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LOCAL EXISTENCE OF SOLUTIONS FOR AN AGGREGATION EQUATION IN BESOV SPACES

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Abstract. We prove the local in time existence of solutions for an aggregation equation in Besov spaces. The Fourier localization technique and Littlewood–Paley theory are the main tools used in the proof.

1. Introduction. In this paper, we consider the following aggregation equation

$$\begin{cases} u_t + \nabla\cdot(u(\nabla K * u)) = 0, \\ u(x,0) = u_0(x), \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

with a given kernel $K:\mathbb{R}^n\to\mathbb{R}$. The unknown function u is either the population density of a species or the density of particles in a granular medium. Aggregation equations of the form (1.1) arise in many problems in biology, chemistry and population dynamics and describe a collective motion and aggregation phenomena in biology and in mechanics of continuous media. From the mathematical point of view, equation (1.1) can be considered as a nonlinear, nonlocal transport equation, and its character depends strongly on the properties of the kernel K.

Laurent [14] has studied problem (1.1) in detail and proved several local and global existence results for a class of kernels K with different regularity. Bertozzi et al. [2–5] have proved finite-time blowup of solutions corresponding to compactly supported radial initial data. Those results can be summarized as follow. Kernels that are smooth (not singular) at the origin $x=0$ lead to the global in time existence of solutions (see e.g. [3, 14]). Nonsmooth kernels (and C^1 off the origin, like $K(x)=e^{-|x|}$) may lead to blowup of solutions either in finite or infinite time [2–4, 14–16].

Equation (1.1) has also been intensively considered in the viscous case, i.e. with the dissipative term $(-\Delta)^\gamma u$. The authors of [6, 7, 8, 15, 16] studied the problem (1.1) with fractional dissipation $(-\Delta)^{\gamma/2}u$, and proved finite blowup of solutions or global wellposedness for a certain class of kernels.

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Recently, Karch and Suzuki [13] have classified kernels which lead either to the blowup or global existence of solutions to (1.1) with the classical dissipation Δu.

The goal of this work is to generalize the recent result by Bertozzi et al. [5] who considered the local well-posedness of problem (1.1) in L^p spaces with kernels which are less singular at the origin than the Newtonian potential, i.e., in the form $K(x) = |x|^\alpha$, $\alpha > 2 - n$, and $K(x) = |x|$ at the origin. In addition, they also require the initial datum to have a finite second moment. The Besov framework adopted in this paper allows us to make two significant advances in the understanding of the aggregation equation. First, it allows us to consider potentials which are more general than those considered in previous papers, namely we require $\nabla K \in W^{1,1}(\mathbb{R}^n)$, which extends to the endpoint the case $\nabla K \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, considered in [5]. Second, Besov spaces are important spaces which contain L^p spaces, Sobolev spaces and Hölder spaces and are applied to many different models. In this paper, using the Fourier localization technique and Littlewood–Paley theory, we prove the local in time existence of solutions for the aggregation equation in Besov spaces. We follow the ideas introduced in [6, 9, 11, 17, 18]. Our main result reads as follows.

THEOREM 1.1. *Let $\nabla K \in W^{1,1}(\mathbb{R}^n)$, $1 < p < \infty$ and $s = 1 + n/p$. Assume that $u_0 \in B^s_{p,1}(\mathbb{R}^n)$. Then there exists $T = T(\|u_0\|_{B^s_{p,1}})$ such that the initial value problem* (1.1) *has a unique solution $u \in C([0,T]; B^s_{p,1}(\mathbb{R}^n)) \cap C^1([0,T]; B^{s-1}_{p,1}(\mathbb{R}^n))$.*

We recall the definition of the Besov space $B^s_{p,1}(\mathbb{R}^n)$ in the next section. Here, we only point out the embedding $B^s_{2,1} \hookrightarrow B^s_{2,2}$ where $B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ is the usual Sobolev space.

Following the reasoning from [5], one can directly complete the result stated in Theorem 1.1 by showing that if $u_0 \in B^s_{p,1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the corresponding solution u of (1.1) satisfies $u \in C([0,T]; B^s_{p,1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$. Moreover, if $u_0 \geq 0$ then $u(t,x) \geq 0$ almost everywhere. For proof that solutions to (1.1) may blowup in finite time, we refer the readers to [2–5, 7].

2. Preliminaries. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform is defined by $\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$. Now let us recall the Littlewood–Paley decomposition (see e.g. [1]). We choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, supported respectively in the ball $\{\xi \in \mathbb{R}^n : |\xi| \leq 4/3\}$ and in the shell $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ such that $\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n$, and $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n\backslash\{0\}$. For $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, the frequency localization operators are defined by

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y)f(x-y)\,dy,$$

$$S_j f = \sum_{-1\le k\le j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y)f(x-y)\,dy.$$

We now define Besov spaces by means of the Littlewood–Paley projections Δ_j and S_j:

Definition 2.1. For $s \in \mathbb{R}$, $1 \le p,q \le \infty$, the *inhomogeneous Besov space* $B^s_{p,q}(\mathbb{R}^n)$ is defined by

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \Big(\sum_{j=-1}^{\infty} 2^{jsq}\|\Delta_j f\|^q_{L^p}\Big)^{1/q} & \text{for } q<\infty, \\ \sup_{j\ge -1} 2^{js}\|\Delta_j f\|_{L^p} & \text{for } q=\infty. \end{cases}$$

The following lemmas will be used in the proof of the main result.

Lemma 2.1 ([12, Lemma A.2]). *Let u be a solution of the transport equation*

$$\begin{cases} u_t + v\cdot\nabla u = 0, \\ u(x,0) = u_0, \end{cases}$$

define $R_q := v\cdot\nabla\Delta_q u - \Delta_q(v\cdot\nabla u)$, and let $1\le p\le p_1\le\infty$, $1\le r\le\infty$ and $s\in\mathbb{R}$ be such that

$$s > -n\min\Big(\frac{1}{p_1},\frac{1}{p'}\Big) \quad \Big(\text{or } s > -1-n\min\Big(\frac{1}{p_1},\frac{1}{p'}\Big) \text{ if } \operatorname{div} v = 0\Big).$$

There exists a sequence $(c_q)\in\ell^r(\mathbb{Z})$ such that $\|(c_q)\|_{\ell^r}=1$ and a constant C depending only on n, r, s, p and p_1, which satisfy

$$\forall q\in\mathbb{Z},\qquad 2^{qs}\|R_q\|_{L^p} \le Cc_q Z'(t)\|u\|_{B^s_{p,r}}$$

with

$$Z'(t) := \begin{cases} \|\nabla v\|^{n/p_1}_{B_{p_1,\infty}\cap L^\infty} & \text{if } s<1+n/p_1, \\ \|\nabla v\|_{B^{s-1}_{p_1,r}} & \text{if either } s>1+n/p_1, \\ & \text{or } s=1+n/p_1 \text{ for } r=1. \end{cases} \tag{2.1}$$

Lemma 2.2 ([10, Lemma 2.2]). *Let $s>0$, $q\in[1,\infty]$. There exists a constant C such that*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot{B}^s_{r_2,q}}), \tag{2.2}$$

where $p_1, r_1 \in [1,\infty]$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$. An analogous inequality is valid when the homogeneous space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is replaced by its inhomogeneous counterpart.

Notice that under the assumptions either $s > n/p$ or $s \geq n/p$ for $q = 1$, the space $B^s_{p,q}(\mathbb{R}^n)$ is a Banach algebra.

3. Local existence of solutions. We now prove Theorem 1.1.

STEP 1. *A priori estimates.* We first derive estimates of solutions to equation (1.1), which we rewrite as follows:

$$\begin{cases} u_t + v \cdot \nabla u + u(\Delta K * u) = 0, \\ v = \nabla K * u, \\ u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+. \end{cases} \tag{3.1}$$

Applying the operation Δ_q with $q \geq -1$ on both sides of the first equation of (3.1), we have

$$\partial_t \Delta_q u + v \cdot \nabla \Delta_q u = R_q - f_q \tag{3.2}$$

with $R_q := v \cdot \nabla \Delta_q u - \Delta_q(v \cdot \nabla u)$ and $f_q = \Delta_q(u(\Delta K * u))$.

Multiplying equality (3.2) by $|\Delta_q u|^{p-2} \Delta_q u$ yields

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\Delta_q u|\, |\Delta_q u|^{p-2} \Delta_q u + \int_{\mathbb{R}^n} v \cdot \nabla \Delta_q u |\Delta_q u|^{p-2} \Delta_q u = \int_{\mathbb{R}^n} R_q |\Delta_q u|^{p-2} \Delta_q u - \int_{\mathbb{R}^n} R_q |\Delta_q u|^{p-2} \Delta_q u. \tag{3.3}$$

Integrating by parts, by the Hölder inequality, we have

$$\frac{d}{dt} \|\Delta_q u\|^p_{L^p} \leq C(\|R_q\|_{L^p} + \|f_q\|_{L^p} + \|\operatorname{div} v\|_{L^\infty} \|\Delta_q u\|_{L^p}) \|\Delta_q u\|^{p-1}_{L^p}. \tag{3.4}$$

Then we get

$$\frac{d}{dt} \|\Delta_q u\|_{L^p} \leq C(\|R_q\|_{L^p} + \|f_q\|_{L^p} + \|\operatorname{div} v\|_{L^\infty} \|\Delta_q u\|_{L^p}). \tag{3.5}$$

Multiplying both sides of the above inequality by 2^{qs} with $q \geq -1$ and computing the ℓ^1 norm, we obtain

$$\frac{d}{dt} \big\|2^{qs} \|\Delta_q u\|_{L^p}\big\|_{\ell^1} \leq C\big(\big\|2^{qs} \|R_q\|_{L^p}\big\|_{\ell^1} + \big\|2^{qs} \|f_q\|_{L^p}\big\|_{\ell^1} + \|\operatorname{div} v\|_{L^\infty} \big\|2^{qs} \|\Delta_q u\|_{L^p}\big\|_{\ell^1}\big). \tag{3.6}$$

By Lemma 2.1, we have

$$\frac{d}{dt} \|u\|_{B^s_{p,1}} \leq C(\|\nabla v\|_{B^{n/p_1}_{p_1,1}} \|u\|_{B^s_{p,1}} + \|\Delta K\|_{L^1} \|u^2\|_{B^s_{p,1}} + \|\nabla v\|_{B^{s-1}_{p,1}} \|u\|_{B^s_{p,1}}) \tag{3.7}$$

$$\le C(\|\nabla v\|_{B^{s-1}_{p,1}}\|u\|_{B^s_{p,1}} + \|u\|^2_{B^s_{p,1}})$$
$$\le C(\|v\|_{B^s_{p,1}}\|u\|_{B^s_{p,1}} + \|u\|^2_{B^s_{p,1}}).$$

Integrating (3.7) with respect to t, we get

$$\|u\|_{L^\infty_t(B^s_{p,1})} \le \|u_0\|_{B^s_{p,1}} + C\int_0^t \|v(\tau)\|_{B^s_{p,1}}\|u\|_{L^\infty_\tau(B^s_{p,1})}\,d\tau + C\int_0^t \|u(\tau)\|^2_{B^s_{p,1}}\,d\tau. \tag{3.8}$$

Let us show that this inequality leads to the estimate

$$\|u\|_{L^\infty_t(B^s_{p,1})} \le Ce^{C\int_0^t \|v(\tau)\|_{B^s_{p,1}}d\tau} \times \Big(\|u_0\|_{B^s_{p,1}} + \int_0^t e^{-C\int_0^t \|v(\tau')\|_{B^s_{p,1}}d\tau'}\|u(\tau)\|^2_{B^s_{p,1}}\,d\tau\Big). \tag{3.9}$$

Indeed, if we denote the right-hand side of inequality (3.8) by $F(t)$, we obtain

$$F'(t) \le \|u\|^2_{B^s_{p,1}} + \|v\|_{B^s_{p,1}}\|u\|_{B^s_{p,1}} \le \|u\|^2_{B^s_{p,1}} + C\|v\|_{B^s_{p,1}}F(t).$$

Thus, we have the inequality

$$\big(e^{-C\int_0^t \|v(\tau)\|_{B^s_{p,1}}d\tau}F\big)' \le C\|u\|^2_{B^s_{p,1}}e^{-C\int_0^t \|v(\tau)\|_{B^s_{p,1}}d\tau},$$

which implies (3.9). This completes the derivation of the a priori estimate for the solutions of equation (1.1).

Step 2. *Approximate solutions and uniform estimates.* In order to establish the local in time existence of solution we construct a sequence $\{u^{(m+1)}\}$, defined recursively by solving the linear equations

$$\begin{cases} \partial_t u^{(m+1)} + v^{(m)}\cdot\nabla u^{(m+1)} + u^{(m)}(\Delta K * u^{(m)}) = 0,\\ v^{(m)} = \nabla K * u^{(m)},\\ u(x,0) = S_{m+1}u_0(x), \end{cases} \tag{3.10}$$

where we set $u^{(0)} = 0$. The existence of solutions of the above system in $C([0,T];B^s_{p,1})$ is proved in [1, Ch. 3.2]. By the same procedure as in estimates leading to (3.9), we obtain

$$\|u^{(m+1)}(t)\|_{B^s_{p,1}} \le Ce^{CV^{(m)}(t)}\Big(\|u_0\|_{B^s_{p,1}} + \int_0^t e^{-CV^{(m)}(\tau)}\|u^{(m)}(\tau)\|^2_{B^s_{p,1}}\,d\tau\Big) \tag{3.11}$$

with $V^{(m)} = \int_0^t \|v^{(m)}(\tau)\|_{B^s_{p,1}}d\tau$.

Let us fix a $T>0$ such that $2C^2\|u_0\|_{B^s_{p,1}}T<1$ and suppose that

$$\|u^{(m)}(t)\|_{B^s_{p,1}} \le \frac{C\|u_0\|_{B^s_{p,1}}}{1-2C^2\|u_0\|_{B^s_{p,1}}t} \quad \text{for all } t\in[0,T]. \tag{3.12}$$

Plugging (3.12) into (3.11) yields

$$\begin{aligned}\|u^{(m+1)}(t)\|_{B^s_{p,1}} &\le \frac{1}{(1-2C^2\|u_0\|_{B^s_{p,1}}t)^{1/2}}\biggl(\|u_0\|_{B^s_{p,1}} \\ &\qquad + C^2\|u_0\|^2_{B^s_{p,1}}\int_0^t \frac{d\tau}{(1-2C^2\|u_0\|_{B^s_{p,1}}t)^{3/2}}\biggr) \\ &\le \frac{C\|u_0\|_{B^s_{p,1}}}{1-2C^2\|u_0\|_{B^s_{p,1}}t}.\end{aligned} \tag{3.13}$$

Therefore, $\{u^{(m)}\}_{m\in\mathbb{N}}$ is bounded in $L^\infty([0,T];B^s_{p,1})$. This clearly entails that $v^{(m)}\nabla u^{(m)}$ is bounded in $L^\infty([0,T];B^{s-1}_{p,1})$. As the third term of the first equation of (3.10) is bounded in $L^\infty([0,T];B^s_{p,1})$, we can conclude that the sequence $\{u^{(m)}\}_{m\in\mathbb{N}}$ is bounded in $C([0,T];B^s_{p,1})\cap C^1([0,T];B^{s-1}_{p,1})$.

Step 3. *Existence of solutions.* We will show that $\{u^{(m)}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,1})$. For all $(m,k)\in\mathbb{N}^2$, using (3.10), it is easy to verify that the difference $u^{(m+k+1)}-u^{(m+1)}$ satisfies

$$\begin{aligned}&\partial_t(u^{(m+k+1)}-u^{(m+1)})+v^{(m+k)}\cdot\nabla(u^{(m+k+1)}-u^{(m+1)}) \\ &\quad+(v^{(m+k)}-v^{(m)})\cdot\nabla u^{(m+1)}+u^{(m+k)}(\Delta K*(u^{(m+k)}-u^{(m)})) \\ &\quad+(u^{(m+k)}-u^{(m)})(\Delta K*u^{(m)})=0.\end{aligned} \tag{3.14}$$

Let $\omega_q=\Delta_q(u^{(m+k+1)}-u^{(m+1)})$, $U_q=\Delta_q((v^{(m+k)}-v^{(m)})\cdot\nabla u^{(m+1)})$, $V_q=\Delta_q(u^{(m+k)}(\Delta K*(u^{(m+k)}-u^{(m)})))$ and $W_q=\Delta_q((u^{(m+k)}-u^{(m)})(\Delta K*u^{(m)}))$. Applying the operation Δ_q on both sides of equation (3.14), we have

$$\partial_t\omega_q+v^{(m+k)}\cdot\nabla\omega_q=T_q-U_q-V_q-W_q \tag{3.15}$$

with $T_q:=v^{(m+k)}\cdot\nabla\omega_q-\Delta_q(v^{(m+k)}\cdot\nabla\omega_q)$. In the same way as in the proof of (3.9), we get

$$\begin{aligned}&\|(u^{(m+k+1)}-u^{(m+1)})(t)\|_{B^{s-1}_{p,1}} \\ &\quad\le Ce^{CV^{(m+k)}(t)}\biggl(\|u_0^{(m+k+1)}-u_0^{(m+1)}\|_{B^{s-1}_{p,1}} \\ &\qquad+\int_0^t e^{-CV^{(m+k)}(\tau)}(\|U_q\|_{B^{s-1}_{p,1}}+\|V_q\|_{B^{s-1}_{p,1}}+\|W_q\|_{B^{s-1}_{p,1}})\,d\tau\biggr).\end{aligned} \tag{3.16}$$

We now estimate the right-hand side terms of the above inequality. Since

$$u_0^{(m+k+1)} - u_0^{(m+1)} = \sum_{q=m+2}^{m+k+1} \Delta_q u_0,$$

we have

$$\|u_0^{(m+k+1)} - u_0^{(m+1)}\|_{B^{s-1}_{p,1}} \le C2^{-m} \|\nabla \Delta_q u_0\|_{B^{s-1}_{p,1}} \le C2^{-m}\|u_0\|_{B^s_{p,1}}. \tag{3.17}$$

Using the fact that $B^{s-1}_{p,1}$ is a Banach algebra, we get

$$\begin{aligned} \|U_q\|_{B^{s-1}_{p,1}} &\le C\|v^{(m+k)} - v^{(m)}\|_{B^{s-1}_{p,1}} \|\nabla u^{(m+1)}\|_{B^{s-1}_{p,1}} \\ &\le C\|\nabla K\|_{L^1} \|u^{(m+1)}\|_{B^s_{p,1}} \|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}} \\ &\le C\|u^{(m+1)}\|_{B^s_{p,1}} \|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}}. \end{aligned} \tag{3.18}$$

Similarly, we obtain

$$\begin{aligned} \|V_q\|_{B^{s-1}_{p,1}} &\le C\|u^{(m+k)}\|_{B^{s-1}_{p,1}} \|\Delta K * (u^{(m+k)} - u^{(m)})\|_{B^{s-1}_{p,1}} \\ &\le C\|u^{(m+k)}\|_{B^{s-1}_{p,1}} \|\Delta K\|_{L^1} \|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}} \\ &\le C\|u^{(m+k)}\|_{B^s_{p,1}} \|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}}, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \|W_q\|_{B^{s-1}_{p,1}} &\le C\|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}} \|\Delta K * u^{(m)}\|_{B^{s-1}_{p,1}} \\ &\le C\|u^{(m)}\|_{B^s_{p,1}} \|u^{(m+k)} - u^{(m)}\|_{B^{s-1}_{p,1}}. \end{aligned} \tag{3.20}$$

Plugging (3.17)–(3.20) into (3.16) and using the uniform estimates (3.13), we finally get a constant C_T independent of m, k and such that for all $t \in [0,T]$,

$$\begin{aligned} &\|(u^{(m+k+1)} - u^{(m+1)})(t)\|_{B^{s-1}_{p,1}} \\ &\qquad \le C_T\Big(2^{-m} + \int_0^t \|(u^{(m+k)} - u^{(m)})(\tau)\|_{B^{s-1}_{p,1}}\, d\tau\Big). \end{aligned} \tag{3.21}$$

Proceeding by induction, one can easily prove that

$$\begin{aligned} &\|u^{(m+k+1)} - u^{(m+1)}\|_{L^\infty([0,T];B^{s-1}_{p,1})} \\ &\qquad \le \frac{(TC_T)^{m+1}}{(m+1)!} \|u^{(k)}\|_{L^\infty([0,T];B^s_{p,1})} + C_T \sum_{l=0}^{m} 2^{-(m-l)} \frac{(TC_T)^l}{l!}. \end{aligned} \tag{3.22}$$

As $\|u^{(k)}\|_{L^\infty([0,T];B^s_{p,1})}$ can be bounded independently of k, we conclude that there exists a new constant C'_T such that

$$\|u^{(m+k+1)} - u^{(m+1)}\|_{L^\infty([0,T];B^{s-1}_{p,1})} \le C'_T 2^{-m}. \tag{3.23}$$

Consequently, the sequence $\{u^{(m)}\}$ converges to a function $u \in C([0,T]; B^{s-1}_{p,1})$.

Step 4. *Passage to the limit.* Now using the definition of weak solutions to problem (1.1), we find that the limit $u \in C([0,T]; B^{s-1}_{p,1})$ is a solution of (1.1) with the initial datum $u_0 \in B^s_{p,1}$. Indeed, for every test function $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$, we have

$$\int_0^t \int_{\mathbb{R}^n} u^{(m+1)} \varphi_t \, dx \, dt - \int_{\mathbb{R}^n} u^{(m+1)}(x,0)\varphi(x,0)\, dx$$
$$= -\int_0^t \int_{\mathbb{R}^n} v^{(m)} \cdot \nabla u^{(m+1)} \varphi \, dx \, dt - \int_0^t \int_{\mathbb{R}^n} u^{(m)} (\Delta K * u^{(m)}) \varphi \, dx \, dt.$$

The passage to the limit in the linear terms on the left-hand side is completely standard. To treat the right-hand side, we have to use the Sobolev embedding $B^{s-1}_{p,1} \hookrightarrow L^p$ for every $p \in [1,\infty]$ and the estimates (3.13) of the sequence $\{u^{(m)}\}$ in the following way:

$$\Big| \int_0^t \int_{\mathbb{R}^n} v^{(m)} \cdot \nabla u^{(m+1)} \varphi \, dx \, dt - \int_0^t \int_{\mathbb{R}^n} v \cdot \nabla u \varphi \, dx \, dt \Big|$$
$$\le C\Big(\int_0^t (\|\operatorname{div} v^{(m)}\|_{L^\infty} \|u^{(m+1)} - u\|_{L^p} \|\varphi\|_{L^{p'}}$$
$$+ \|v^{(m)}\|_{L^\infty} \|u^{(m+1)} - u\|_{L^p} \|\nabla\varphi\|_{L^{p'}})\, dt$$
$$+ \int_0^t \|\nabla u\|_{L^\infty} \|v^{(m)} - v\|_{L^p} \|\varphi\|_{L^{p'}} \, dt\Big)$$
$$\le C\Big(\|\Delta K\|_{L^1} \int_0^t \|u^{(m)}\|_{L^\infty} \|u^{(m+1)} - u\|_{B^{s-1}_{p,1}} \|\varphi\|_{L^{p'}} \, dt$$
$$+ \|\nabla K\|_{L^1} \int_0^t \|u^{(m)}\|_{L^\infty} \|u^{(m+1)} - u\|_{B^{s-1}_{p,1}} \|\nabla\varphi\|_{L^{p'}} \, dt$$
$$+ \|\nabla K\|_{L^1} \int_0^t \|\nabla u\|_{L^\infty} \|u^{(m)} - u\|_{B^{s-1}_{p,1}} \|\varphi\|_{L^{p'}} \, dt\Big)$$
$$\le C \int_0^t (\|u^{(m+1)} - u\|_{B^{s-1}_{p,1}} + \|u^{(m)} - u\|_{B^{s-1}_{p,1}})\, dt,$$

and, similarly,

$$\Big| \int_0^t \int_{\mathbb{R}^n} u^{(m)} (\Delta K * u^{(m)}) \varphi \, dx \, dt - \int_0^t \int_{\mathbb{R}^n} u (\Delta K * u) \varphi \, dx \, dt \Big|$$

$$\leq C\Big(\int_0^t \|u^{(m)} - u\|_{L^p}\|\Delta K * u\|_{L^\infty}\|\varphi\|_{L^{p'}}\,dt$$
$$+\int_0^t \|u^{(m)}\|_{L^\infty}\|\Delta K * (u^{(m)} - u)\|_{L^p}\|\varphi\|_{L^{p'}}\,dt\Big)$$
$$\leq C\int_0^t \|u^{(m)} - u\|_{B^{s-1}_{p,1}}\,dt.$$

STEP 5. *Uniqueness.* Consider two solutions $u_1, u_2 \in C([0,T]; B^s_{p,1})$ with the same initial data. Let $\omega = u_1 - u_2$. Then ω satisfies the equation

$$\partial_t\omega + v_1 \cdot \nabla\omega + (v_1 - v_2)\cdot\nabla u_2 + u_1(\Delta K * \omega) + \omega(\Delta K * u_2) = 0. \tag{3.24}$$

In the same way as in deriving (3.21), we obtain the estimate

$$\|\omega\|_{C([0,T];B^{s-1}_{p,1})} \leq C_2 T\|\omega\|_{C([0,T];B^{s-1}_{p,1})}. \tag{3.25}$$

Thus, for sufficiently small T, we have $\omega \equiv 0$, i.e., $u_1 = u_2$. This completes the proof of Theorem 1.1.

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