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EMBEDDING A TOPOLOGICAL GROUP INTO A CONNECTED GROUP

BY

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Abstract. It was proved in [HM] that each topological group $(G,\cdot,\tau)$ may be embedded into a connected topological group $(\widehat{G},\bullet,\widehat{\tau})$. In fact, two methods of introducing $\widehat{\tau}$ were given. In this note we show relations between them.

At the beginning let us recall Hartman and Mycielski's construction of the embedding of a topological group G into a connected group $\widehat{G}$ ([HM]).

If $(G,\cdot)$ is a group, then all functions from the interval $[0,1)$ to G form a group with the pointwise multiplication $f \bullet g(x) = f(x)\cdot g(x)$. Of course G is isomorphic to the subgroup of all constant functions. If $\widehat{G}$ denotes the subset of all step functions, i.e. $f \in \widehat{G}$ if there are points $a_0 = 0 < a_1 < \cdots < a_n = 1$ such that $f|_{[a_{i-1},a_i)}$ is a constant, for $i = 1,\dots,n$, then $(\widehat{G},\bullet)$ is a group.

Given a topological group $(G,\cdot,\tau)$ one can construct a topology $\widehat{\tau}$ making $(\widehat{G},\bullet,\widehat{\tau})$ a topological group. Namely, let β denote a symmetric open basis of the neutral element $e \in G$ and let m stand for the Lebesgue measure on $[0,1)$. Put $N(V,\varepsilon) = \{f \in \widehat{G} : m(f^{-1}(G\setminus V)) < \varepsilon\}$ for $\varepsilon > 0$ and $V \in \beta$. Then $\{g \bullet N(V,\varepsilon) : g \in \widehat{G},\ \varepsilon > 0,\ V \in \beta\}$ is a basis for the topology $\widehat{\tau}$.

If d is a metric on G, then $\widehat{d}(f,g) = \int_0^1 d(f(x),g(x))\,dx$ defines a metric $\widehat{d}$ on $\widehat{G}$ and if $(G,\cdot,\tau_d)$ is a topological group, then $(\widehat{G},\bullet,\tau_{\widehat{d}})$ is a topological group as well, where $\tau_d,\tau_{\widehat{d}}$ denote the topologies generated by $d,\widehat{d}$.

For $x_0,x_1 \in G$ the family $\{f_t\}_{t\in[0,1]} \subset \widehat{G}$, where $f_t([0,t)) = x_0$ and $f_t([t,1)) = x_1$, is an arc in both topologies $\tau_{\widehat{d}}$ and $\widehat{\tau}$ on $\widehat{G}$. Moreover, in both topologies $\widehat{G}$ is arcwise and locally arcwise connected.

A natural question arises: if the topology τ in $(G,\cdot,\tau)$ is generated by a metric d, is then $\widehat{\tau}$ the same topology as the one generated by $\widehat{d}$? In other words: is it true that $\widehat{\tau} = \tau_{\widehat{d}}$? We show in Proposition 1 that it

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is not the case in general, however for bounded metrics it holds true (see Theorem 2).

PROPOSITION 1. *Let* $(\mathbb{Z}, +, \tau)$ *be the additive group of integers with the discrete topology* τ *generated by the Euclidean metric* d. *Then* $\widehat{\tau} \subsetneq \tau_{\widehat{d}}$, *i.e. the metric* $\widehat{d}$ *generates a stronger topology* $\tau_{\widehat{d}}$ *than* $\widehat{\tau}$ *on* $\widehat{\mathbb{Z}}$.

Proof. First we will show that $\widehat{\tau} \subset \tau_{\widehat{d}}$. Let $N(V, \varepsilon) \in \widehat{\tau}$ be a neighborhood of the neutral element $\widehat{0} \in \widehat{\mathbb{Z}}$. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Consider a ball $B(\widehat{0}, 1/n^2) \in \tau_{\widehat{d}}$, i.e. the set of all functions in $\widehat{\mathbb{Z}}$ with distance $< 1/n^2$ to the constant function $\widehat{0}(x) = 0$. For $f \in B(\widehat{0}, 1/n^2)$ we have $\widehat{d}(f, \widehat{0}) = \int_0^1 |f(x) - \widehat{0}(x)|\, dx = \int_0^1 |f(x)|\, dx < 1/n^2$. Since $\{x \in [0,1) : f(x) \notin V\} \subset \{x \in [0,1) : f(x) \neq 0\} = \{x \in [0,1) : |f(x)| \geq 1/n\}$, we obtain $m(\{x \in [0,1) : f(x) \notin V\}) \leq m(\{x \in [0,1) : |f(x)| \geq 1/n\}) < 1/n < \varepsilon$. This shows $B(\widehat{0}, 1/n^2) \subset N(V, \varepsilon)$.

Conversely, we consider a ball $B(\widehat{0}, 1/n) \in \tau_{\widehat{d}}$. Suppose for contradiction that $\tau_{\widehat{d}} \subset \widehat{\tau}$, hence $N(V, \varepsilon) \subset B(\widehat{0}, 1/n)$ for some $V \in \beta$ and $\varepsilon > 0$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence defined by $f_k = k^2 \chi_{[0,1/k)}$. Then $f_k \in N(V, \varepsilon)$ for $k > 1/\varepsilon$. On the other hand, $\widehat{d}(f_k, \widehat{0}) = \int_0^1 |f_k(x)|\, dx = k$, thus $f_k \notin B(\widehat{0}, 1/n)$ for all $k \in \mathbb{N}$. ■

We can improve the statement of Proposition 1 as follows:

THEOREM 2. *Let* $(G, \cdot, \tau_\varrho)$ *be a topological group with the topology* τ_ϱ *generated by a metric* ϱ. *If* ϱ *is bounded, then the metric* $\widehat{\varrho}$ *generates the topology* $\tau_{\widehat{\varrho}}$ *coinciding with* $\widehat{\tau_\varrho}$ *on* $\widehat{G}$. *In particular, if* G *is compact, then* $\widehat{\tau_\varrho} = \tau_{\widehat{\varrho}}$.

Proof. Let e and $\widehat{e}$ denote the neutral elements of G and $\widehat{G}$ (respectively), where $\widehat{e}$ is the constant function $\widehat{e}(x) = e$. Let $B_r(e) \subset G$ and $\widehat{B_r(\widehat{e})} \subset \widehat{G}$ denote the open balls of radius r centered at e in (G, ϱ) and centered at $\widehat{e}$ in $(\widehat{G}, \widehat{\varrho})$, respectively. Let $\lambda \in \mathbb{R}$ be such that $\varrho(x, y) \leq \lambda$ for all $x, y \in G$.

First we will show $\widehat{\tau_\varrho} \subset \tau_{\widehat{\varrho}}$. Suppose $N(V, \varepsilon) \in \widehat{\tau_\varrho}$ is given, where $e \in V \in \tau_\varrho$ and $\varepsilon > 0$. There is $0 < r < \varepsilon$ such that $B_r(e) \subset V$. For $\delta = r^2$, we show that $\widehat{B_\delta(\widehat{e})}$ is a subset of $N(V, \varepsilon)$, so we suppose $g \in \widehat{B_\delta(\widehat{e})}$.

Put $E = \{x \in [0,1) : \varrho(g(x), \widehat{e}(x)) \geq r\}$ and $F = [0,1) \setminus E$.

Observe that $m(E) < \varepsilon$. Indeed, since $m(E) = (1/r) \int_E r\, dx$, we have

$$m(E) \leq \frac{1}{r} \int_E \varrho(g(x), \widehat{e}(x))\, dx \leq \frac{1}{r} \int_0^1 \varrho(g(x), \widehat{e}(x))\, dx = \frac{1}{r}\, \widehat{\varrho}(g, \widehat{e}) < \varepsilon.$$

Further note that $g(F) \subset V$. Indeed, suppose $y \in g(F)$, hence $y = g(x)$ for some $x \in F$. Thus, $\varrho(g(x), \widehat{e}(x)) < r$, which implies $y = g(x) \in B_r(e) \subset V$. Therefore $m(\{x \in [0,1) : g(x) \notin V\}) \leq m(E) < \varepsilon$ and $g \in N(V, \varepsilon)$, which proves $\widehat{B_\delta(\widehat{e})} \subset N(V, \varepsilon)$. Notice that this part of proof does not depend on the boundedness assumption on ϱ, hence $\widehat{\tau}_\varrho \subset \tau_{\widehat{\varrho}}$ is always true.

Next, we will show $\widehat{\tau}_\varrho \supset \tau_{\widehat{\varrho}}$. Suppose $\widehat{B_\varepsilon(\widehat{e})} \in \tau_{\widehat{\varrho}}$ is given. There is $V \in \tau_\varrho$ containing e such that $V \subset B_{\varepsilon/2}(e)$. Put $\delta = \varepsilon/2\lambda$. We prove that $N(V, \delta) \subset \widehat{B_\varepsilon(\widehat{e})}$. Suppose $g \in N(V, \delta)$. Let E and F be subsets of $[0,1)$ defined by $E = \{x \in [0,1) : g(x) \in V\}$ and $F = \{x \in [0,1) : g(x) \notin V\}$.

Since $V \subset B_{\varepsilon/2}(e)$, we have

$$\int_E \varrho(g(x), \widehat{e}(x))\, dx \leq \int_E \frac{\varepsilon}{2}\, dx \leq \int_0^1 \frac{\varepsilon}{2}\, dx = \frac{\varepsilon}{2}.$$

Since $g \in N(V, \delta)$ implies $m(F) < \delta$, we have

$$\int_F \varrho(g(x), \widehat{e}(x))\, dx \leq \int_F \lambda\, dx = \lambda m(F) < \lambda\delta = \frac{\varepsilon}{2}.$$

Therefore

$$\widehat{\varrho}(g, \widehat{e}) = \int_E \varrho(g(x), \widehat{e}(x))\, dx + \int_F \varrho(g(x), \widehat{e}(x))\, dx < \varepsilon,$$

and this implies $g \in \widehat{B_\varepsilon(\widehat{e})}$ which completes the proof. ■

EXAMPLE 3. Consider a function $\varrho : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by $\varrho(m,n) = 1/2^k$, where k is the highest power of 2 that divides $|n-m|$ for $n \neq m$, and $\varrho(m,m) = 0$. Then ϱ is a metric on $\mathbb{Z}$, called the "2-adic" metric, which is bounded. Therefore, we have $(\widehat{\mathbb{Z}}, \widehat{\tau}_\varrho) = (\widehat{\mathbb{Z}}, \tau_{\widehat{\varrho}})$.

Finally, I would like to remark that any metric ϱ on a metric space (G, τ) is equivalent to a bounded metric ϱ'. Such a bounded metric can be obtained by defining either $\varrho' = \varrho/(\varrho+1)$ or $\varrho' = \min\{\varrho, 1\}$. Thus, ϱ and ϱ' generate the same topology on G. If we use such a bounded metric ϱ' on the group G, then the $\widehat{\varrho'}$-topology and $\widehat{\tau_{\varrho'}}$ topology are the same.

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