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ON A DECOMPOSITION OF BANACH SPACES

BY

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Abstract. By using D. Preiss' approach to a construction from a paper by J. Matoušek and E. Matoušková, and some results of E. Matoušková, we prove that we can decompose a separable Banach space with modulus of convexity of power type p as a union of a ball small set (in a rather strong symmetric sense) and a set which is Aronszajn null. This improves an earlier unpublished result of E. Matoušková. As a corollary, in each separable Banach space with modulus of convexity of power type p, there exists a closed nonempty set A and a Borel non-Haar null set Q such that no point from Q has a nearest point in A. Another corollary is that ℓ_1 and L_1 can be decomposed as unions of a ball small set and an Aronszajn null set.

1. Introduction. The aim of this paper is to construct decompositions of certain separable Banach spaces into a ball small set and an Aronszajn null set. Let X be a separable Banach space. A set $E \subset X$ is called *porous* if there is $c \in (0,1)$ such that for every $x \in E$ and every $\delta > 0$ there is $z \in X$ such that $0 < \|z-x\| < \delta$ and $E \cap B(z, c\|z-x\|) = \emptyset$. D. Preiss and S. Tišer proved in [9] that every infinite-dimensional separable Banach space X may be decomposed into two sets U and V such that U is of linear measure zero on every line, and V is a countable union of closed porous sets. In particular, the set U is negligible in the sense of Aronszajn (see the definition below).

D. Preiss and L. Zajíček introduced in [10] the notion of a ball small set, which is a subclass of σ-porous sets, and is related to Fréchet differentiability in Hilbert spaces. J. Matoušek and E. Matoušková [6] used the notion of ball smallness to construct an equivalent norm on a separable Hilbert space which is Fréchet differentiable almost nowhere in the sense of Aronszajn. This was achieved by decomposing the Hilbert space as a union of a ball small set and an Aronszajn null set. Using an idea of D. Preiss (see [6]) and results of E. Matoušková from [8] we produce such a decomposition for separable superreflexive spaces whose modulus of convexity is of some power type (actually, we can even take a "symmetric ball small set" in the

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decomposition; see Remark 3.3). Application of Proposition 3.2 from [4] yields a counterexample to a conjecture due to J. P. R. Christensen from [2] in such spaces.

After finishing most of this manuscript (which is part of the author's Ph.D. thesis [3]), the author has learned that the following is implicit in E. Matoušková's unpublished manuscript [7]: every separable infinite-dimensional superreflexive space with modulus of convexity of some power type and with a basis allows a decomposition into a ball small set and an Aronszajn null set. Because [7] is not easily available, we include the full proof which also includes Matoušková's result.

Our approach has another interesting corollary: the spaces ℓ_1 and L_1 can be decomposed as unions of a ball small set and an Aronszajn null set (see Corollary 3.5). This indicates that such "paradoxical" decompositions are also possible for non-reflexive spaces. Our decomposition results are related to earlier results due to Preiss and Tišer [9] in the following way: we replace the notion of σ-porous sets by a (more restrictive) notion of ball small sets; on the other hand, instead of being null on every line, the complements of our sets in the decomposition are Aronszajn null.

2. Preliminaries. Let X be a real Banach space, $x \in X$, and $r > 0$. We denote by $B(x,r)$ the open ball with center x and radius r, and $S(x,r) := \overline{B(x,r)} \setminus B(x,r)$. For $C \subset X$ and $r > 0$ we define $B(C,r) := \bigcup_{c\in C} B(c,r)$. If Y is a subspace of X, then X/Y denotes the quotient of X by Y; it is the set of equivalence classes $\hat{x} = x + Y$ for $x \in X$ (the canonical quotient map $q\colon X \to X/Y$ is defined as $q(x) = \hat{x} = x + Y$), which is a Banach space endowed with the norm $\|\hat{x}\| = \inf\{\|x+y\|\colon y \in Y\}$.

We will need the *modulus of convexity* δ_X which is defined for $\varepsilon \in (0,2]$ as

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2\colon x, y \in S_X,\ \|x-y\| \geq \varepsilon\}.$$

For more information about the modulus of convexity see [1]. We shall say that δ_X is of *power type* p (for some $p \geq 2$) provided there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^p$ for $\varepsilon \in (0,2]$. Note that if X has modulus of convexity of power type p, then X is superreflexive (and thus reflexive); see for example [1].

Let X be a separable Banach space, and let A be a Borel subset of X. The set A is called *Haar null* if there is a Borel probability measure μ on X such that $\mu(x + A) = 0$ for every $x \in X$. Let $B \subset X$ be Borel. We say that B is *Aronszajn null* if for every sequence $(x_i)_{i=1}^{\infty}$ in X whose closed linear span is X, there exist Borel sets $B_i \subset X$ such that $B = \bigcup_i B_i$ and for each $i \in \mathbb{N}$ the intersection of B_i with any line in direction x_i has one-dimensional Lebesgue measure zero. Note that Aronszajn null sets are Haar null, but not conversely. For more details about these notions see [1].

The following notion was defined by D. Preiss and L. Zajíček in [10]. Let X be a normed linear space and $A \subset X$, $r > 0$. We say that A is *r-ball porous* if for each $x \in A$ and $\varepsilon > 0$ there exists $y \in X$ such that $\|x - y\| = r$ and $B(y, r - \varepsilon) \cap A = \emptyset$. We say that $B \subset X$ is *ball small* if $B = \bigcup_n A_n$ where each A_n is r_n-ball porous for some $r_n > 0$. It is a very restrictive porosity-like property. Ball-smallness depends on the equivalent norm (see Example 2.5 in [4]). Note that if for each $x \in A$ and $\varepsilon > 0$ there exists $y \in B(x, r + \varepsilon)$ with $B(y, r) \cap A = \emptyset$, then A is r-ball porous.

We shall need the following lemma from [6].

LEMMA 2.1 ([6, Lemma 2.2]). *Let X be a separable Banach space, let $A \subset X$ be a Borel set, and let Y be a closed subspace of X of finite codimension. Let $n \in \mathbb{N}$ be such that the intersection of A with any n-dimensional affine subspace of X parallel to Y is of n-dimensional measure zero. Then A is Aronszajn null.*

The next lemma comes from [8].

LEMMA 2.2 ([8, Lemma 2.3]). *Let X, Y be separable infinite-dimensional Banach spaces, and $T\colon X \to Y$ a continuous linear surjective mapping. Let $A \subset Y$ be Aronszajn null. Then $T^{-1}(A)$ is Aronszajn null.*

PROPOSITION 2.3. *Let X be a Banach space and Y be a subspace of X.*

(i) *Let $q\colon X \to X/Y$ be the canonical quotient map. Then for each $x \in X$ and $r > 0$ we have*

$$q^{-1}(B_{X/Y}(q(x), r)) = B_X(x, r) + Y.$$

Thus, if $A \subset X/Y$ is a ball small subset of X/Y, then $q^{-1}(A)$ is a ball small subset of X.

(ii) *If X has modulus of convexity of power type p, then so does X/Y.*

Proof. To prove (i), take $z \in q^{-1}(B_{X/Y}(q(x), r))$. Then we have $q(z) \in B_{X/Y}(q(x), r)$, and so $\|q(z) - q(x)\| < r$. Thus there exists $y \in Y$ such that $\|z - y - x\|_X < r$. Then $z = z - y + y$, where $z - y \in B(x, r)$ and $y \in Y$.

For the other inclusion, consider $z + y$ where $z \in B(x, r)$ and $y \in Y$. Then

$$\|q(x) - q(z + y)\| = \|q(x) - q(z)\| \le \|x - z\| < r$$

and thus $z + y \in q^{-1}(B_{X/Y}(q(x), r))$. The rest follows easily.

For (ii), take $\widetilde{x}, \widetilde{z} \in X/Y$ with $\|\widetilde{x}\|_{X/Y} = \|\widetilde{z}\|_{X/Y} = 1$ and $\|\widetilde{x} - \widetilde{z}\|_{X/Y} \ge \varepsilon$ for some $\varepsilon \in (0, 2]$. Pick $x, z \in X$ with $q(x) = \widetilde{x}$, $q(z) = \widetilde{z}$. Because X is reflexive, by weak compactness and weak lower semicontinuity of the norm there exist $y_x, y_z \in Y$ such that $\|x + y_x\|_X = 1$, and $\|z + y_z\|_X = 1$. It follows that $\|x+y_x-z-y_z\|_X \ge \varepsilon$. Now

$$1-\left\|\frac{\widetilde{x}+\widetilde{z}}{2}\right\|_{X/Y} \ge 1-\left\|\frac{x+z}{2}+\frac{y_x+y_z}{2}\right\|_X \ge C\varepsilon^p,$$

since X has modulus of convexity of power type p. To conclude the proof, we take the infimum over all such $\widetilde{x},\widetilde{z}\in X/Y$. ∎

The following theorem can be found in e.g. [1, Theorem E.3(ii)].

Theorem 2.4 (Gurarii–Gurarii). *Let E be a superreflexive space. Then there are $1<t<q<\infty$ and a constant $\gamma=\gamma_E>0$ such that every normalized basic sequence $\{x_n\}$ in E satisfies*

$$\gamma^{-1}\Big(\sum|a_n|^q\Big)^{1/q} \le \Big\|\sum a_nx_n\Big\| \le \gamma\Big(\sum|a_n|^t\Big)^{1/t} \tag{2.1}$$

for every choice of scalars $\{a_n\}$ for which $\sum a_nx_n$ converges.

Remark 2.5. Let E be a superreflexive space with a normalized basis $(f_n)_n$ and let $(e_n)_n$ be its dual basis (see [5]). Let $\gamma=\gamma_E>0$ be the constant from the previous theorem. Then it follows easily from (2.1) (applied to $(f_n)_n$) that $\gamma^{-1}\le\|e_n\|_{E^*}\le\gamma$. Set $\widetilde{e}_n=e_n/\|e_n\|$. Let γ_1,t_1,q_1 be the constants from Theorem 2.4 for E^*. Then (2.1) also holds for $(e_n)_n$ with t_1,q_1, and $\gamma_2=\gamma\cdot\gamma_1$.

Let X be a Banach space. For each n-dimensional subspace Y of X fix an isomorphism $M_Y\colon Y\to\mathbb{R}^n$ ($\mathbb{R}^n$ is taken with the Euclidean norm) with $\|M_Y\|\le 1$. This isomorphism induces a measure λ_Y on Y which is the image of the Lebesgue measure on $\mathbb{R}^n$ under M_Y^{-1}. For each n-dimensional affine subspace $W\subset X$ parallel to Y, fix a vector $o_W\in W$. We define $\lambda_W(C)=\lambda_Y(C-o_W)$ for any Borel subset C of W.

Lemma 2.6. *Suppose $A\subset W$ is Borel. The induced measures have the following properties*:

(i) $\lambda_W(A)=\lambda_W(A+y)$ *for any* $y\in Y$,

(ii) *if Z is an n-dimensional affine subspace of X parallel to W, then*

$$\lambda_W(A)=\lambda_Z(A-o_W+o_Z),$$

(iii) $\eta^n\lambda_W(A)=\lambda_{W\eta}(\eta A)$ *for any* $\eta>0$,

(iv) $\lambda_W(B(s,t)\cap W)\le vt^n$ *for any $s\in W$ and $t>0$, where*

$$v=\sup_n \operatorname{vol}_n(B_{\mathbb{R}^n}(0,1))<\infty.$$

An N-dimensional *test cube* U will be any set of the form

$$U=\Big\{x+\sum_{i=1}^N\alpha_iu_i\colon \alpha_i\in[0,1]\Big\},$$

where $u_i\in\overline{B}_X(0,1)$, and u_i's are linearly independent.

The following lemma gives us an estimate of the measure of sections of balls by affine finite-dimensional subspaces; it is an analogue of Claim 4.6′ from [6] for spaces with modulus of convexity of power type p.

LEMMA 2.7. *Suppose that a Banach space X has modulus of convexity of power type p. Then for each $N \in \mathbb{N}$ there exists $\beta = \beta(p, N, X) > 0$ such that whenever Z is an N-dimensional affine subspace of X, and $x \in X$ satisfies* $\operatorname{dist}(x, Z) \geq 1 - \varrho$ *for $0 < \varrho < 1$, then $\lambda_Z(B(x, 1) \cap Z) \leq \beta\varrho^{N/p}$.*

Proof. Without any loss of generality we can assume that $x = 0$. We know that for some $C > 0$ we have $\delta_X(\varepsilon) \geq C\varepsilon^p$ for $\varepsilon \in (0, 2]$, where δ_X is the modulus of convexity for X. This implies the following:

(∗) If $y, z \in B_X$ and $C\varepsilon^p \geq 1 - \|(y + z)/2\|$ for some $\varepsilon \in (0, 2]$, then $\|y - z\| < \varepsilon$.

Suppose that $\varrho < C \cdot 2^p$. Take $s, y \in B(x, 1) \cap Z$. Then $(y+s)/2 \in B(x, 1) \cap Z$ and thus $\|(y + s)/2\| > 1 - \varrho$. We have $\varrho = C\varepsilon^p$ for some $\varepsilon \in (0, 2)$ and by (∗) we obtain $\|y - s\| < \varepsilon \leq C_1\varrho^{1/p}$. Write $Z = o_Z + Y$, where Y is an N-dimensional linear subspace of X. We get

$$(B(x, 1) \cap Z) - o_Z \subset (B(s, C_1\varrho^{1/p}) \cap Z) - o_Z.$$

This inclusion together with Lemma 2.6 implies that

$$\lambda_Z(B(x, 1) \cap Z) \leq \lambda_Z(B(s, C_1\varrho^{1/p}) \cap Z) \leq C_2\varrho^{N/p},$$

where $C_2 = C_1^N v$, $s - o_Z \in Y$, and $v = \sup_n \operatorname{vol}_n(B_{\mathbb{R}^n}(0, 1))$. Set $\beta_1 = C_2$.

When $\varrho \geq C \cdot 2^p$, we can estimate $\lambda_Z(B(x, 1) \cap Z)$ by $v \cdot 2^N$. In this case, set $\beta_2 = v \cdot 2^N(\max(1, C \cdot 2^p))^{-N/p}$. To conclude the proof, define $\beta := \max(\beta_1, \beta_2)$. ∎

The following is an analogue of Proposition 3.2 from [6] for spaces with modulus of convexity of power type p.

PROPOSITION 2.8. *Let X be an infinite-dimensional separable Banach space with a basis and with modulus of convexity of power type p. Then there exists an $N \in \mathbb{N}$ such that for each $\varepsilon > 0$ we can find $r > 0$, and a countable $C \subset X$, such that*

(A) *for each $x \in X$ and $\psi, \xi > 0$ there exist infinitely many $c_n, \widetilde{c}_n \in C$ with*

  (i) $\|\widetilde{c}_n - (2x - c_n)\| < \xi$,
  (ii) $\|c_n - c_m\| \geq r - \psi$, $\|\widetilde{c}_n - c_m\| \geq r - \psi$ *for $n \neq m$,*
  (iii) $\|\widetilde{c}_n - c_n\| = 2r$,
  (iv) $\|x - c_n\| \leq r + \xi$, $\|x - \widetilde{c}_n\| \leq r + \xi$ *(and thus $B(C, r + \delta) = X$ for any $\delta > 0$),*

(B) $\lambda_{\operatorname{aff}(U)}(U \cap B(C, r)) < \varepsilon$ *for any N-dimensional test cube U.*

REMARK 2.9. For $X = \ell_s$ with $1 < s < \infty$, we can strengthen condition (ii) of Proposition 2.8(A) as follows:

(ii) $\|c_n - c_m\| \geq 2^{1/s} r$, $\|\widetilde{c}_n - c_m\| \geq 2^{1/s} r$ for $n \neq m$.

Proof of Proposition 2.8. Let $(e_k)_k$ be the basis of X and let $(f_k)_k$ be the dual basis. For $X = \ell_s$, take $(e_k)_k$ to be the standard basis of ℓ_s and $(f_k)_k$ the standard basis of $\ell_{s'}$, where $1/s + 1/s' = 1$. We can suppose that $\|f_k\| = 1$. Let $(x_k)_k$ be a dense finitely-supported sequence in X with each point repeated infinitely many times. Construct sequences $n_k \in \mathbb{N}$ and $s_k \in S_X$ so that

- $\max\{\max(\operatorname{supp} x_k), n_{k-1}\} < n_k$,
- $\|s_k\| = \langle f_{n_k}, s_k \rangle = 1$,
- $|f_{n_k}(s_l)| \leq 1/k$ for $l < k$.

This can be achieved using Remark 2.5, because for any $x \in X$ we see that $f_j(x) \to 0$ as $j \to \infty$. For $X = \ell_s$, just choose $s_k = e_{n_k}$. Define $c_k = x_k + s_k$, $\widetilde{c}_k = x_k - s_k$. Note that $\langle f_{n_k}, c_k \rangle = 1$, and put $C_1 = \{c_k \colon k \in \mathbb{N}\}$, $\widetilde{C}_1 = \{c_k \colon k \in \mathbb{N}\}$, and finally $C = C_1 \cup \widetilde{C}_1$.

We will show that for $K > 0$ large enough the following holds:

(A′) for each $x \in X$ and $\psi, \xi > 0$ there exist infinitely many $c_n, \widetilde{c}_n \in C$ with

(i) $\|\widetilde{c}_n - (2x - c_n)\| < \xi$,
(ii) $\|c_n - c_m\| \geq 1 - \psi$, $\|\widetilde{c}_n - c_m\| \geq 1 - \psi$ for $m \neq n$,
(iii) $\|\widetilde{c}_n - c_n\| = 2$ for $n \in \mathbb{N}$,
(iv) $\|x - c_n\| \leq 1 + \xi$, $\|x - \widetilde{c}_n\| \leq 1 + \xi$ (and thus $B(C, 1 + \delta) = X$ for any $\delta > 0$),

(B′) $K^{-N} \lambda_{\mathrm{aff}\{T\}}(T \cap B(C, r)) < \varepsilon$ for any T which is a K times enlarged test cube U.

For $X = \ell_s$, just modify (A) in an obvious way. Once this is established, our proposition follows by taking $r = 1/K$ and renaming $(1/K)C$ as C. Condition (A) follows easily from (A′): just apply (A′) to x/r, ψ/r, and ξ/r. Condition (B) can be obtained from (B′). To see this, choose a test cube $U = \{x + \sum_{i=1}^{N} \alpha_i u_i \colon \alpha_i \in [0, 1]\}$ and define $Y = \operatorname{span}\{u_i\}$ and $V = x + Y$. Take

$$T = KU = \Big\{Kx + \sum_{i=1}^{N} \alpha_i u_i \colon \alpha_i \in [0, K]\Big\}$$

to be our enlarged test cube and define $W = \operatorname{aff}\{T\} = Kx + Y$ (thus $(1/K)W = V$). Then by Lemma 2.6(iii) we obtain

$$K^{-N} \lambda_W(T \cap B(C, 1)) = \lambda_V(U \cap B(C/K, 1/K)).$$

Pick $N > pq$, where $q = q_1$ comes from Remark 2.5. We now prove (B′). First, we only work with C_1. Consider $K > 1/N$. Let U be a test cube and let T be the K times enlarged copy of U (i.e. $T = \{x + \sum_{i=1}^{N} \alpha_i u_i \colon \alpha_i \in [0, K]\}$). For $j = 0, 1, \dots$ let

$$I_j = \{k \in \mathbb{N}\colon 1 - 2^{-j} \le \operatorname{dist}(T, c_k) < 1 - 2^{-j-1}\}.$$

Take $w_k \in T$ such that $\|w_k - c_k\| = \operatorname{dist}(T, c_k)$. If $\operatorname{dist}(c_k, T) < 1$, then $k \in I_j$ for some j and we have

$$\lambda_{\mathrm{aff}(T)}(T \cap B(c_k, 1)) \le \beta 2^{-Nj/p}.$$

Hence

$$K^{-N} \lambda_{\mathrm{aff}\{T\}}(B(C_1, 1) \cap T) \le \frac{\beta}{K^N} \sum_{j=0}^{\infty} 2^{-Nj/p} |I_j|.$$

To estimate $|I_j|$ note that for almost all $k \in I_j$ the vector f_{n_k} is "almost orthogonal" to $\operatorname{span} u_j$. Indeed, take $\eta = 1/(NK\, 2^{j+2})$ and define

$$I'_j = \{k \in I_j \colon |\langle u_i, f_{n_k}\rangle| < \eta \text{ for all } i = 1, \dots, N\}.$$

Because u_i's are unit vectors, by Remark 2.5 we obtain

$$|\{k\colon |\langle u_i, f_{n_k}\rangle| \ge \eta\}| \le \sum_{k\,:\,|\langle u_i, f_{n_k}\rangle| \ge \eta} |\langle u_i, f_{n_k}\rangle|^q / \eta^q \le \gamma^q \|u_i\|^q / \eta^q \le \gamma^q \eta^{-q}.$$

Hence

$$|I_j \setminus I'_j| \le N \gamma^q \eta^{-q} = \gamma^q N^{q+1} K^q 2^{q(j+2)}.$$

We need to bound $|I'_j|$ so suppose that $k \in I'_j$. Then obviously

$$\|w_k - c_k\| \ge |\langle f_{n_k}, w_k\rangle - \langle f_{n_k}, c_k\rangle| = |\langle f_{n_k}, w_k\rangle - 1|$$

and from $\|w_k - c_k\| < 1 - 2^{-j-1}$ it follows that $\langle f_{n_k}, w_k\rangle > 2^{-j-1}$. Write w_k as $x + \sum_{i=1}^{N} \alpha_i u_i$ where $0 \le \alpha_i \le K$. We obtain

$$\langle f_{n_k}, x\rangle \ge \langle f_{n_k}, w_k\rangle - NK\eta \ge 2^{-j-2}.$$

For $k \in I'_j$ we have

$$\begin{aligned}
\gamma^{-1} \Big(\sum_{i=n_k}^{\infty} |\langle f_i, x\rangle|^q \Big)^{1/q} &\le \gamma^{-1} \Big(\sum_{i=1}^{\infty} |\langle f_i, x - x_k\rangle|^q \Big)^{1/q} \\
&\le \|x - x_k\| \le \|x - c_k\| + \|s_k\| \le \|x - w_k\| + \|w_k - c_k\| + 1 < 4NK,
\end{aligned}$$

because $\operatorname{supp} x_k \subset [0, n_k)$. Take the first $l \in \mathbb{N}$ so that $(\sum_{i=l}^{\infty} |\langle f_i, x\rangle|^q)^{1/q} \le 4\gamma NK$, and then observe that

$$(4\gamma NK)^q \ge \sum_{i=l}^{\infty} |\langle f_i, x\rangle|^q \ge \sum_{k \in I'_j\,:\, n_k > l} \langle f_{n_k}, x\rangle^q \ge (|I'_j| - 1) 2^{-q(j+2)}.$$

From this we obtain $|I'_j| \le \gamma^q N^q K^q 2^{q(j+4)+1}$ for sufficiently large K. The final estimate is

$$K^{-N}\lambda_{\mathrm{aff}\{T\}}(B(C_1,1)\cap T) \le \frac{\beta}{K^N}\sum_{j=0}^{\infty} 2^{-Nj/p}|I_j|$$
$$\le \frac{\beta}{K^N}\Big(\sum_{j=0}^{\infty} 2^{-Nj/p}|I'_j| + \sum_{j=0}^{\infty} 2^{-Nj/p}|I_j\setminus I'_j|\Big) \le \frac{C(N,p,q,X)}{K^{N-q}}.$$

Now take K large enough to make the last quantity less than $\varepsilon/2$.

The same argument as above works also for $\widetilde{C}_1$, provided we replace f_{n_k} by $-f_{n_k}$, c_k by $\widetilde{c}_k$, and C_1 by $\widetilde{C}_1$. Thus, altogether we obtain

$$K^{-N}\lambda_{\mathrm{aff}\{T\}}(B(C,1)\cap T) < \varepsilon.$$

To see that (A′) holds, fix $\psi,\xi>0$ and $x\in X$. Select $k\in\mathbb{N}$ such that $2\,\|x_k-x\|<\xi$ and $1/k<\psi$. By the choice of $(x_k)_k$ there exists a sequence $(m_j)_j\subset\mathbb{N}$ such that $m_j>k$ and $x_k=x_{m_j}$ for each $j\in\mathbb{N}$. We shall see that the sequences $(c_{m_j})_j$ and $(\widetilde{c}_{m_j})_j$ satisfy the conclusion of (A′). To see that (iv) holds, note that $\|x-c_{m_j}\|\le\|x-x_{m_j}\|+\|s_{m_j}\|\le 1+\xi$, and similarly for $\widetilde{c}_{m_j}$. Notice that $\|\widetilde{c}_{m_j}-(2x-c_{m_j})\| = 2\,\|x_{m_j}-x\|<\xi$, and this implies (i). To prove (ii), for $i>j$ we get

$$\|c_{m_i}-c_{m_j}\|\ge\|s_{m_i}-s_{m_j}\|\ge f_{n_{m_i}}(s_{m_i}-s_{m_j})\ge 1-1/m_i\ge 1-\psi,$$

where the penultimate inequality follows from the construction of n_k. Again for $i>j$ we get

$$\|\widetilde{c}_{m_i}-c_{m_j}\|\ge\|s_{m_i}+s_{m_j}\|\ge f_{n_{m_i}}(s_{m_i}+s_{m_j})\ge 1-1/m_i\ge 1-\psi,$$

and an analogous argument works in the case $i<j$. Finally, (iii) follows from $\|\widetilde{c}_{m_i}-c_{m_i}\|=2\|s_{m_i}\|$. If $X=\ell_p$, we get $\|c_{m_i}-c_{m_j}\|\ge\|e_{m_i}-e_{m_j}\|\ge 2^{1/p}$, and $\|\widetilde{c}_{m_i}-c_{m_j}\|\ge\|e_{m_i}+e_{m_j}\|\ge 2^{1/p}$ for $i\ne j$. ■

3. Spaces with power type modulus of convexity. We are now ready to prove the main theorem:

THEOREM 3.1. *Let X be an infinite-dimensional separable Banach space with modulus of convexity of power type p. Then there exists a Borel set $A\subset X$ which is ball small and whose complement is Aronszajn null.*

REMARK 3.2. It follows from results of Hanner (see e.g. [1]) that spaces L_p and ℓ_p for $1<p<\infty$ have modulus of convexity of power type $\max(2,p)$, and thus they satisfy the assumptions of the above theorem.

Proof. Choose a subspace $Y\subset X^*$ with a basis; such a subspace exists according to Theorem 1.a.5 from [5]. Then by reflexivity $W=X/Y^*$ also has a basis (see [5]). Proposition 2.3(ii) implies that X/Y^* has modulus of convexity of power type p. Let $D\subset W$ be an Aronszajn null set whose complement is ball small. It follows from Lemma 2.2 that $q^{-1}(D)$ is an Aronszajn null subset of X, and Proposition 2.3(i) that $q^{-1}(W \setminus D) = X \setminus q^{-1}(D)$ is a ball small subset of X. Thus if we can construct a set D which satisfies the conclusion of the theorem for W, then $q^{-1}(D)$ is the desired set for X. These observations imply that we can suppose that our space has a basis.

For each $m \in \mathbb{N}$ apply Proposition 2.8 with $\varepsilon = 1/m$ obtaining $r_m > 0$ and a set C_m. Define $E = \bigcap_m B(C_m, r_m)$. To see that E is Aronszajn null, it is enough to see that $U \cap E$ has Lebesgue measure zero for any test cube U, as any N-dimensional affine subspace $Z \subset X$ can be written as a countable union of such cubes. If U is such a test cube, then $\lambda_U(U \cap B(C_m, r_m)) \le 1/m$ and thus $\lambda_U(U \cap E) = 0$. Now application of Lemma 2.1 shows that E is Aronszajn null.

We have to establish that $A = X \setminus E = \bigcup_m (X \setminus B(C_m, r_m))$ is ball small. It suffices to observe that condition (A) of Proposition 2.8 implies that $X \setminus B(C_m, r_m)$ is r_m-ball porous. ∎

Remark 3.3. In fact, the set A from the previous theorem is ball small in a very strong symmetric sense. It can be decomposed as $A = \bigcup_n A_n$, where $A_n = X \setminus B(C_n, r_n)$ and for each A_n we have (take $r := r_n$):

(†) for each $x \in A_n$ and $\alpha, \beta > 0$ there exist countably many c_j with

- (i) $\|(2x - c_j) - x\| = \|x - c_j\| \le r + \alpha$,
- (ii) $B(c_j, r) \cap A_n = \emptyset$, $B(2x - c_j, r - \alpha) \cap A_n = \emptyset$,
- (iii) $\|c_j - c_k\| \ge r - \beta$, $\|(2x - c_j) - c_k\| \ge r - \beta$ for $j \ne k$,
- (iv) $\|(2x - c_j) - c_j\| \ge 2r - \beta$.

To see this, choose $\psi, \xi > 0$ such that $\max(\psi, \psi + \xi) < \beta$ and $\xi < \alpha$. Then condition (A) of Proposition 2.8 yields countably many $c_j, \widetilde{c}_j$. Condition (i) follows from condition (iv) of Proposition 2.8. To prove (ii), note that it follows immediately from the definition of A_n that $B(c_j, r) \cap A_n = \emptyset$ for all $j \in \mathbb{N}$. Since $\|\widetilde{c}_j - (2x - c_j)\| < \xi < \alpha$ it follows that $B(2x - c_j, r - \alpha) \subset B(\widetilde{c}_j, r)$, and thus $B(2x - c_j, r - \alpha) \cap A_n = \emptyset$. Now estimate (for any $j, k \in \mathbb{N}$)

$$\|(2x - c_j) - c_k\| \ge \|(2x - c_j) - \widetilde{c}_j + \widetilde{c}_j - c_k\| \ge r - \psi - \xi \ge r - \beta,$$

and so conditions (iii) and (iv) hold. That concludes the proof of (†).

Note that (†) also implies that A_n is r-ball porous. To see this, take $x \in A_n$ and $\varepsilon > 0$. We can assume by shifting and rescaling that $x = 0$ and $r = 1$. Take $\alpha := \varepsilon/2$ and $y := c_k/\|c_k\|$, where $(c_j)_j$ is the sequence from (†) and $k \in \mathbb{N}$ is arbitrary. Assume that $\|z - y\| < 1 - \varepsilon$ for some $z \in X$. Then

$$\|z - c_j\| \le \|z - c_j/\|c_j\|\,\| + \|c_j - c_j/\|c_j\|\,\| < 1 - \varepsilon + \varepsilon/2 = 1 - \alpha.$$

Thus $B(y, 1 - \varepsilon) \subset B(c_j, r)$ and so $B(y, 1 - \varepsilon) \cap A_n = \emptyset$. We have established that A_n is r-ball porous.

For $X = \ell_p$ with $1 < p < \infty$, by Remark 2.9 it is easy to see that condition (iii) from (†) can be replaced with

(iii) $\|c_j - c_k\| \geq 2^{1/p}r - \beta$, $\|(2x - c_j) - c_k\| \geq 2^{1/p}r - \beta$ for $j \neq k$.

COROLLARY 3.4. *Let X be an infinite-dimensional separable superreflexive space. Then there exists an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ has modulus of convexity of power type p for some $p \geq 2$ and there exists a ball small set A whose complement is Aronszajn null.*

Proof. It is well known that for superreflexive spaces, there exists an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ has modulus of convexity of power type p for some $p \geq 2$ (see e.g. [1, Theorem A.6]). Apply Theorem 3.1 to $(X, |\cdot|)$. ∎

We get the following decomposition for the spaces ℓ_1 and L_1:

COROLLARY 3.5. *There exist Borel ball small subsets $A \subset \ell_1$ and $B \subset L_1$ whose complements are Aronszajn null.*

Proof. Let $q : \ell_1 \to \ell_2$ be a linear quotient map (it exists by e.g. [5, p. 108]). Let $\ell_2 = B \cup D$, where B is ball small and D is Aronszajn null, and let $A := q^{-1}(B)$. Then A is ball small by Proposition 2.3(i), and $\ell_1 \setminus A$ is Aronszajn null by Lemma 2.2.

Let $Y \subset L_1$ be a closed complemented subspace of L_1 isometric to ℓ_1 (call the isometry $T : Y \to \ell_1$); existence of such a space Y is well known. Let $P : L_1 \to Y$ be the projection onto Y. Then Proposition 2.3(i) and Lemma 2.2 imply that $B := P^{-1}(T^{-1}(A)) \subset L_1$ (where $A \subset \ell_1$ is as in the previous paragraph) is a Borel ball small set with an Aronszajn null complement. ∎

The following proposition was proved in [4].

PROPOSITION 3.6 ([4, Proposition 3.2]). *Let X be a separable Banach space and $D \subset X$ be a Borel ball small set. Suppose that $X \setminus D$ is Aronszajn null. Then there exists a nonempty closed set A and a Borel set Q which is not Haar null such that the metric projection $P_A(x)$ is empty for each $x \in Q$.*

By combining Theorem 3.1 with Proposition 3.6, we obtain the following corollary, which shows that Christensen's conjecture [2] concerning almost-everywhere existence of nearest points fails also in separable spaces with modulus of convexity of power type p (for some $p \geq 2$).

COROLLARY 3.7. *Let X be an infinite-dimensional separable superreflexive space such that X has modulus of convexity of power type p for some $p \geq 2$. Then there exists a nonempty closed set A and a Borel set Q which is not Haar null such that $P_A(x) = \emptyset$ for all $x \in Q$.*

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