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SPACES OF COMPACT OPERATORS ON $C(\mathbf{2^m} \times [0,\alpha])$ *SPACES*

BY

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Abstract. We classify, up to isomorphism, the spaces of compact operators $\mathcal{K}(E,F)$, where E and F are the Banach spaces of all continuous functions defined on the compact spaces $\mathbf{2^m} \times [0,\alpha]$, the topological products of Cantor cubes $\mathbf{2^m}$ and intervals of ordinal numbers $[0,\alpha]$.

1. Introduction. Let X be a Banach space and K a compact Hausdorff space. By $C(K,X)$ we denote the Banach space of all continuous X-valued functions defined on K and equipped with the supremum norm. This space will be denoted by $C(K)$ in the case $X = \mathbb{R}$. For a set Γ, $c_0(\Gamma)$ is the Banach space of all scalar-valued maps f on Γ with the property that for every $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f(\gamma)| \geq \varepsilon\}$ is finite, and equipped with the supremum norm. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when the cardinality of Γ is equal to τ. As usual, this space will be denoted by c_0 when $\tau = \aleph_0$. Given Banach spaces X and Y, $\mathcal{K}(X,Y)$ denotes the Banach space of compact operators from X to Y. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic. An ordinal α is said to be *regular* if its cofinality is α. Other notation and terminology may be found in [13] and [14].

This paper is a continuation of [10], where an isomorphic classification of the spaces of compact operators on $C(\mathbf{2^m} \oplus [0,\alpha])$ spaces was presented. Here $\mathbf{2^m} \oplus [0,\alpha]$ denotes the *topological sum* of the Cantor cube $\mathbf{2^m}$ and the interval of ordinals $[0,\alpha]$. The main result of [10] states that under certain conditions on the cardinals $\mathfrak{m}$ and $\mathfrak{n}$ and the ordinals λ and μ, for any uncountable ordinals ξ and η the following statements are equivalent:

(a) $\mathcal{K}(C(\mathbf{2^m} \oplus [0,\lambda]), C(\mathbf{2^n} \oplus [0,\xi])) \sim \mathcal{K}(C(\mathbf{2^m} \oplus [0,\mu]), C(\mathbf{2^n} \oplus [0,\eta]))$.
(b) either
- $C([0,\xi])$ is isomorphic to $C([0,\eta])$, or
- $C([0,\xi])$ is isomorphic to $C([0,\alpha p])$ and $C([0,\eta])$ is isomorphic to $C([0,\alpha q])$ for some regular ordinal α and finite ordinals $p \neq q$.

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This time we turn our attention to the isomorphic classification of the spaces of compact operators on $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces, where $\mathbf{2}^{\mathfrak{m}} \times [0,\alpha]$ denotes the *topological product* of the Cantor cube $\mathbf{2}^{\mathfrak{m}}$ and the interval of ordinals $[0,\alpha]$. Of course, this new situation is much more complicated than that consider in [10] due to the nature of topological products.

Surprisingly the main result of the present paper (Theorem 3.1) shows that the isomorphic classification of spaces of compact operators on $C(\mathbf{2}^{\mathfrak{m}} \times [\mathbf{0},\alpha])$ spaces is very similar to that obtained in [10]. Essentially we should replace the word "finite" by the word "countable" in the above mentioned result. However, the proofs are different. We will need for instance to state a new relation between non-sequential cardinals, non-real-valued measurable cardinals and a certain Banach space having the Mazur property (see Proposition 2.1).

2. Preliminary results. Before stating and proving our main theorem we need some auxiliary results. We recall that a cardinal $\mathfrak{m}$ is said to be *sequential* if there exists a sequentially continuous but not continuous real-valued function on $\mathbf{2}^{\mathfrak{m}}$. Recall that a function $f : \mathbf{2}^{\mathfrak{m}} \to \mathbb{R}$ is said to be *sequentially continuous* when $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to k in $\mathbf{2}^{\mathfrak{m}}$ (see [1] and [20]). The first sequential cardinal will be denoted by $\mathfrak{s}$. We also recall that an uncountable cardinal $\mathfrak{m}$ is *real-valued measurable* if there exists a non-trivial $\mathfrak{m}$-additive measure μ on $\mathfrak{m}$ [13, p. 300]. We will denote by $\mathfrak{m}_{\mathbb{R}}$ the least real-valued measurable cardinal.

We recall that a Banach space X is said to have the *Mazur property* (for short MP) if every element of X^{**} (the bidual space of X) which is weak* sequentially continuous is weak* continuous and thus is an element of X. Such spaces were investigated in [6], [18] and also in [15] and [28] where they were called d-complete and μB-spaces, respectively.

Let E be a Banach space and Γ a set. By $l_1(\Gamma, E)$ we will denote the Banach space of all families of elements $x = (x_\gamma)_{\gamma\in\Gamma}$ in E such that $\|x\| = \sum_{\gamma\in\Gamma} \|x_\gamma\| < \infty$.

The following result plays an important role in the proof of our main result.

PROPOSITION 2.1. *Suppose that $\mathfrak{m}$, $\mathfrak{n}$ and $\aleph_\gamma$ are cardinals satisfying $2^{\mathfrak{m}}\aleph_\gamma < \mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{n} < \mathfrak{s}$. If $\lambda < \omega$ or $\lambda, \mu \in [\omega_\gamma, \omega_{\gamma+1}[$, then $C(\mathbf{2}^{\mathfrak{n}}, X)$ has the MP, where*

$$X = l_1\Big([0,\lambda], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big).$$

Proof. Observe that $L_1[0,1]^{\mathfrak{m}}$ has the MP since it is weakly compactly generated [6, p. 564]. Moreover, since $\mathfrak{n} < \mathfrak{s}$, we know that $C(\mathbf{2}^{\mathfrak{n}})$ has the

MP [21] (see also [22, Proposition 5.2.c]). Therefore by [15, Corollary 5.2.2] the injective tensor product $L_1[0,1]^{\mathfrak{m}} \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}})$ has the MP. Next, denote by $|\lambda|$ the cardinality of λ and notice that by [27, Theorem 20.5.6] we have

$$C(\mathbf{2}^{\mathfrak{n}}, X) \sim l_1\Big([0,\lambda], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big) \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}}) \sim \Big(\sum_{2^{\mathfrak{m}} \times |\lambda|} L_1[0,1]^{\mathfrak{m}}\Big)_1 \hat{\hat{\otimes}} C(\mathbf{2}^{\mathfrak{n}}).$$

Since $2^{\mathfrak{m}}\ \aleph_\gamma < \mathfrak{m}_{\mathbb{R}}$, it follows from [15, Theorem 5.3] that $C(\mathbf{2}^{\mathfrak{n}}, X)$ has the MP. ∎

The following proposition follows directly from [9, Theorem 1.7].

Proposition 2.2. *Let X be a Banach space isomorphic to its c_0-sum such that X has the MP and contains no copy of $c_0(\aleph_1)$. Then for all uncountable ordinals ξ and η, the following are equivalent:*

(a) $C([0,\xi], X) \sim C([0,\eta], X)$.
(b) *either*
- $C([0,\xi]) \sim C([0,\eta])$, *or*
- $C([0,\xi]) \sim C([0,\alpha\xi'])$ *and* $C([0,\eta]) \sim C([0,\alpha\eta'])$ *for some regular ordinal α and countable ordinals ξ' and η'.*

We will also need the following recent result about the $c_0(\Gamma)$ subspaces of $C(K,X)$ spaces [11, Theorem 3.1].

Theorem 2.3. *Let K be a compact Hausdorff space, X a Banach space and $\tau > \aleph_0$. If $C(K,X)$ contains a copy of $c_0(\tau)$, then either $C(K)$ contains a copy of $c_0(\tau)$, or X contains a copy of c_0.*

Finally, recall that if X and Y are Banach spaces such that X has the approximation property, then the space $\mathcal{K}(X,Y)$ of compact operators is isomorphic to the injective tensor product $X^* \hat{\hat{\otimes}} Y$ [4, Proposition 5.3].

3. On spaces of compact operators on $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces. First of all notice that if $\mathfrak{m}$ is an infinite cardinal and α is a countable ordinal, then by the classical Milyutin theorem [27, Theorem 21.5.10] about the isomorphic classification of $C(K)$ spaces, with K an uncountable compact metric space, we have

$$C(\mathbf{2}^{\aleph_0} \times [0,\alpha]) \sim C(\mathbf{2}^{\aleph_0}).$$

Therefore

$$C(\mathbf{2}^{\mathfrak{m}}, C(\mathbf{2}^{\aleph_0} \times [0,\alpha])) \sim C(\mathbf{2}^{\mathfrak{m}}, C(\mathbf{2}^{\aleph_0})) \sim C(\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\aleph_0}) \sim C(\mathbf{2}^{\mathfrak{m}}).$$

Consequently,

$$C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}} \times \mathbf{2}^{\aleph_0} \times [0,\alpha]) \sim C(\mathbf{2}^{\mathfrak{m}}, C(\mathbf{2}^{\aleph_0} \times [0,\alpha])) \sim C(\mathbf{2}^{\mathfrak{m}}).$$

Thus, we will only consider the cases where α is an uncountable ordinal. Our main result is as follows.

THEOREM 3.1. *Suppose that $\mathfrak{m}$, $\mathfrak{n}$ and $\aleph_\gamma$ are cardinals satisfying $2^{\mathfrak{m}}\aleph_\gamma < \mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{n} < \mathfrak{s}$. Then for any ordinals ξ, η, λ and μ with $\xi \geq \omega_1$, $\eta \geq \omega_1$, $\lambda = \mu < \omega$ or $\lambda, \mu \in [\omega_\gamma, \omega_{\gamma+1}[$, the following assertions are equivalent:*

(a) $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi])) \sim \mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\mu]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta]))$.

(b) *either*

- $C([0,\xi])$ *is isomorphic to* $C([0,\eta])$, *or*
- $C([0,\xi])$ *is isomorphic to* $C([0,\alpha\xi'])$ *and* $C([0,\eta])$ *is isomorphic to* $C([0,\alpha\eta'])$ *for some regular ordinal* α *and countable ordinals* ξ' *and* η'.

Proof. First assume that (b) holds. Since $\mathfrak{n} < \mathfrak{s}$, $C(\mathbf{2}^{\mathfrak{n}})$ has the MP [21]. Furthermore, recall that a topological space K is said to satisfy the *countable chain condition* (ccc) if every uncountable family of open subsets of K contains two distinct sets with non-empty intersection. By [7, Theorem 2.3.17], $\mathbf{2}^{\mathfrak{n}}$ satisfies the ccc and thus according to [24, Theorem 4.5], $C(\mathbf{2}^{\mathfrak{n}})$ contains no copy of $c_0(\aleph_1)$. It is also clear that $C(\mathbf{2}^{\mathfrak{n}})$ is isomorphic to its c_0-sum. Then by Proposition 2.2,

$$C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi]) \sim C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta]). \tag{1}$$

On the other hand, by [23, Proposition 5.2] we know that

$$(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]))^* \sim l_1\Big([0,\lambda], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big).$$

As is well known [5, Example 11, p. 245], $L_1[0,1]^{\mathfrak{m}}$ has the approximation property. Hence by [3, Proposition 2.14], the spaces $(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]))^*$ also have the approximation property. Thus, as mentioned at the end of the preliminary results we can identify the spaces of compact operators which we are considering as an injective tensor product of Banach spaces as follows:

$$\begin{aligned}\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi])) \\ \sim l_1\Big([0,\lambda], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big) \hat{\hat{\otimes}}\, C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi]).\end{aligned} \tag{2}$$

Similarly we have

$$\begin{aligned}\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\mu]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta])) \\ \sim l_1\Big([0,\mu], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big) \hat{\hat{\otimes}}\, C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta]).\end{aligned} \tag{3}$$

Since $\lambda = \mu < \omega$ or $\lambda, \mu \in [\omega_\gamma, \omega_{\gamma+1}[$, it follows from (1)–(3) that (a) holds.

Suppose now that (a) holds. We denote

$$X = l_1\Big([0,\lambda], \Big(\sum_{2^{\mathfrak{m}}} L_1[0,1]^{\mathfrak{m}}\Big)_1\Big).$$

Then (2) can be rewritten in the form

$$\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi])) \sim C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi], X). \tag{4}$$

But by our hypotheses on λ and μ it follows that

$$l_1[0,\lambda] \sim l_1[0,\mu].$$

Thus similarly to (4) we infer that

$$\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta])) \sim C(\mathbf{2}^{\mathfrak{n}} \times [0,\eta], X). \tag{5}$$

Hence by (4), (5), [27, Theorem 20.5.6 and p. 358], it follows that

$$C([0,\xi], C(\mathbf{2}^{\mathfrak{n}}, X)) \sim C([0,\eta], C(\mathbf{2}^{\mathfrak{n}}, X)).$$

Since X contains no copy of c_0 and as we have already mentioned above, $C(\mathbf{2}^{\mathfrak{m}})$ contains no copy of $c_0(\aleph_1)$, it follows from Theorem 2.3 that $C(\mathbf{2}^{\mathfrak{n}}, X)$ contains no copy of $c_0(\aleph_1)$. Moreover, by Proposition 2.1, $C(\mathbf{2}^{\mathfrak{n}}, X)$ has the MP. Hence by Proposition 2.2, we conclude that (b) holds. ∎

Concerning Theorem 3.1, recall that a cardinal $\mathfrak{m}$ is a *two-valued measurable cardinal* if there is a non-trivial two-valued measure defined on all subsets of a set of cardinality $\mathfrak{m}$ for which points have measure zero [6, p. 560]. Let $\mathfrak{m}_2$ denote the least two-valued measurable cardinal. It is well-known that $\mathfrak{s} \le \mathfrak{m}_{\mathbb{R}}$; $\mathfrak{s} \le 2^{\aleph_0}$ or $\mathfrak{s} = \mathfrak{m}_2$; and $\mathfrak{s} = \mathfrak{m}_2$ under Martin's axiom [1], [8], [19]. Thus, it is relatively consistent with ZFC that there exists no sequential cardinal [22] and therefore no real-valued measurable cardinal as well.

Hence it is also consistent with ZFC that Theorem 3.1 furnishes a complete isomorphic classification of the spaces $\mathcal{K}(C(\mathbf{2}^{\mathfrak{m}} \times [0,\lambda]), C(\mathbf{2}^{\mathfrak{n}} \times [0,\xi]))$ with $\lambda < \omega$ or $\lambda \in [\omega_\gamma, \omega_{\gamma+1}[$ and $\xi \ge \omega_1$.

Finally, regarding statement (b) of Theorem 3.1 it is worth recalling that the isomorphic classification of $C([0,\xi])$ spaces was accomplished by Bessaga and Pełczyński [2] in the case where $\omega \le \xi < \omega_1$; by Semadeni [26] in the case where $\omega_1 < \xi \le \omega_1\omega$; by Labbé [17] in the case where $\omega_1\omega < \xi < \omega_1^\omega$; and independently by Kislyakov [16] and by Gul'ko and Os'kin [12] in the general case.

Problem 3.2. *Does the above isomorphic classification of the spaces of compact operators on $C(\mathbf{2}^{\mathfrak{m}} \times [0,\alpha])$ spaces remain true without the hypotheses on the cardinal numbers?*

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