COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 1

CHEN'S INEQUALITY IN THE LAGRANGIAN CASE

BY

TEODOR OPREA (Bucureşti)

Abstract. In the theory of submanifolds, the following problem is fundamental: *establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of submanifolds.* The basic relationships discovered until now are inequalities. To analyze such problems, we follow the idea of C. Udrişte that the method of constrained extremum is a natural way to prove geometric inequalities. We improve Chen's inequality which characterizes a totally real submanifold of a complex space form. For that we suppose that the submanifold is Lagrangian and we formulate and analyze a suitable constrained extremum problem.

1. Optimization on Riemannian submanifolds. Let $(N,\widetilde{g})$ be a Riemannian manifold of dimension m, M be a Riemannian submanifold of it, g be the metric induced on M by $\widetilde{g}$, and $f : N \to \mathbb{R}$ be a differentiable function.

In [6] we considered the constrained extremum problem and proved the following theorem:

THEOREM 1. *If* $x_0 \in M$ *is such that* $f(x_0) = \min_{x\in M} f(x)$, *then*

(i) $(\operatorname{grad} f)(x_0) \in T^{\perp}_{x_0}M$,
(ii) *the bilinear form*

$$\alpha : T_{x_0}M \times T_{x_0}M \to \mathbb{R},$$
$$\alpha(X,Y) = \operatorname{Hess}_f(X,Y) + \widetilde{g}(h(X,Y),(\operatorname{grad} f)(x_0))$$

is positive semidefinite, where h *is the second fundamental form of the submanifold* M *in* N.

We shall use this theorem in order to find an inequality satisfied by the Chen invariant of a Lagrangian submanifold in a complex space form.

2. An estimation of Chen's invariant. Let (M,g) be a Riemannian manifold of dimension n, and x a point in M. We consider an orthonormal

2000 *Mathematics Subject Classification*: 53C21, 53C24, 53C25, 49K35.
Key words and phrases: constrained maximum, Chen's inequality, Lagrangian submanifold.

frame $\{e_1, \ldots, e_n\}$ in T_xM. The scalar curvature at x is defined by

$$\tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j),$$

where R is the Riemann curvature tensor of (M, g). We define $\delta_M = \tau - \min(k)$, where k is the sectional curvature at x. The invariant δ_M is called the *Chen invariant* of the Riemannian manifold (M, g).

Let $(\widetilde{M}, \widetilde{g}, J)$ be a Kähler manifold of real dimension $2m$. A submanifold M of dimension n of $(\widetilde{M}, \widetilde{g}, J)$ is called *totally real* if $J(T_xM) \subset T_x^{\perp}M$ for any x in M.

If, in addition, $n = m$, then M is called a *Lagrangian submanifold*. For a Lagrangian submanifold, the relation $J(T_xM) = T_x^{\perp}M$ holds.

A Kähler manifold with constant holomorphic sectional curvature c is called a *complex space form* and is denoted by $\widetilde{M}(c)$. The Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z = \frac{c}{4}\{&\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y \\ &+ \widetilde{g}(JY, Z)JX - \widetilde{g}(JX, Z)JY + 2\widetilde{g}(X, JY)JZ\}.\end{aligned}$$

B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken showed in [5] that every totally real submanifold M of real dimension n in a complex space form $\widetilde{M}(c)$ of real dimension $2m$ satisfies Chen's inequality

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)\frac{c}{4}\right\},$$

where H is the mean curvature vector of M.

Remarks 1. If M is a totally real submanifold of real dimension n in a complex space form $\widetilde{M}(c)$ of real dimension $2m$, then

$$A_{JY}X = -Jh(X, Y) = A_{JX}Y$$

for any vector fields X and Y on M, where A_X is the Weingarten operator.

2. Let $m = n$ (M is Lagrangian in $\widetilde{M}(c)$). If we consider a point $x \in M$ and orthonormal frames $\{e_1, \ldots, e_n\}$ in T_xM and $\{Je_1, \ldots, Je_n\}$ in $T_x^{\perp}M$, then

$$h^i_{jk} = h^j_{ik}, \quad \forall i, j, k \in \overline{1, n},$$

where h^i_{jk} is the coefficient of Je_i in the expansion of the vector $h(e_j, e_k)$.

With these ingredients we prove the next result, which is an improved version of Chen's inequality in the Lagrangian case.

Theorem 2. *Let M be a Lagrangian submanifold in a complex space form $\widetilde{M}(c)$ of real dimension $2n$, $n \geq 3$. Then*

$$\delta_M \leq \frac{(n-2)(n+1)}{2}\frac{c}{4} + \frac{n^2}{2}\frac{2n-3}{2n+3}\|H\|^2.$$

Proof. We consider a point $x \in M$ and orthonormal frames $\{e_1, \ldots, e_n\}$ in T_xM and $\{Je_1, \ldots, Je_n\}$ in $T_x^{\perp}M$, $\{e_1, e_2\}$ being an orthonormal frame in the 2-plane which minimizes the sectional curvature at x.

By using Gauss's equation and the fact that $\widetilde{M}(c)$ is a complex space form, we obtain

$$\tau = \frac{n(n-1)}{2}\,\frac{c}{4} + \sum_{r=1}^{n} \sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r - \sum_{r=1}^{n} \sum_{1\le i<j\le n} (h_{ij}^r)^2 \tag{1}$$

and

$$R(e_1, e_2, e_1, e_2) = \frac{c}{4} + \sum_{r=1}^{n} h_{11}^r h_{22}^r - \sum_{r=1}^{n} (h_{12}^r)^2. \tag{2}$$

By subtracting (1) and (2), we find

$$\begin{aligned}\delta_M = \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \sum_{r=1}^{n}\Big(&\sum_{3\le j\le n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r \\ &- \sum_{3\le j\le n} (h_{1j}^r)^2 - \sum_{2\le i<j\le n} (h_{ij}^r)^2\Big).\end{aligned} \tag{3}$$

By using the symmetry in the three indices of h_{ij}^k, we can rewrite (3) as

$$\begin{aligned}\delta_M \le{}& \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \sum_{r=1}^{n}\Big(\sum_{3\le j\le n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r\Big) \\ &- \sum_{3\le j\le n} (h_{1j}^1)^2 - \sum_{3\le j\le n} (h_{1j}^j)^2 - \sum_{2\le i<j\le n} (h_{ij}^i)^2 - \sum_{2\le i<j\le n} (h_{ij}^j)^2 \\ ={}& \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \sum_{r=1}^{n}\Big(\sum_{3\le j\le n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r\Big) \\ &- \sum_{3\le j\le n} (h_{11}^j)^2 - \sum_{3\le j\le n} (h_{jj}^1)^2 - \sum_{2\le i<j\le n} (h_{ii}^j)^2 - \sum_{2\le i<j\le n} (h_{jj}^i)^2 \\ ={}& \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \sum_{r=1}^{n}\Big(\sum_{3\le j\le n} (h_{11}^r + h_{22}^r) h_{jj}^r + \sum_{3\le i<j\le n} h_{ii}^r h_{jj}^r\Big) \\ &- \sum_{3\le j\le n} (h_{11}^j)^2 - \sum_{3\le j\le n} (h_{jj}^1)^2 - \sum_{\substack{i,j\in\overline{2,n}\\ i\ne j}} (h_{jj}^i)^2.\end{aligned} \tag{4}$$

Let us consider the quadratic forms f_1, f_2, $f_r : \mathbb{R}^n \to \mathbb{R}$, $r \in \overline{3,n}$, defined by

$$f_1(h_{11}^1, h_{22}^1, \ldots, h_{nn}^1) = \sum_{3\le j\le n} (h_{11}^1 + h_{22}^1) h_{jj}^1 + \sum_{3\le i<j\le n} h_{ii}^1 h_{jj}^1 - \sum_{3\le j\le n} (h_{jj}^1)^2,$$

$$f_2(h^2_{11}, h^2_{22}, \dots, h^2_{nn}) = \sum_{3\le j\le n} (h^2_{11} + h^2_{22})h^2_{jj} + \sum_{3\le i<j\le n} h^2_{ii}h^2_{jj} - \sum_{3\le j\le n} (h^2_{jj})^2,$$

$$f_r(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) = \sum_{3\le j\le n} (h^r_{11} + h^r_{22})h^r_{jj} + \sum_{3\le i<j\le n} h^r_{ii}h^r_{jj} - (h^r_{11})^2 - \sum_{\substack{j\in\overline{2,n}\\ j\neq r}} (h^r_{jj})^2.$$

First, we find an upper bound for f_1, subject to $P : h^1_{11} + h^1_{22} + \dots + h^1_{nn} = k^1$, where k^1 is a real constant.

The first three partial derivatives of the function f_1 are

$$\frac{\partial f_1}{\partial h^1_{11}} = \sum_{3\le j\le n} h^1_{jj}, \tag{5}$$

$$\frac{\partial f_1}{\partial h^1_{22}} = \sum_{3\le j\le n} h^1_{jj}, \tag{6}$$

$$\frac{\partial f_1}{\partial h^1_{33}} = h^1_{11} + h^1_{22} + \sum_{4\le j\le n} h^1_{jj} - 2h^1_{33}. \tag{7}$$

Since for a solution $(h^1_{11}, h^1_{22}, \dots, h^1_{nn})$ of the problem in question, the vector $\operatorname{grad} f_1$ is normal to P, from (5)–(7) we obtain

$$h^1_{11} + h^1_{22} = 3h^1_{jj} = 3a^1, \quad \forall j \in \overline{3,n}. \tag{8}$$

By using the relation $h^1_{11} + h^1_{22} + h^1_{33} + \dots + h^1_{nn} = k^1$, from (8) we obtain $3a^1 + (n-2)a^1 = k^1$. Consequently,

$$a^1 = \frac{k^1}{n+1}. \tag{9}$$

As f_1 is obtained from the function involved in Chen's inequality (see [6]) by subtracting some square terms, the Hessian of $f_1|P$ is negative definite. Consequently, the point $(h^1_{11}, h^1_{22}, \dots, h^1_{nn})$ given by the relations (8) and (9) is a maximum point, and hence

$$f_1 \le 3a^1(n-2)a^1 + C^2_{n-2}(a^1)^2 - (n-2)(a^1)^2 = \frac{(a^1)^2}{2}(n+1)(n-2). \tag{10}$$

From (9) and (10), it follows that

$$f_1 \le \frac{(k^1)^2}{2}\,\frac{n-2}{n+1} = \frac{n^2}{2}\,\frac{n-2}{n+1}\,(H^1)^2. \tag{11}$$

In a similar manner, we show that

$$f_2 \le \frac{n^2}{2}\,\frac{n-2}{n+1}\,(H^2)^2. \tag{12}$$

Next, we find an upper bound for f_3, subject to $P : h^3_{11} + h^3_{22} + \cdots + h^3_{nn} = k^3$, where k^3 is a real constant.

The first four partial derivatives of the function f_3 are

$$\frac{\partial f_3}{\partial h^3_{11}} = \sum_{3\le j\le n} h^3_{jj} - 2h^3_{11}, \tag{13}$$

$$\frac{\partial f_3}{\partial h^3_{22}} = \sum_{3\le j\le n} h^3_{jj} - 2h^3_{22}, \tag{14}$$

$$\frac{\partial f_3}{\partial h^3_{33}} = h^3_{11} + h^3_{22} + \sum_{4\le j\le n} h^3_{jj}, \tag{15}$$

$$\frac{\partial f_3}{\partial h^3_{44}} = h^3_{11} + h^3_{22} + \sum_{\substack{3\le j\le n\\ j\ne 4}} h^3_{jj} - 2h^3_{44}. \tag{16}$$

For a solution $(h^1_{11}, h^1_{22}, \ldots, h^1_{nn})$ of the problem in question, the vector grad f_3 is colinear to $(1, 1, \ldots, 1)$. By using (13)–(16) we obtain

$$h^3_{11} = h^3_{22} = 3a^3, \tag{17}$$

$$h^3_{33} = 12a^3, \tag{18}$$

$$h^3_{jj} = 4a^3, \qquad \forall j \in \overline{4, n}. \tag{19}$$

As $h^3_{11} + h^3_{22} + h^3_{33} + \cdots + h^3_{nn} = k^3$, from (17)–(19), one gets

$$a^3 = \frac{k^3}{4n+6}. \tag{20}$$

By an argument similar to that above, the point $(h^3_{11}, h^3_{22}, \ldots, h^3_{nn})$ given by (17)–(20) is a maximum point. Therefore

$$\begin{aligned} f_3 &\le 6a^3 12a^3 + 6a^3(n-3)4a^3 + 12b(n-3)4a^3 \\ &\quad + C^2_{n-3}16(a^3)^2 - 18(a^3)^2 - (n-3)16(a^3)^2 \\ &= 2(a^3)^2(2n-3)(2n+3). \end{aligned} \tag{21}$$

From (20) and (21) we obtain

$$f_3 \le \frac{(k^3)^2}{2}\,\frac{2n-3}{2n+3} = \frac{n^2}{2}\,\frac{2n-3}{2n+3}\,(H^3)^2.$$

In a similar manner, we prove that

$$f_r \le \frac{n^2}{2}\,\frac{2n-3}{2n+3}\,(H^r)^2, \qquad \forall r \in \overline{3, n}. \tag{22}$$

As $\frac{n-2}{n+1} < \frac{2n-3}{2n+3}$, from (11), (12) and (22) it follows that

$$f_r \le \frac{n^2}{2}\,\frac{2n-3}{2n+3}\,(H^r)^2, \qquad \forall r \in \overline{1, n}. \tag{23}$$

By using (4) and (23), we have

$$\delta_M \le \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \frac{n^2}{2}\,\frac{2n-3}{2n+3}\sum_{r=1}^{n}(H^r)^2 = \frac{(n-2)(n+1)}{2}\,\frac{c}{4} + \frac{n^2}{2}\,\frac{2n-3}{2n+3}\,\|H\|^2, \tag{24}$$

wich completes the proof.

Remarks. 1. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken showed in [5] that every Lagrangian submanifold, of real dimension $2n$, $n \ge 3$, of a complex space form $\widetilde{M}(c)$, satisfying the equality

$$\delta_M = \frac{n-2}{2}\left\{\frac{n^2}{n-1}\,\|H\|^2 + (n+1)\,\frac{c}{4}\right\},$$

is minimal. Now, this result is an immediate consequence of our Theorem 2.

2. If $n = 3$, in Theorem 2 equality occurs if and only if there is an orthonormal frame $\{e_1, e_2, e_3\}$ in T_xM in which the Weingarten operators take the following form:

$$A_{Je_1} = \begin{pmatrix} a & b & c \\ b & -a & 0 \\ c & 0 & 0 \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} b & -a & 0 \\ -a & -b & c \\ 0 & c & 0 \end{pmatrix}, \quad A_{Je_3} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 4c \end{pmatrix},$$

where a, b and c are real numbers.

3. In [1] J. Bolton and L. Vrancken showed how to construct all 3-dimensional non-minimal submanifolds in $\mathbb{CP}^3(4)$ attaining equality in Theorem 2 at all points. The classification in the minimal case can be found in [5].

Acknowledgments. I would like to thank Professor C. Udrişte, who has always been generous with his time and advice.

REFERENCES

[1] J. Bolton and L. Vrancken, *Lagrangian submanifolds attaining equality in the improved Chen's inequality*, arXiv:math.DG/0604543 v1 (2006).

[2] B. Y. Chen, *Some pinching classification theorems for minimal submanifolds*, Arch. Math. (Basel) 60 (1993), 568–578.

[3] —, *Some new obstructions to minimal Lagrangian isometric immersions*, Japan. J. Math. 26 (2000), 105–127.

[4] —, *Ideal Lagrangian immersions in complex space forms*, Math. Proc. Cambridge Philos. Soc. 128 (2000), 511–533.

[5] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of CP^n satisfying a basic equality*, Arch. Math. (Basel) 63 (1994), 553–564.

[6] T. Oprea, *Optimization methods on Riemannian submanifolds*, An. Univ. Bucureşti Mat. 54 (2005), 127–136.
[7] —, *On a Riemannian invariant of Chen type*, Rocky Mountain J. Math., to appear.
[8] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer, Dordrecht, 1994.
[9] C. Udrişte, O. Dogaru and I. Ţevy, *Extrema with Nonholonomic Constraints*, Geometry Balkan Press, Bucharest, 2002.

Faculty of Mathematics and Informatics
University of Bucharest
Str. Academiei 14
010014 Bucureşti, Romania
E-mail: teodoroprea@yahoo.com

Received 15 September 2005;
revised 15 October 2006 (4667)