $I = \mathrm{D}(\Lambda)$. If A is not a summand of $\operatorname{soc} I$, i.e. A is not simple, then the degeneration $B <_{\deg} A \oplus C$ is top-stable, and if A is not a summand of $\operatorname{soc}^i I$ for any i, then the degeneration $B <_{\deg} A \oplus C$ is layer-stable.

We say that a module M has a *square-free top* if $M/\mathfrak{r}M = \bigoplus_i S_i$ where S_i's are simple modules such that $S_i \not\simeq S_j$ when $i \neq j$.

Using purely geometrical methods, Huisgen-Zimmermann shows in Theorem 4.2 and Corollary 4.3 of [H-Z1] the following results.

THEOREM 2.1.

(a) *If M is a Λ-module with simple top, then M does not have any proper layer-stable degenerations.*

(b) *Given a Λ-module M with square-free top, the following conditions are equivalent:*

  (1) *M does not have any proper top-stable degenerations.*

  (2) *$[M,M] = [P,M]$, where P is a projective cover of M.*

Using the hom-order and the algebraic characterization of degenerations by the existence of Riedtmann sequences, these results can be proven more elegantly and in more general form, as we shall see here. But first note the following.

PROPOSITION 2.2. *Let P be a projective cover of M. Then $[M,M] \leq [P,M]$ and $[M,M] = [P,M]$ if and only if M is projective as a $\Lambda/\operatorname{ann} M$-module.*

Proof. Let

$$0 \to K \to P \to M \to 0$$

be an exact sequence where the last map is a projective cover of M. Applying the functor $(\ ,M)$ to this sequence, it is easy to see that $[M,M] \leq [P,M]$.

Assume that M is a projective $\Lambda/\operatorname{ann} M$-module, i.e. $M = P/\operatorname{ann} M \cdot P$. Applying the functor $(\ ,M)$ to the exact sequence

$$0 \to \operatorname{ann} M \cdot P \to P \to P/\operatorname{ann} M \cdot P \to 0,$$

we get the exact sequence

$$0 \to (P/\operatorname{ann} M \cdot P, M) \to (P, M) \to (\operatorname{ann} M \cdot P, M).$$

The last map in this sequence is non-zero if and only if there is a map $h : P \to M$ such that $h(\operatorname{ann} M \cdot P) \neq 0$. But $h(\lambda x) = \lambda h(x) = 0$ for all $\lambda \in \operatorname{ann} M$ and all $x \in P$. Hence, the last map is 0 and it follows that $[P, M] = [P/\operatorname{ann} M \cdot P, M]$ for all M. By assumption, $M = P/\operatorname{ann} M \cdot P$ and hence $[M, M] = [P, M]$.

Assume now that $[M, M] = [P, M]$ and that M is not a projective $\Lambda/\operatorname{ann} M$-module. Let

$$0 \to C \to P/\operatorname{ann} M \cdot P \to M \to 0$$

be an exact sequence of $\Lambda/\operatorname{ann} M$-modules, where the last map is a projective cover. Since M is a faithful $\Lambda/\operatorname{ann} M$-module, there is a monomorphism $\Lambda/\operatorname{ann} M \to M^r$ for some $r \in \mathbb{N}$ and so there is a monomorphism $P/\operatorname{ann} M \cdot P \to M^s$ for some $s \in \mathbb{N}$. Hence there is some $u : P/\operatorname{ann} M \cdot P \to M$ such that $u(C) \neq 0$. From the above, $[P/\operatorname{ann} M \cdot P, M] = [P, M]$ and hence, by assumption, $[P/\operatorname{ann} M \cdot P, M] = [M, M]$. Applying the functor $(\ ,M)$ to the exact sequence above, we then infer that $u(C) = 0$ for each map $u : P/\operatorname{ann} M \cdot P \to M$. Thus, we get the required contradiction. ∎

Definition. We define top-stable and layer-stable hom-order in a natural way: we call the hom-order $M \leq_{\text{hom}} N$ *top-stable* if $M/\mathfrak{r}M \simeq N/\mathfrak{r}M$, and *layer-stable* if $\mathfrak{r}^t M/\mathfrak{r}^{t+1} M \simeq \mathfrak{r}^t N/\mathfrak{r}^{t+1} N$ for all t.

Theorem 2.3. *Given a d-dimensional Λ-module M with simple top, if there is some d-dimensional module N such that $M \leq_{\text{hom}} N$ is layer-stable, then $M \simeq N$.*

Proof. Assume that $M \leq_{\text{hom}} N$ is layer-stable and $M/\mathfrak{r}M$ is a simple module, say S. We proceed by induction on the Loewy length of M to show that M and N must be isomorphic.

If the Loewy length of M is 1, the claim clearly holds. Assume that it holds for modules of Loewy length less than the Loewy length of M. Consider the modules $M' = M/\mathfrak{r}^n M$ and $N' = N/\mathfrak{r}^n N$ where $n+1$ is the Loewy length of M. These two modules have the same dimension since $\mathfrak{r}^t M/\mathfrak{r}^{t+1} M \simeq \mathfrak{r}^t N/\mathfrak{r}^{t+1} N$ for all t. Moreover, $\mathfrak{r}^t M'/\mathfrak{r}^{t+1} M' \simeq \mathfrak{r}^t N'/\mathfrak{r}^{t+1} N'$ for all t, and $M'/\mathfrak{r}M' \simeq S$. It is not difficult to see that $M/\mathfrak{r}^n M \leq_{\text{hom}} N/\mathfrak{r}^n N$ as we shall see here. Given a Λ-module X with Loewy length less than $n+1$, the Loewy length of M, we have

$$[M/\mathfrak{r}^n M, X] = [M, X] \le [N, X] = [N/\mathfrak{r}^n N, X]$$

since $M \le_{\text{hom}} N$. If X is a Λ-module with Loewy length greater than or equal to $n+1$, then $[M/\mathfrak{r}^n M, X] = [M/\mathfrak{r}^n M, \operatorname{soc}^n X]$ (and $[N/\mathfrak{r}^n N, X] = [N/\mathfrak{r}^n N, \operatorname{soc}^n X]$) and $\operatorname{soc}^n X$ has Loewy length n. Therefore, by the previous remark, we will again have $[M/\mathfrak{r}^n M, X] \le [N/\mathfrak{r}^n N, X]$. Thus we have this inequality for each module X and consequently $M/\mathfrak{r}^n M \le_{\text{hom}} N/\mathfrak{r}^n N$. By induction, it follows that $M/\mathfrak{r}^n M \simeq N/\mathfrak{r}^n N$.

We have $M \le_{\text{hom}} N$ by assumption. In particular, $[M, M] \le [N, M]$. Assume that M and N are not isomorphic. Then

$$[M, M] - 1 = [M, \mathfrak{r}M] \quad \text{and} \quad [N, M] = [N, \mathfrak{r}M]$$

since $M/\mathfrak{r}M$ is a simple module. Moreover,

$$[M, \mathfrak{r}M] = [M/\mathfrak{r}^n M, \mathfrak{r}M] \quad \text{and} \quad [N, \mathfrak{r}M] = [N/\mathfrak{r}^n N, \mathfrak{r}M].$$

Since $M/\mathfrak{r}^n M \simeq N/\mathfrak{r}^n N$, we see that $[M/\mathfrak{r}^n M, \mathfrak{r}M] = [N/\mathfrak{r}^n N, \mathfrak{r}M]$ and it follows that $[M, M] > [N, M]$, a contradiction. Therefore M and N must be isomorphic. ∎

Since the degeneration-order implies the hom-order between two modules, we get the result (a) of Theorem 2.1 as a consequence.

Note that Theorem 2.3 above also holds when k is not algebraically closed. The only difference we have to make in the proof is that one has to observe that $[M, M] - a = [M, \mathfrak{r}M]$ where $a \ge 1$.

THEOREM 2.4. *Let M be a d-dimensional Λ-module and let P be its projective cover.*

(1) *If $[M, M] = [P, M]$ and N is a d-dimensional module such that $M \le_{\text{hom}} N$ is top-stable, then $M \simeq N$.*
(2) *If M is an indecomposable module or has a square-free top and $[M, M] < [P, M]$, then M has a proper top-stable degeneration.*

Proof. (1) Assume that $[M, M] = [P, M]$ and that there is some d-dimensional module N such that $M <_{\text{hom}} N$ is top-stable. Since $M/\mathfrak{r}M \simeq N/\mathfrak{r}N$, we know that P is also a projective cover of N. It follows by Proposition 2.2 that $[N, N] \le [P, N]$. Since M and N have the same composition factors by assumption, $[P, N] = [P, M]$, hence $[N, N] \le [M, M]$, which is a contradiction since $[M, M] < [N, N]$ when $M <_{\text{hom}} N$.

(2) Assume first that M is an indecomposable module such that $[M, M] < [P, M]$. Let

$$\eta : \ 0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$$

be an exact sequence where g is a projective cover of M. Since $[M, M] < [P, M]$ by assumption, there is a map $h : P \to M$ such that $hf \ne 0$. Moreover, we can assume that h is not an epimorphism. Indeed, assume that it is, and denote by π the natural epimorphism from M to $M/\mathfrak{r}M$. Then both πg and πh are epimorphisms from P to $M/\mathfrak{r}M$, and they induce isomorphisms

$$\overline{\pi g}, \overline{\pi h} : P/\mathfrak{r}P \to M/\mathfrak{r}M.$$

Since k is an algebraically closed field, $\operatorname{Hom}(S_i, S_j) \simeq k$ if $S_i \simeq S_j$ and $\operatorname{Hom}(S_i, S_j) = 0$ otherwise. Hence, the maps $\overline{\pi g}$ and $\overline{\pi h}$ are given by $r \times r$-matrices A and B over k, respectively, where r is the number of indecomposable summands of the top of M (and P). By changing the basis if necessary, we can assume that $\overline{\pi g}$ is the identity, i.e. $A = I$, the identity matrix. We want to find some $\alpha \in k$ such that $h + \alpha g$ is not an epimorphism. In other words, we want $\alpha \in k$ such that $\det(B + \alpha I) = 0$. Let $a \in k$ be one of the eigenvalues of B and let $\alpha = -a$. Clearly, $\det(B + \alpha I) = 0$ and $h + \alpha g$ is not an epimorphism. Also, $(h + \alpha g)f = hf \neq 0$ and therefore, if h is an epimorphism, we can use $h + \alpha g$ instead.

Since P is a projective module, there is a map $g' : P \to P$ such that $gg' = h$. Let $K_0 = \operatorname{Ker} g'$ and assume that $K \cap K_0 = (0)$. Then the map $\binom{g'}{g} : P \to P \oplus M$ is a monomorphism and it is not split since M is indecomposable and h is not an epimorphism. Consequently, $M <_{\deg} \operatorname{Coker} \binom{g'}{g} = N$ since we get a (non-split) Riedtmann sequence

$$0 \to P \xrightarrow{\binom{g'}{g}} P \oplus M \xrightarrow{(u\ v)} N \to 0.$$

Note that u is an epimorphism (since g is) and also $\operatorname{Ker} u \simeq \operatorname{Ker} g \subseteq \mathfrak{r}P$. Hence M and N have isomorphic tops and the degeneration is top-stable.

Assume now that $K \cap K_0 \neq (0)$. Since $hf \neq 0$, $K \cap K_0$ is a proper submodule of K. Let

$$\pi_0 : P \to P/K \cap K_0$$

be the natural epimorphism and let $g_0 : P/K \cap K_0 \to M$ be such that $g = g_0\pi_0$. We get an induced exact sequence

$$\eta_0 : \ 0 \to K/K \cap K_0 \xrightarrow{f_0} P/K \cap K_0 \xrightarrow{g_0} M \to 0.$$

For $i > 0$, let

$$\pi_i : P \to P/K \cap K_i$$

be the natural epimorphism, where $\phi_{i-1} := \pi_{i-1}g'$ and $K_i := \operatorname{Ker} \phi_{i-1}$. By construction, we see that $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$, hence

$$K \cap K_0 \subseteq K \cap K_1 \subseteq K \cap K_2 \subseteq \cdots.$$

Since $hf \neq 0$, $K \cap K_i$ is a proper submodule of K and hence $K/K \cap K_i \neq 0$ for each i.

Now since K is noetherian, there must be some t such that $K \cap K_t = K \cap K_{t+1}$. Consider the exact sequence

$$\eta_t:\ 0 \to K/K \cap K_t \xrightarrow{f_t} P/K \cap K_t \xrightarrow{g_t} M \to 0.$$

By the definition, $\phi_t = \pi_t g' : P \to P/K \cap K_t$ is such that $g_t \phi_t = h$ and $\operatorname{Ker} \phi_t = K_{t+1}$. This map induces a map $\bar{\phi}_t : P/K \cap K_{t+1} = P/K \cap K_t \to P/K \cap K_t$. Consider the map

$$\begin{pmatrix} \bar{\phi}_t \\ g_t \end{pmatrix} : P/K \cap K_t \to (P/K \cap K_t) \oplus M.$$

It is a monomorphism since

$$(\operatorname{Ker} g_t) \cap (\operatorname{Ker} \bar{\phi}_t) = (K/K \cap K_t) \cap (K_{t+1}/K \cap K_t) = (0).$$

Now, M is indecomposable by assumption and $\bar{\phi}_t$ is not an epimorphism since h is not. It follows that the map $\binom{\bar{\phi}_t}{g_t}$ is not split. Let $N = \operatorname{Coker} \binom{\bar{\phi}_t}{g_t}$. We have a non-split exact sequence

$$0 \to P/K \cap K_t \xrightarrow{\binom{\bar{\phi}_t}{g_t}} (P/K \cap K_t) \oplus M \xrightarrow{(u,v)} N \to 0$$

and hence $M <_{\deg} N$. Also, u is an epimorphism since g_t is and $\operatorname{Ker} u \simeq \operatorname{Ker} g_t \subseteq \mathfrak{r}(P/K \cap K_t)$. Hence, M and N have isomorphic tops and we are done.

Assume now that M is a (possibly decomposable) module with square-free top and with $[M, M] < [P, M]$. We then show that there is some proper top-stable degeneration of M. Let $M = \bigoplus_i M_i$ where M_i is an indecomposable module for each i. If there is some j such that the top of M_j is not simple, then, clearly, M_j is not a projective $\Lambda/\operatorname{ann} M_j$-module and it has a proper top-stable degeneration N_j as we have seen above. But then the degeneration

$$M = \bigoplus_{i=1}^{r} M_i <_{\deg} M_1 \oplus \cdots \oplus M_{j-1} \oplus N_j \oplus M_{j+1} \oplus \cdots \oplus M_r$$

is proper and top-stable and we are done.

So, assume that $M = \bigoplus_i M_i$, where M_i is an indecomposable module with simple top for each i, i.e. M is a direct sum of local modules. To prove that there is some proper top-stable degeneration of M, we can proceed as in the case of M being indecomposable above. But, to be sure that the Riedtmann sequence arising is not split, we need some stronger condition on the map $h : P \to M$. Before it was enough that h is not an epimorphism, now we need that $\operatorname{Im} h \subseteq \mathfrak{r}M$.

Assume that $\operatorname{Im} h \not\subseteq \mathfrak{r}M$. Let g be, as before, a projective cover of M and $\pi : M \to M/\mathfrak{r}M$ a natural epimorphism. The maps $\pi h, \pi g : P \to M/\mathfrak{r}M$

are then non-zero and they induce maps

$$\overline{\pi h}, \overline{\pi g} : P/\mathfrak{r}P \to M/\mathfrak{r}M.$$

Since k is an algebraically closed field, for two simple modules S and S' we have $\mathrm{Hom}(S,S') \simeq k$ if $S \simeq S'$, and $\mathrm{Hom}(S,S') = 0$ otherwise. Thus, we can represent the maps $\overline{\pi h}, \overline{\pi g}$ by two $r \times r$-matrices over k, A and B respectively, where r is the number of indecomposable summands of $M/\mathfrak{r}M$. Now, $\overline{\pi g}$ is an isomorphism and, changing the basis if necessary, we can assume that $B = I$, the identity matrix. Since the top of M is square-free, the matrix A is diagonal with $a_{11}, \ldots, a_{rr} \in k$ on the diagonal.

Let $\phi : M/\mathfrak{r}M \to M/\mathfrak{r}M$ be given by the matrix $-A$ and let $\phi' : M = \bigoplus_i M_i \to M = \bigoplus_i M_i$ be the map induced by ϕ, i.e.

$$\phi'(m_1, \ldots, m_r) = (-a_{11}m_1, \ldots, -a_{rr}m_r)$$

(ϕ' is well defined since M_i is a local module for each i). Since $\pi(h+\phi' g) = 0$, it follows that $\mathrm{Im}(h + \phi' g) \subseteq \mathfrak{r}M$. Also, $(h + \phi' g)f = hf \neq 0$. Hence, if $\mathrm{Im}\, h \nsubseteq \mathfrak{r}M$, there is some $\phi' : M \to M$ such that $\mathrm{Im}(h + \phi' g) \subseteq \mathfrak{r}M$ and $(h + \phi' g)f = hf \neq 0$. Hence we can use the map $h + \phi' g$ instead of h. ∎

Thus, by the theorem above, Theorem 2.1(b) holds not just for modules with square-free top, but also for indecomposable modules.

Part (2) of Theorem 2.4 does not hold for an arbitrary module M. Here is a counterexample.

Example. Given the k-algebra $\Lambda = k[x]/(x^2)$, let $M = \Lambda \oplus S$, where S is a simple Λ-module. A projective cover of M is $P = \Lambda^2$ and $5 = [M,M] < [P,M] = 6$. But it is easy to see that there is no 3-dimensional module N such that $M <_{\mathrm{hom}} N$ (or/and $M <_{\mathrm{deg}} N$) is top-stable.

Note that part (1) of the theorem above also holds when the field k is not algebraically closed.

However, part (2) of the theorem (if M is an indecomposable module or has a square-free top and $[M,M] < [P,M]$, then there is a d-dimensional module N such that $M <_{\mathrm{hom}} N$ is top-stable) does not hold in general if the field k is not algebraically closed, as the following example shows.

Example. Consider the $\mathbb{R}$-algebra $\Lambda = \left(\begin{smallmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{smallmatrix}\right)$, and see [ARS, III.2] for the following: Let $\mathcal{C}$ be the category whose objects are triples (A, B, f), where $A \in \mathrm{mod}\,\mathbb{C}$, $B \in \mathrm{mod}\,\mathbb{R}$ and $f : A \to B$ is an $\mathbb{R}$-morphism, and the morphisms between two objects (A, B, f) and (A', B', f') are pairs of morphisms (α, β) where $\alpha : A \to A'$ is a $\mathbb{C}$-morphism, $\beta : B \to B'$ is an $\mathbb{R}$-morphism and $f'\alpha = \beta f$. There is an equivalence

$$F : \mathrm{mod}\,\Lambda \to \mathcal{C}.$$

Let P be a Λ-module given by the triple $(\mathbb{C}, \mathbb{C}, 1_{\mathbb{C}})$ and let K be a module given by the triple $(0, \mathbb{R}, 0)$; P is an indecomposable projective module and K is a simple module. If $\beta : \mathbb{R} \to \mathbb{C}$ is given by $\beta(x) = x$, it is easy to see that $(0, \beta)$ gives a monomorphism from K to P. Let M be the Λ-module given by $(\mathbb{C}, \mathbb{C}, 1_{\mathbb{C}})/\mathrm{Im}(0, \beta)$, i.e. M is isomorphic to the module given by the triple $(\mathbb{C}, \mathbb{R}, f)$ where $f : \mathbb{C} \to \mathbb{R}$ is given by $f(x + yi) = y$. Now, P is a projective cover of M and

$$1 = [M, M] < [P, M] = 2.$$

Note that M is a 3-dimensional module with simple top given by the triple $(\mathbb{C}, 0, 0)$ and simple radical given by the triple $(0, \mathbb{R}, 0)$ and in fact M is an injective envelope of $(0, \mathbb{R}, 0)$. Let N be a Λ-module which is 3-dimensional as an $\mathbb{R}$-space with top $(\mathbb{C}, 0, 0)$. It follows that N must be given by a triple $(\mathbb{C}, \mathbb{R}, g)$ where $g : \mathbb{C} \to \mathbb{R}$ is non-zero, i.e. $g(x + yi) = ax + by$ for some $(a, b) \neq (0, 0)$ in $\mathbb{R} \times \mathbb{R}$. The map $(\alpha, 1_{\mathbb{R}})$, where $\alpha : \mathbb{C} \to \mathbb{C}$ is given by $\alpha(x + yi) = (x + yi)(b + ai)$, gives an isomorphism between N and M. In other words, if M and N have isomorphic tops, then they are isomorphic.

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Department of Mathematical Sciences
University of Science and Technology
N-7491 Trondheim, Norway
E-mail: sverresm@math.ntnu.no
anita.valenta@math.ntnu.no

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