COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 1

INDUCED MODULES OF STRONGLY GROUP-GRADED ALGEBRAS

BY

TH. THEOHARI-APOSTOLIDI and H. VAVATSOULAS (Thessaloniki)

Abstract. Various results on the induced representations of group rings are extended to modules over strongly group-graded rings. In particular, a proof of the graded version of Mackey's theorem is given.

1. Introduction. Let G be a group and $\Lambda = \bigoplus_{g\in G} \Lambda_g$ a strongly G-graded ring that is an algebra over an artinian commutative ring R. For a subgroup H of G we consider the ring

$$\Lambda_H = \bigoplus_{h\in H} \Lambda_h,$$

which is a strongly H-graded R-algebra. Let V be a left Λ_H-module and W a left Λ-module.

In the first section of this paper we examine the properties of the injective hulls, projective covers and the functor Hom under the induction and restriction functors.

In the second section we give the graded version of Mackey's theorem. A proof of this theorem was given by Boisen in [3] but the functions defined there do not have the required properties.

The reader is referred to [1], [4] and [6] for basic facts and notation of group representation theory, to [2] for background on modules over artinian algebras, and to [7]–[10] for graded rings theory.

2. Induction and restriction functors. Let G be a group and H a subgroup of G of finite index. Let R be a commutative artinian ring and

$$\Lambda = \bigoplus_{g\in G} \Lambda_g$$

a *strongly G-graded R-algebra*, that is, $\Lambda_g\Lambda_h = \Lambda_{gh}$ for all $g, h \in G$. Moreover, since $\Lambda_g\Lambda_{g^{-1}} = \Lambda_1$ for all g in G, where 1 is the unity of G, there exist elements $a_g^{(i)} \in \Lambda_g$, $b_{g^{-1}}^{(i)} \in \Lambda_{g^{-1}}$ and a positive integer n_g depending on g

2000 *Mathematics Subject Classification*: Primary 16W50, 16G30, 16G10.

Key words and phrases: strongly group-graded rings, induced representations.

such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1. \tag{2.1}$$

Consider the strongly H-graded R-algebra $\Lambda_H = \bigoplus_{h\in H} \Lambda_h$. If V is a left Λ_H-module and W a left Λ-module we denote by $V^G = \Lambda \otimes_{\Lambda_H} V$ the induced Λ-module and by W_H the restriction of W viewed as a left Λ_H-module. We denote by T a left transversal of H in G. It is clear that

$$V^G = \bigoplus_{t\in T} \Lambda_t \otimes_{\Lambda_H} V.$$

Moreover we set $V^g = \Lambda_g \otimes_{\Lambda_H} V$ for $g \in G$. Note that V^g is a left $\Lambda_{gHg^{-1}}$-module.

Finally, for a ring S we denote by $\operatorname{mod} S$ the category of finitely generated left S-modules. We recall that a left Λ-module W is *H-projective* if the exact sequence of left Λ-modules

$$0 \to X \to Y \to W \to 0$$

for which the associated sequence of Λ_H-modules

$$0 \to X_H \to Y_H \to W_H \to 0$$

splits, is also a splitting sequence of Λ-modules. Equivalently, W is H-projective if and only if $W \mid (W_H)^G$ ([5]), where the notation $X \mid Y$ means that the module X is isomorphic to a direct summand of the module Y.

For a module V, we denote by $I(V)$ and $P(V)$ the injective hull and projective cover of V, respectively.

Using the above notation we prove the following result, which is known for group rings.

LEMMA 2.1. *Let V be a left Λ_H-module, W a left Λ-module and $\sigma \in G$. Then the following hold*:

(i) *$I(V^G)$ (resp. $P(V^G)$) is isomorphic to a direct summand of $[I(V)]^G$ (resp. $[P(V)]^G$).*
(ii) *$I(W_H)$ (resp. $P(W_H)$) is isomorphic to a direct summand of $[I(W)]_H$ (resp. $[P(W)]_H$).*
(iii) *If $H \trianglelefteq G$, then $I(V^\sigma)$ (resp. $P(V^\sigma)$) is isomorphic to a direct summand of $[I(V)]^\sigma$ (resp. $[P(V)]^\sigma$).*
(iv) *If W is H-projective, then $I(W)$ is isomorphic to a direct summand of $[I(W_H)]^G$.*
(v) *$P(W)$ is isomorphic to a direct summand of $[P(W_H)]^G$ and $P(W)$ is isomorphic to a direct summand of $\{[P(W)]_H\}^G$.*

Proof. We prove (iii). The proofs of the remaining statements are analogous to those in the group ring case (see [6, Ch. 1, Prop. 12.5]).

Let H be a normal subgroup of G. Since the sequence

$$0 \to V \xrightarrow{f} I(V)$$

is exact, so is

$$0 \to V^\sigma \xrightarrow{f^\sigma} [I(V)]^\sigma,$$

where f^σ is the restriction of $1 \otimes f$ to V^σ, i.e. $f^\sigma(\lambda_\sigma \otimes v) = \lambda_\sigma \otimes f(v)$ for $\lambda_\sigma \in \Lambda_\sigma$ and $v \in V$. Moreover, by [8, Section 3, Prop. 2], the module $[I(V)]^\sigma$ is also injective. Therefore for the first part of (iii) it remains to prove that f^σ is *essential*, that is, if X' is any nonzero Λ_H-submodule of $[I(V)]^\sigma$, then $f^\sigma(V^\sigma) \cap X' \neq \{0\}$. For this, let X be the Λ_H-submodule of $I(V)$ generated by the elements $\lambda_{H\sigma^{-1}} y$, where $\lambda_{H\sigma^{-1}} \in \Lambda_{H\sigma^{-1}}$ and

$$y = \sum_{k=1}^{\nu} \lambda_\sigma^{(k)} x^{(k)}, \quad \text{with} \quad \sum_{k=1}^{\nu} \lambda_\sigma^{(k)} \otimes x^{(k)} \in X',\ \nu \in \mathbb{N}.$$

Since f is essential, it follows from the relation $f(V) \cap X \neq \{0\}$ that there exists a nonzero element $x \in X$ such that $x = f(v)$ for some nonzero $v \in V$. Since $v \neq 0$, there exists $\mu \in \{1, \ldots, n_\sigma\}$ such that $b_\sigma^{(\mu)} \otimes v \neq 0$, because otherwise $\sum_{i=1}^{n_\sigma} a_{\sigma^{-1}}^{(i)} b_\sigma^{(i)} \otimes v = 0$, where $a_{\sigma^{-1}}^{(i)}$, $b_\sigma^{(i)}$ are as in (2.1), and so $v = 0$. Write

$$x = \sum_{j=1,k=1}^{\varrho,\nu_j} \lambda_{H\sigma^{-1}}^{(j)} \lambda_\sigma^{(k)(j)} x^{(k)(j)}$$

for some $\varrho \in \mathbb{N}$, where $\lambda_{H\sigma^{-1}}^{(j)} \in \Lambda_{H\sigma^{-1}}$ and $\sum_{k=1}^{\nu_j} \lambda_\sigma^{(k)(j)} x^{(k)(j)} \in X'$ for $j \in \{1, \ldots, \varrho\}$. Then

$$0 \neq f^\sigma(b_\sigma^{(\mu)} \otimes v) = b_\sigma^{(\mu)} \otimes f(v) = \sum_{j=1,k=1}^{\varrho,\nu_j} b_\sigma^{(\mu)} \lambda_{H\sigma^{-1}}^{(j)} \lambda_\sigma^{(k)(j)} \otimes x^{(k)(j)}$$

and we get

$$f^\sigma(b_\sigma^{(\mu)} \otimes v) \in X' \cap f^\sigma(V^\sigma) \neq \{0\}.$$

This proves that f^σ is essential and therefore $I(V^\sigma) \cong [I(V)]^\sigma$.

The second part of (iii) is proved analogously. ∎

Theorem 2.2. *Let H be a normal subgroup of G. Assume that the Krull–Schmidt–Azumaya theorem holds in* $\operatorname{mod} \Lambda$. *Then the following hold for a left Λ_H-module V and a left Λ-module W:*

(i) $I(V^G) \cong [I(V)]^G$ *and* $P(V^G) \cong [P(V)]^G$ *as left Λ-modules.*
(ii) $I[(V^G)_H] \cong [I(V^G)]_H$ *and* $P[(V^G)_H] \cong [P(V^G)]_H$ *as left Λ_H-modules.*
(iii) *If W is H-projective then*

$$I(W_H) \cong [I(W)]_H \quad \text{and} \quad P(W_H) \cong [P(W)]_H$$

as left Λ_H-modules.

Proof. We will prove the injective hull case. The projective cover case is analogous.

(i), (ii). It follows from Lemma 2.1(i),(ii), that there exist Z and Y in $\operatorname{mod}\Lambda$ such that

$$I(V^G) \oplus Z \cong [I(V)]^G, \tag{2.2}$$

$$I[(V^G)_H] \oplus Y \cong [I(V^G)]_H. \tag{2.3}$$

Moreover, by Lemma 2.1(iii),

$$([I(V)]^G)_H \cong \bigoplus_{t\in T} [I(V)]^t \cong \bigoplus_{t\in T} I(V^t) \cong I\Big(\bigoplus_{t\in T} V^t\Big) \cong I[(V^G)_H]. \tag{2.4}$$

Now, using (2.2) and (2.3), the relation (2.4) becomes

$$[I(V^G)]_H \oplus Z_H \cong I[(V^G)_H]$$

and so

$$I[(V^G)_H] \oplus Y \oplus Z_H \cong I[(V^G)_H].$$

By applying the Krull–Schmidt–Azumaya theorem to the above relation, it follows that $Y = Z_H = 0$, and parts (i) and (ii) of the theorem follow from (2.2) and (2.3).

(iii) If W is H-projective, then there exists a Λ-module X such that

$$W \oplus X \cong (W_H)^G.$$

Then, by (ii),

$$\begin{aligned} &[I(W)]_H \oplus [I(X)]_H \\ &\quad \cong [I(W) \oplus I(X)]_H \cong [I(W \oplus X)]_H \cong [I(W_H)^G]_H \cong I([(W_H)^G]_H) \\ &\quad \cong I[(W \oplus X)_H] \cong I(W_H \oplus X_H) \cong I(W_H) \oplus (X_H). \end{aligned} \tag{2.5}$$

Now, from Lemma 2.1(ii), there exist U and M in $\operatorname{mod}\Lambda$ such that

$$[I(W)]_H \cong I(W_H) \oplus U, \qquad [I(X)]_H \cong I(X_H) \oplus M.$$

Combining the above relations with (2.5) and using the Krull–Schmidt–Azumaya theorem, we deduce that $U = M = 0$, and (iii) follows. ■

THEOREM 2.3. *Let W be a left Λ-module and H a subgroup of G of finite index. Then there exists an isomorphism*

$$\Theta : \operatorname{Hom}_\Lambda(W, \Lambda) \to \operatorname{Hom}_{\Lambda_H}(W_H, \Lambda_H)$$

of right Λ_H-modules. This isomorphism is natural. In particular, if W is a Λ-Λ-bimodule, then Θ is a Λ-Λ_H-bimodule isomorphism.

Proof. As W is a left Λ-module, $\operatorname{Hom}_\Lambda(W, \Lambda)$ becomes a right Λ-module under the rule $(f \cdot \lambda)(w) = f(w)\lambda$ for $f \in \operatorname{Hom}_\Lambda(W, \Lambda)$, $\lambda \in \Lambda$ and $w \in W$. Similarly $\operatorname{Hom}_{\Lambda_H}(W_H, \Lambda_H)$ becomes a right Λ_H-module. Let $f \in \operatorname{Hom}_\Lambda(W, \Lambda)$ and $w \in W$. Then $f(w) = \sum_{t\in T} f(w)_{tH}$ for $f(w) \in \Lambda_{tH}$. We define

$$\Theta : \mathrm{Hom}_{\Lambda}(W, \Lambda) \to \mathrm{Hom}_{\Lambda_H}(W_H, \Lambda_H) \quad \text{by} \quad \Theta(f)(w) = f(w)_H.$$

It is easy to see that Θ is a right Λ_H-homomorphism. To prove that Θ is onto, for each $\mu \in \mathrm{Hom}_{\Lambda_H}(W_H, \Lambda_H)$ we define

$$\widetilde{\mu} : W \to \Lambda \quad \text{by} \quad \widetilde{\mu}(w) = \sum_{t\in T} \sum_i a_t^{(i)} \mu(b_{t^{-1}}^{(i)} w),$$

where $a_t^{(i)} \in \Lambda_t$ and $b_{t^{-1}}^{(i)} \in \Lambda_{t^{-1}}$ are as in (2.1). It is easy to see that $\widetilde{\mu} \in \mathrm{Hom}_{\Lambda}(W, \Lambda)$. Moreover, $\Theta(\widetilde{\mu})(w) = \widetilde{\mu}(w)_H = \mu(w)$ for $w \in W$. To prove that Θ is a monomorphism, let $w \in W$ and $f \in \ker \Theta$. Since

$$f(w) = \sum_{t\in T} f(w)_{tH} = \sum_{t\in T} \sum_i a_t^{(i)} b_{t^{-1}}^{(i)} f(w)_{tH},$$

for any $g \in T$ we get $0 = \Theta(f)(b_{g^{-1}}^{(i)} w) = b_{g^{-1}}^{(i)} f(w)_{gH}$ and hence $f(w) = 0$, for all $w \in W$. The fact that Θ is a Λ-Λ_H-homomorphism is proved by straightforward calculations. ■

Corollary 2.4. *Let W be a left Λ-module and H a subgroup of G of finite index. Then*

$$\mathrm{Ext}^n_{\Lambda}(W, \Lambda) \cong \mathrm{Ext}^n_{\Lambda_H}(W_H, \Lambda_H)$$

as abelian groups, for all $n \in \mathbb{N}$.

Theorem 2.5. *Let G be a group, H a subgroup of G of finite index and V a left Λ_H-module. Then*

$$[\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)]^G \cong \mathrm{Hom}_{\Lambda}(V^G, \Lambda)$$

as right Λ-modules. This isomorphism is natural.

Proof. Since $\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)$ is a right Λ_H-module, it follows that each element of $[\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)]^G$ is of the form $\sum_{t\in T} f_{t^{-1}} \otimes \lambda_{t^{-1}}$ for some $f_{t^{-1}} \in \mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)$ and $\lambda_{t^{-1}} \in \Lambda_{t^{-1}}$. We consider the map

$$\Delta : [\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)]^G \to \mathrm{Hom}_{\Lambda}(V^G, \Lambda)$$

defined by

$$\Delta\Big(\sum_{t\in T} f_{t^{-1}} \otimes \lambda_{t^{-1}}\Big) = f, \text{ where } f\Big(\sum_{s\in S} \mu_s \otimes v_s\Big) = \sum_{s\in S} \sum_{t\in T} \mu_s f_{t^{-1}}(v_s) \lambda_{t^{-1}}$$

for another set S of left transversals of H in G, $\mu_s \in \Lambda_s$ and $v_s \in V$. Let us prove that Δ is independent of the choice of T and S. Let T' and S' be any two sets of left transversals of H in G. For any $s' \in S'$, there exists a unique $s \in S$ and a unique $h_s \in H$ such that $s' = sh_s$. Similarly for any $t' \in T'$, there exist unique $t \in T$ and $h_t \in H$ such that $t' = th_t$. Let $\lambda_{s'} = \sum_i \lambda_s^{(i)} \lambda_{h_s}^{(i)}$ and $\lambda_{t'^{-1}} = \sum_j \lambda_{h_t^{-1}}^{(j)} \lambda_{t^{-1}}^{(j)}$ for some $\lambda_{s'} \in \Lambda_{s'}$ and $\lambda_{t'^{-1}} \in \Lambda_{t'^{-1}}$, where $\lambda_x^{(i)}, \lambda_x^{(j)} \in \Lambda_x$ for $x \in G$. Then

$$f\Big(\sum_{s'\in S'}\lambda_{s'}\otimes v_{s'}\Big) = f\Big(\sum_i\sum_{s\in S}\lambda_s^{(i)}\otimes\lambda_{h_s}^{(i)}v_{s'}\Big)$$
$$= \sum_i\sum_{s\in S}\sum_{t\in T}\lambda_s^{(i)}f_{t^{-1}}(\lambda_{h_s}^{(i)}v_{s'})\lambda_{t^{-1}} = \sum_{s'\in S'}\sum_{t\in T}\lambda_{s'}f_{t^{-1}}(v_{s'})\lambda_{t^{-1}}.$$

Moreover,

$$\Delta\Big(\sum_{t'\in T'}f_{t'^{-1}}\otimes\lambda_{t'^{-1}}\Big)\Big(\sum_{s\in S}\lambda_s\otimes v_s\Big) = \sum_j\sum_{s\in S}\sum_{t\in T}\lambda_s(f_{t'^{-1}}\cdot\lambda_{h_t^{-1}}^{(j)})(v_s)\lambda_{t^{-1}}^{(j)}$$
$$= \sum_j\sum_{s\in S}\sum_{t\in T}\lambda_s f_{t'^{-1}}(v_s)\lambda_{h_t^{-1}}^{(j)}\lambda_{t^{-1}}^{(j)} = \sum_{s\in S}\sum_{t\in T}\lambda_s f_{t'^{-1}}(v_s)\lambda_{t'^{-1}}.$$

The map f is a Λ-homomorphism. Indeed, let $g\in G$ and $S'=\{gs : s\in S\}$. Then

$$f\Big(\lambda_g\sum_{s\in S}\lambda_s\otimes v_s\Big) = f\Big(\sum_{gs\in S'}\lambda_g\lambda_s\otimes v_s\Big) = \sum_{t\in T}\sum_{gs\in S'}\lambda_g\lambda_s f_{t^{-1}}(v_s)\lambda_{t^{-1}}$$
$$= \lambda_g f\Big(\sum_{s\in S}\lambda_s\otimes v_s\Big).$$

We now prove that Δ is a Λ-homomorphism of right Λ-modules. Let $g\in G$, and consider the set $\{g^{-1}t : t\in T\}$ of right transversals of H in G and elements $a_{g^{-1}t}^{(i)}\in\Lambda_{g^{-1}t}$ and $b_{t^{-1}g}^{(i)}\in\Lambda_{t^{-1}g}$ with $\sum_i a_{g^{-1}t}^{(i)}b_{t^{-1}g}^{(i)}=1$, as in (2.1). If

$$\varphi=\sum_{t\in T}f_{t^{-1}}\otimes\lambda_{t^{-1}}\in[\mathrm{Hom}_{\Lambda_H}(V,\Lambda_H)]^G,$$

then

$$\varphi\cdot\lambda_g=\sum_i\sum_{t\in T}f_{t^{-1}}(\lambda_{t^{-1}}\lambda_g a_{g^{-1}t}^{(i)})\otimes b_{t^{-1}g}^{(i)}$$

and

$$\Delta(\varphi\cdot\lambda_g)\Big(\sum_{s\in S}\lambda_s\otimes v_s\Big) = \sum_j\sum_{t\in T}\sum_{s\in S}\lambda_s[f_{t^{-1}}(\lambda_{t^{-1}}\lambda_g a_{g^{-1}t}^{(i)})](v_s)b_{t^{-1}g}^{(i)}$$
$$= \sum_{t\in T}\sum_{s\in S}\lambda_s f_{t^{-1}}(v_s)\lambda_{t^{-1}}\lambda_g = [\Delta(\varphi)\cdot\lambda_g]\Big(\sum_{s\in S}\lambda_s\otimes v_s\Big).$$

We now define the inverse of Δ. Let $f\in\mathrm{Hom}_\Lambda(V^G,\Lambda)$. Then $f(1\otimes v)\in\Lambda=\bigoplus_{t\in T}\Lambda_H\Lambda_{t^{-1}}$ for $v\in V$, and for each $t\in T$ there exists a unique $f_{Ht^{-1}}^v\in\Lambda_H\Lambda_{t^{-1}}$ such that

$$f(1\otimes v)=\sum_{t\in T}f_{Ht^{-1}}^v.$$

Let $a_t^{(i)}\in\Lambda_t$ and $b_{t^{-1}}^{(i)}\in\Lambda_{t^{-1}}$ be as in (2.1). We define

$$f_{i,t^{-1},a}: V\to\Lambda_H,\qquad v\mapsto f_{Ht^{-1}}^v a_t^{(i)}.$$

Since f is a Λ-homomorphism it follows that, for $x \in \Lambda_H$ and $v \in V$,

$$f(1 \otimes xv) = xf(1 \otimes v)$$

and we get

$$f^{xv}_{Ht^{-1}} = xf^{v}_{Ht^{-1}},$$

because of the unique decomposition, and so $f_{i,t^{-1},a}$ is a Λ_H-homomorphism. Now we define

$$\Psi : \mathrm{Hom}_\Lambda(V^G, \Lambda) \to [\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)]^G$$

by

$$f \mapsto \sum_i \sum_{t \in T} f_{i,t^{-1},a} \otimes b^{(i)}_{t^{-1}}.$$

We remark that Ψ is independent of the choice of $a^{(i)}_t$, $b^{(i)}_{t^{-1}}$. Indeed, let $c^{(j)}_t \in \Lambda_t$, $d^{(j)}_{t^{-1}} \in \Lambda_{t^{-1}}$ with

$$\sum_j c^{(j)}_t d^{(j)}_{t^{-1}} = 1$$

as in (2.1). We have to show that

$$\sum_i \sum_{t \in T} f_{i,t^{-1},a} \otimes b^{(i)}_{t^{-1}} = \sum_j \sum_{t \in T} f_{j,t^{-1},c} \otimes d^{(j)}_{t^{-1}}.$$

Since the right hand side is equal to

$$\sum_{i,j} \sum_{t \in T} f_{j,t^{-1},c} d^{(j)}_{t^{-1}} a^{(i)}_t \otimes b^{(i)}_{t^{-1}},$$

it is enough to note that, for all $v \in V$,

$$f_{i,t^{-1},a}(v) = \sum_j f_{j,t^{-1},c} d^{(j)}_{t^{-1}} a^{(i)}_t(v) \;\Leftrightarrow\; f^v_{Ht^{-1}} a^{(i)}_t = \sum_j f^v_{Ht^{-1}} c^{(j)}_t d^{(j)}_{t^{-1}} a^{(i)}_t,$$

and the latter holds. Now we prove that Ψ is independent of the choice of T. Let S be another set of left transversals of H in G. Then, for every $s \in S$, there exist unique elements $t \in T$ and $h \in H$ such that $s = th$, so $\Lambda_H \Lambda_{t^{-1}} = \Lambda_H \Lambda_{s^{-1}}$. Moreover, for $v \in V$,

$$f(1 \otimes v) = \sum_{t \in T} f^v_{Ht^{-1}} = \sum_{t \in T} f^v_{Hs^{-1}} \quad \text{and} \quad f^v_{Ht^{-1}} = f^v_{Hs^{-1}}.$$

Let $\sum_i \gamma^{(i)}_s \delta^{(i)}_{s^{-1}} = 1$, as in (2.1). We remark that, for $f \in \mathrm{Hom}_\Lambda(V^G, \Lambda)$ and $v \in V$,

$$\begin{aligned}\Psi(f)(v) &= \sum_i \sum_{s \in S} f_{i,s^{-1},\gamma}(v) \otimes \delta^{(i)}_{s^{-1}} \\ &= \sum_j \sum_i \sum_{s \in S} f_{i,s^{-1},\gamma}(v) \delta^{(i)}_{s^{-1}} a^{(i)}_t \otimes b^{(i)}_{t^{-1}} = \sum_i \sum_{t \in T} f^v_{Ht^{-1}} a^{(i)}_t \otimes b^{(i)}_{t^{-1}}.\end{aligned}$$

It follows that Ψ is independent of the choice of T.

Finally, to prove that $\Psi \circ \Delta$ and $\Delta \circ \Psi$ are identity maps, let

$$x = \sum_{t\in T} f_{t^{-1}} \otimes \lambda_{t^{-1}} \in [\mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)]^G$$

with $f_{t^{-1}} \in \mathrm{Hom}_{\Lambda_H}(V, \Lambda_H)$ and $\lambda_{t^{-1}} \in \Lambda_{t^{-1}}$, and let $\Delta(x) = f \in \mathrm{Hom}_\Lambda(V^G, \Lambda)$. Then

$$\Psi \circ \Delta(x) = \Psi(f) = \sum_i \sum_{t\in T} f_{i,t^{-1},a} \otimes b^{(i)}_{t^{-1}}.$$

Since

$$x = \sum_i \sum_{t\in T} f_{t^{-1}} \cdot (\lambda_{t^{-1}} a^{(i)}_t) \otimes b^{(i)}_{t^{-1}},$$

it is enough to note that

$$\sum_{t\in T} f_{i,t^{-1},a}(v) = \sum_{t\in T} f_{t^{-1}} \cdot (\lambda_{t^{-1}} a^{(i)}_t)(v) \ \Leftrightarrow\ \sum_{t\in T} f^v_{Ht^{-1}} a^{(i)}_t = \sum_{t\in T} f_{t^{-1}}(v)\lambda_{t^{-1}} a^{(i)}_t$$
$$\Leftrightarrow\ \sum_{t\in T} (f^v_{Ht^{-1}} - f_{t^{-1}}(v)\lambda_{t^{-1}}) a^{(i)}_t = 0 \ \Leftrightarrow\ \sum_{t\in T} f^v_{Ht^{-1}} = \sum_{t\in T} f_{t^{-1}}(v)\lambda_{t^{-1}},$$

and the latter holds, since both sides are equal to $f(1\otimes v)$.

Let now $f \in \mathrm{Hom}_\Lambda(V^G, \Lambda)$ be such that $f(1\otimes v) = \sum_{t\in T} f^v_{Ht^{-1}}$ for $v \in V$. Then

$$\begin{aligned}\Delta \circ \Psi(f)\Big(\sum_{s\in S} \lambda_s \otimes v_s\Big) &= \Delta\Big(\sum_i \sum_{t\in T} f_{i,t^{-1},a} \otimes b^{(i)}_{t^{-1}}\Big)\Big(\sum_{s\in S} \lambda_s \otimes v_s\Big)\\ &= \sum_i \sum_{s\in S} \sum_{t\in T} \lambda_s f_{i,t^{-1},a}(v_s) b^{(i)}_{t^{-1}} = \sum_i \sum_{s\in S} \sum_{t\in T} \lambda_s f^{v_s}_{Ht^{-1}} a^{(i)}_t b^{(i)}_{t^{-1}}\\ &= \sum_{s\in S} \sum_{t\in T} \lambda_s f^{v_s}_{Ht^{-1}} = \sum_{s\in S} \lambda_s f(1\otimes v_s) = f\Big(\sum_{s\in S} \lambda_s \otimes v_s\Big).\end{aligned}$$

Finally, it is a routine matter to prove that Δ is a natural homomorphism. ∎

COROLLARY 2.6. *Let V be a left Λ_H-module and H a subgroup of G of finite index. Then*

$$[\mathrm{Ext}^n_{\Lambda_H}(V, \Lambda_H)]^G \cong \mathrm{Ext}^n_\Lambda(V^G, \Lambda)$$

as additive groups, for $n = 1, 2, \ldots$.

THEOREM 2.7. *Let G be a group, Λ a strongly G-graded ring, H a subgroup of G of finite index and V a left Λ_H-module. Then there is an isomorphism*

$$\mathrm{Hom}_{\Lambda_H}(\Lambda, V) \cong V^G$$

of left Λ-modules, which is functorial with respect to homomorphisms $V \to V'$.

Proof. We remark that $\mathrm{Hom}_{\Lambda_H}(\Lambda, V)$ is a left Λ-module, by the rule $\lambda \cdot f(x) = f(x\lambda)$ for $x, \lambda \in \Lambda$ and $f \in \mathrm{Hom}_{\Lambda_H}(\Lambda, V)$. Moreover, we can write

$$\Lambda = \bigoplus_{t\in T} \Lambda_H \Lambda_{t^{-1}},$$

so an element λ of Λ can be written as $\lambda = \sum_{t\in T} y_{t^{-1}}$ with $y_{t^{-1}} \in \Lambda_H \Lambda_{t^{-1}}$. Hence

$$y_{t^{-1}} = \sum_i \mu_{h_t}^{(i)} l_{t^{-1}}^{(i)}$$

with $\mu_{h_t}^{(i)} \in \Lambda_H$, $l_{t^{-1}}^{(i)} \in \Lambda_{t^{-1}}$ and i running over a finite index set depending on t. Using the above notation, we consider the map

$$\Theta : V^G \to \mathrm{Hom}_{\Lambda_H}(\Lambda, V), \qquad v \mapsto \Theta(v),$$

for $v = \sum_{t\in T} \lambda_t \otimes v_t$ and $\lambda_t \in \Lambda_t$, $v_t \in V$, with $\Theta(v) : \Lambda \to V$ defined by

$$\Theta(v)\Big(\sum_{t\in T}\sum_i \mu_{h_t}^{(i)} l_{t^{-1}}^{(i)}\Big) = \sum_{t\in T}\sum_i \mu_{h_t}^{(i)} l_{t^{-1}}^{(i)} \lambda_t v_t.$$

It is easy to see that the definition of Θ is independent of the decomposition of $y_{t^{-1}}$.

We prove that Θ is independent of the choice of T. Let S be another set of left transversals of H in G. Then for each $t \in T$ there exist unique elements $s \in S$ and $h \in H$ such that $t = sh$. Hence

$$y_{t^{-1}} \in \Lambda_H\Lambda_{t^{-1}} = \Lambda_H\Lambda_{s^{-1}}, \text{ and } y_{t^{-1}} = \sum_j \mu_{h_s}^{(j)} l_{s^{-1}}^{(j)} \text{ and } \lambda = \sum_{s\in S}\sum_j \mu_{h_s}^{(j)} l_{s^{-1}}^{(j)},$$

where j runs over a finite index set depending on S. We now consider the element $v = \sum_{t\in T} \lambda_t \otimes v_t$ of V^G. We remark that $\lambda_t = \sum_i \lambda_s^{(i)} \lambda_h^{(i)}$, since $\lambda_t \in \Lambda_t = \Lambda_s \Lambda_h$, where $\lambda_s^{(i)} \in \Lambda_s$, $\lambda_h^{(i)} \in \Lambda_H$ and i runs over a finite index set depending on s. So

$$v = \sum_{s\in S}\sum_i \lambda_s^{(i)} \otimes \lambda_h^{(i)} v_t$$

and

$$\Theta(v)(\lambda) = \sum_{s\in S}\sum_j\sum_i \mu_{h_s}^{(j)} l_{s^{-1}}^{(j)} \lambda_s^{(i)} \lambda_h^{(i)} v_t = \sum_{t\in T} y_{t^{-1}} \lambda_t v_t.$$

It is easy to see that $\Theta(v)$ is a Λ_H-homomorphism.

Now we prove that Θ is a Λ-homomorphism. For $g \in G$, $\lambda_g \in \Lambda_g$ and $v \in V^G$, it is enough to prove that

$$\Theta(\lambda_g v) = \lambda_g \Theta(v), \quad \text{i.e.} \quad \Theta(\lambda_g v)(\lambda) = \Theta(v)(\lambda\lambda_g).$$

Since gT is another set of left transversals of H in G, let $\lambda = \sum_{t\in T} \omega_{(gt)^{-1}}$

with $\omega_{(gt)^{-1}} \in \Lambda_H \Lambda_{t^{-1}} \Lambda_{g^{-1}}$. Then

$$\lambda_g v = \sum_{t\in T} \lambda_g \lambda_t \otimes v_t = \sum_{t\in T} \lambda_{gt} \otimes v_{gt},$$

where $\lambda_{gt} = \lambda_g \lambda_t$ and $v_{gt} = v_t$. So

$$\Theta(\lambda_g v)(\lambda) = \sum_{t\in T} \omega_{(gt)^{-1}} \lambda_{gt} v_{gt} = \sum_{t\in T} \omega_{(gt)^{-1}} \lambda_g \lambda_t v_t = \Theta(v)(\lambda\lambda_g).$$

We now define the map

$$\Phi : \mathrm{Hom}_{\Lambda_H}(\Lambda, V) \to V^G, \qquad f \mapsto \sum_{t\in T} \sum_i a_t^{(i)} \otimes f(b_{t^{-1}}^{(i)}),$$

where $a_t^{(i)}$ and $b_{t^{-1}}^{(i)}$ are as (2.1). It is a routine matter to prove that Φ is the inverse of Θ. ∎

3. Mackeys's theorem for strongly graded rings. In this section we prove Mackey's theorem for strongly group-graded rings. In [3, Theorem 2.2], Boisen has given a proof of the graded version of Mackey's theorem, but the map

$$\Psi : V \otimes_{R_H} R_{\sigma K} \to V \otimes_{R_H} R_\sigma \otimes_{R_{H^\sigma \cap K}} R_K, \qquad v \otimes x \mapsto \sum v \otimes a_i \otimes b_i x,$$

he has constructed there does not have the required properties, because the second tensor product is over $R_{H^\sigma\cap K}$ and not over R_{H^σ}. For another proof of the graded version of Mackey's theorem see [7, 3.7.3].

We use the notation of the previous sections. Let G be a group and let X, Y be a pair of subgroups of G, both of finite index in G. We denote by $X\backslash G/Y$ the (X,Y) cosets XgY relative to the pair of subgroups X, Y. If $D = X\alpha Y$ is such a double coset, X^α is the conjugate subgroup $\alpha X \alpha^{-1}$. Let

$$Y = \bigcup_{s\in S_\alpha} s(X^\alpha \cap Y)$$

for a finite subset S_α of G. Then it is easy to see that

$$Y\alpha X = \bigcup_{s\in S_\alpha} s\alpha X, \quad \text{for} \quad G = \bigcup_{\alpha\in A} Y\alpha X, \tag{3.6}$$

where A is a complete set of representatives of double (X,Y) cosets in G ([4, 10.13]). If $\alpha \in G$ and V is a left Λ_X-module, it is clear that $\Lambda_\alpha \otimes_{\Lambda_X} V$ is a left Λ_{X^α}-module and so a left $\Lambda_{X^\alpha\cap Y}$-module.

In the following we write $\otimes$ instead of $\otimes_{\Lambda_X}$.

THEOREM 3.1 (Mackey's theorem—graded version). *Let G be a group, Λ a strongly G-graded ring, X, Y a pair of subgroups of G of finite index in G, and V a left Λ_X-module. Then there is an isomorphism of left Λ_Y-*

modules

$$(V^G)_Y \cong \bigoplus_{\alpha\in A} [(\Lambda_\alpha \otimes V)_{\Lambda_{X^\alpha\cap Y}}]^Y,$$

where the sum is taken over $X\backslash G/Y$. The summands are independent of the choice of the double coset representatives, in the sense that

$$[(\Lambda_\alpha \otimes V)_{\Lambda_{X^\alpha\cap Y}}]^Y \cong [(\Lambda_\beta \otimes V)_{\Lambda_{X^\beta\cap Y}}]^Y$$

as left Λ_Y-modules whenever $Y\alpha X = Y\beta X$.

Proof. Let $G = \bigcup_{\alpha\in A} Y\alpha X$ $D = Y\alpha X$ for some $\alpha \in A$, and let

$$V(\alpha) = \bigoplus_{s\in S_\alpha} \Lambda_{s\alpha} \otimes V.$$

It is easy to see that $V(\alpha)$ is a left Λ_Y-module, and

$$(V^G)_Y = \bigoplus_{\alpha\in A} V(\alpha)$$

as left Λ_Y-modules. It is enough to prove that

$$[(\Lambda_\alpha \otimes V)_{\Lambda_{X^\alpha\cap Y}}]^Y \cong V(\alpha)$$

as left Λ_Y-modules. Given $y \in Y$, by (3.6), there exist $x \in X$ and $s \in S_\alpha$ such that

$$y\alpha = s\alpha x. \tag{3.7}$$

We define the Λ_Y-homomorphism

$$F : [(\Lambda_\alpha \otimes V)_{\Lambda_{X^\alpha\cap Y}}]^Y \to V(\alpha)$$

by the rule

$$F(\lambda_y\lambda \otimes_{\Lambda_{X^\alpha\cap Y}} \lambda_\alpha\lambda_X \otimes v) = \lambda_y\lambda\lambda_\alpha\lambda_X \otimes v$$

for $\lambda_y \in \Lambda_y$, $y \in Y$, $\lambda \in \Lambda_{X^\alpha\cap Y}$, $\lambda_\alpha \in \Lambda_\alpha$, $\lambda_X \in \Lambda_X$ and $v \in V$, and the Λ_Y-homomorphism

$$\Phi : V(\alpha) \to [(\Lambda_\alpha \otimes V)_{\Lambda_{X^\alpha\cap Y}}]^Y$$

by the rule

$$\Phi(\lambda_{s\alpha}\lambda_X \otimes v) = \sum_{i,j} a_s^{(i)} \otimes_{\Lambda_{X^\alpha\cap Y}} a_\alpha^{(j)} \otimes b_{\alpha^{-1}}^{(j)} b_{s^{-1}}^{(i)} \lambda_{s\alpha}\lambda_X v$$

for $y \in Y$, $s \in S$ defined by (3.7), $a_s^{(i)} \in \Lambda_s$, $b_{s^{-1}}^{(i)} \in \Lambda_{s^{-1}}$, $a_\alpha^{(j)} \in \Lambda_\alpha$, $b_{\alpha^{-1}}^{(i)} \in \Lambda_{\alpha^{-1}}$ as in (2.1), $\lambda_{s\alpha} \in \Lambda_{s\alpha}$, $\lambda_X \in \Lambda_X$ and $v \in V$. It is easy to see that F is a Λ_Y-homomorphism.

Now we prove that the definition of Φ is independent of the choice of $a_s^{(i)}$, $b_{s^{-1}}^{(i)}$, $a_\alpha^{(j)}$ and $b_{\alpha^{-1}}^{(i)}$. Let

$$\sum_k a_s'^{(k)} b_{s^{-1}}'^{(k)} = 1 \quad\text{and}\quad \sum_\nu a_\alpha'^{(\nu)} b_{\alpha^{-1}}'^{(\nu)} = 1,$$

as in (2.1). Then

$$\begin{aligned}\sum_{i,j} a_s^{(i)} \otimes a_\alpha^{(j)} \otimes b_{\alpha^{-1}}^{(j)} b_{s^{-1}}^{(i)} \lambda_{s\alpha} \lambda_X v & \\ &= \sum_{i,j,k} a_s^{\prime(k)} \otimes b_{s^{-1}}^{\prime(k)} a_s^{(i)} a_\alpha^{(j)} \otimes b_{\alpha^{-1}}^{(j)} b_{s^{-1}}^{(i)} \lambda_{s\alpha} \lambda_X v \\ &= \sum_{i,j,k,\nu} a_s^{\prime(k)} \otimes a_\alpha^{\prime(\nu)} \otimes b_{\alpha^{-1}}^{\prime(\nu)} b_{s^{-1}}^{\prime(k)} a_s^{(i)} a_\alpha^{(j)} b_{\alpha^{-1}}^{(j)} b_{s^{-1}}^{(i)} \lambda_{s\alpha} \lambda_X v \\ &= \sum_{k,\nu} a_s^{\prime(k)} \otimes a_\alpha^{\prime(\nu)} \otimes b_{\alpha^{-1}}^{\prime(\nu)} b_{s^{-1}}^{\prime(k)} \lambda_{s\alpha} \lambda_X v.\end{aligned}$$

Finally, it is easy to see that the Λ_Y-homomorphisms F and Φ are inverse to each other. ∎

Acknowledgments. The second author wishes to thank the Greek National Scholarship Foundation for the financial support of this work.

REFERENCES

[1] I. Assem, D. Simson and A. Skowroński, *Elements of Representation Theory of Associative Algebras, I: Techniques of Representation Theory*, London Math. Soc. Student Texts 65, Cambridge Univ. Press, 2006.

[2] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, 1995.

[3] P. R. Boisen, *The representation theory of fully group-graded algebras*, J. Algebra 151 (1992), 160–179.

[4] C. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. I, Wiley, New York, 1981.

[5] E. C. Dade, *The equivalence of various generalizations of group rings and modules*, Math. Z. 181 (1982), 335–344.

[6] G. Karpilovsky, *Induced Modules over Group Algebras*, North-Holland Math. Stud. 161, North-Holland, 1990.

[7] C. Nastasescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Math. 1836, Springer, 2004.

[8] Th. Theohari-Apostolidi and H. Vavatsoulas, *Induction and Auslander–Reiten sequences over crossed products*, in: Formal Power Series and Algebraic Combinatorics (Moscow, 2000), D. Krob *et al.* (eds.), Springer, 2000, 765–774.

[9] —, —, *On the separability of the restriction functor*, Algebra Discrete Math. 3 (2003), 95–101.

[10] —, —, *On strongly graded Gorenstein orders*, ibid. 2 (2005), 80–89.

Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki 54124, Greece
E-mail: theohari@math.auth.gr
vava@math.auth.gr

Received 30 June 2006 (4780)