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CONSTRUCTING UNIVERSALLY SMALL SUBSETS OF A GIVEN PACKING INDEX IN POLISH GROUPS

BY

TARAS BANAKH (Kielce and Lviv) and NADYA LYASKOVSKA (Lviv)

Abstract. A subset of a Polish space X is called *universally small* if it belongs to each ccc σ-ideal with Borel base on X. Under CH in each uncountable Abelian Polish group G we construct a universally small subset $A_0 \subset G$ such that $|A_0 \cap gA_0| = \mathfrak{c}$ for each $g \in G$. For each cardinal number $\kappa \in [5, \mathfrak{c}^+]$ the set A_0 contains a universally small subset A of G with sharp packing index $\mathrm{pack}^\sharp(A_\kappa) = \sup\{|\mathcal{D}|^+ : \mathcal{D} \subset \{gA\}_{g\in G}$ is disjoint$\}$ equal to κ.

1. Introduction. This paper is motivated by a problem of Dikranjan and Protasov [4] who asked if the group $\mathbb{Z}$ of integers contains a subset A such that the family of shifts $\{x + A\}_{x\in\mathbb{Z}}$ contains a disjoint subfamily of arbitrarily large finite cardinality but does not contain an infinite disjoint subfamily. This problem can be reformulated in the language of packing indices $\mathrm{pack}(A)$ and $\mathrm{pack}^\sharp(A)$, defined for any subset A of a group G by the formulas

$$\mathrm{pack}(A) = \sup\{|\mathcal{D}| : \mathcal{D} \subset \{gA\}_{g\in G} \text{ is a disjoint subfamily}\},$$
$$\mathrm{pack}^\sharp(A) = \sup\{|\mathcal{D}|^+ : \mathcal{D} \subset \{gA\}_{g\in G} \text{ is a disjoint subfamily}\}.$$

So, actually Dikranjan and Protasov asked about the existence of a subset $A \subset \mathbb{Z}$ with $\mathrm{pack}^\sharp(A) = \aleph_0$. This problem was answered affirmatively in [1] and [2]. Moreover, in [7] the second author proved that for any cardinal κ with $2 \le \kappa \le |G|^+$ and $\kappa \notin \{3, 4\}$, in any Abelian group G there is a subset $A \subset G$ with $\mathrm{pack}^\sharp(A) = \kappa$. By Theorem 6.3 of [3], such a set A can be found in any subset $A_0 \subset G$ with $\mathrm{Pack}(A_0) = 1$ where

$$\mathrm{Pack}(A_0) = \sup\{|\mathcal{A}| : \mathcal{A} \subset \{gA_0\}_{g\in G} \text{ is } |G|\text{-almost disjoint}\}.$$

A family $\mathcal{A}$ of subsets of G is called $|G|$-*almost disjoint* if $|A \cap A'| < |G| = |A|$ for any distinct $A, A' \in \mathcal{A}$.

A subset $A \subset G$ with small packing index can be thought of as large in a geometric sense because in this case the group G does not contain

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many disjoint translation copies of A. It is natural to compare this largeness with other largeness properties that have topological or measure-theoretic nature. It turns out that a subset of a group can have small packing index (so can be large in the geometric sense) and simultaneously be small in other senses. In [3] it was proved that each uncountable Polish Abelian group G contains a closed subset $A \subset G$ with $\mathrm{Pack}(A) = 1$ which is nowhere dense and Haar null in G. According to Theorem 16.3 of [9], under CH (the Continuum Hypothesis), each Polish group G contains a subset A with $\mathrm{pack}(A) = 1$, which is *universally null* in the sense that A has measure zero with respect to any atomless Borel probability measure on G.

In this paper we move further in this direction and prove that under CH each uncountable Abelian Polish group G contains a subset $A \subset G$ with $\mathrm{Pack}(A) = 1$ which is *universally small* in the sense that it belongs to any ccc σ-ideal with Borel base on G. This fact, combined with Theorem 6.3 of [3], allows us to construct universally small subsets of a given packing index in uncountable Polish Abelian groups.

Following Zakrzewski [11] we call a subset A of a Polish space X *universally small* if A belongs to each ccc σ-ideal with Borel base on X. By an *ideal* on a set X we understand a family $\mathcal{I}$ of subsets of X such that

- $\bigcup\mathcal{I} = X \notin \mathcal{I}$;
- $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$;
- $A \cap B \in \mathcal{I}$ for any $A \in \mathcal{I}$ and $B \subset X$.

An ideal $\mathcal{I}$ on a Polish space X is called

- a *σ-ideal* if $\bigcup\mathcal{A} \in \mathcal{I}$ for any countable subfamily $\mathcal{A} \subset \mathcal{I}$;
- an *ideal with Borel base* if each set $A \in \mathcal{I}$ is contained in a Borel set $B \in \mathcal{I}$;
- a *ccc ideal* if X contains no uncountable disjoint family of Borel subsets outside $\mathcal{I}$.

Standard examples of ccc Borel σ-ideals are the ideal $\mathcal{M}$ of meager subsets of a Polish space X and the ideal $\mathcal{N}$ of null subsets with respect to an atomless Borel σ-additive measure on X. This implies that a universally small subset A is universally null and universally meager. Following [10] we call a subset A of a Polish space X *universally meager* if for any Borel isomorphism $f : A \to 2^\omega$ the image $f(A)$ is meager in the Cantor cube 2^ω. Universally small sets were introduced by P. Zakrzewski [11] who constructed an uncountable universally small subset in each uncountable Polish space. It should be mentioned that there are models of ZFC [8, §5] in which all universally small sets in Polish spaces have cardinality $\leq \aleph_1 < \mathfrak{c}$. In such models any universally small set A in the real line has maximal possible packing index $\mathrm{pack}(A) = \mathrm{Pack}(A) = \mathfrak{c}$. This fact shows that the following theorem, which is the main result of this paper, necessarily has consistency nature and cannot be proved in ZFC.

THEOREM 1. *Under CH, each uncountable Abelian Polish group G contains a universally small subset $A_0 \subset G$ with* $\mathrm{Pack}(A_0) = 1$.

Combining this theorem with Theorem 6.4 of [3] we get

COROLLARY 1. *Under CH, for any cardinal $\kappa \in [2, \mathfrak{c}^+]$ with $\kappa \notin \{3, 4\}$ any uncountable Polish Abelian group G contains a universally small subset A with* $\mathrm{pack}^\sharp(A) = \kappa$.

2. Universally small sets from coanalytic ranks. In this section we describe a (known) method of constructing universally small sets, based on coanalytic ranks. Let us recall that a subset A of a Polish space X is

- *analytic* if A is a continuous image of a Polish space;
- *coanalytic* if $X \setminus A$ is analytic.

By Suslin's Theorem [5, 14.11], a subset of a Polish space is Borel if and only if it is analytic and coanalytic.

It is known [5, 34.4] that each coanalytic subset K of a Polish space X admits a *rank function* $\mathrm{rank} : K \to \omega_1$ that has the following properties:

(1) for every countable ordinal α the set $B_\alpha = \{x \in K : \mathrm{rank}(x) \le \alpha\}$ is Borel in X;
(2) each analytic subspace $A \subset K$ lies in some set B_α, $\alpha < \omega_1$.

The following fact is known and belongs to mathematical folklore (cf. [8, 5.3]). For the convenience of the reader we supply a short proof.

LEMMA 1. *Let K be a coanalytic non-analytic set in a Polish space X,* $\mathrm{rank} : K \to \omega_1$ *be a rank function, and $B_\alpha = \{x \in K : \mathrm{rank}(x) \le \alpha\}$ for $\alpha < \omega_1$. For any transfinite sequence of points $x_\alpha \in K \setminus B_\alpha$, $\alpha \in \omega_1$, the set $\{x_\alpha\}_{\alpha \in \omega_1}$ is universally small in X.*

Proof. Given any ccc Borel σ-ideal $\mathcal{I}$ on X, use the classical Szpilrajn-Marczewski Theorem [6, §11] to conclude that the coanalytic set K belongs to the completion $\mathcal{B}_{\mathcal{I}}(X) = \{A \subset X : \exists B \in \mathcal{B}(X)\ A \triangle B \in \mathcal{I}\}$ of the σ-algebra of Borel subsets of X by the ideal $\mathcal{I}$. Consequently, there is a Borel subset $B \subset K$ of X such that $K \setminus B \in \mathcal{I}$. By the property of the rank function, the Borel set B lies in B_β for some countable ordinal β. Then the set $\{x_\alpha\}_{\alpha < \omega_1}$ belongs to the σ-ideal $\mathcal{I}$, being the union of the countable set $\{x_\alpha\}_{\alpha \le \beta}$ and the set $\{x_\alpha\}_{\beta < \alpha < \omega_1} \subset K \setminus B_\alpha \subset K \setminus B$ from $\mathcal{I}$. ∎

In order to prove Theorem 1 we shall combine Lemma 1 with the following technical lemma that will be proved in Section 4.

LEMMA 2. *For any uncountable Polish Abelian group G there are a non-empty open set $U \subset G$ and a coanalytic subset K of G such that $U \subset (K \setminus A) - (K \setminus A)$ for any analytic subspace $A \subset K$ of G.*

3. Proof of Theorem 1. Assume the Continuum Hypothesis. Given an uncountable Polish Abelian group G we need to construct a universally small subset $A \subset G$ with $\mathrm{Pack}(A) = 1$. We shall use the additive notation for the group operation on G. So, 0 will denote the neutral element of G. For two subsets $A, B \subset G$ we put $A + B = \{a + b : a \in A,\ b \in B\}$ and $A - B = \{a - b : a \in A,\ b \in B\}$.

By Lemma 2, there are a non-empty open set $U \subset G$ and a coanalytic subset K such that $U \subset (K \setminus B) - (K \setminus B)$ for any Borel subset $B \subset K$ of G. This implies that the coanalytic set K is not Borel in G. Let $\mathrm{rank} : K \to \omega_1$ be a rank function for K. This function induces the decomposition $K = \bigcup_{\alpha<\omega_1} B_\alpha$ into Borel sets $B_\alpha = \{x \in K : \mathrm{rank}(x) \le \alpha\}$, $\alpha < \omega_1$, such that each Borel subset $B \subset K$ of G lies in some set B_α, $\alpha < \omega_1$.

The Continuum Hypothesis allows us to choose an enumeration $U = \{u_\alpha\}_{\alpha<\omega_1}$ of the open set U such that for every $u \in U$ the set $\Omega_u = \{\alpha < \omega_1 : u_\alpha = u\}$ is uncountable. The separability of G yields a countable subset $C \subset G$ such that $G = C + U$.

By induction, for every $\alpha < \omega_1$ find two points $x_\alpha, y_\alpha \in K \setminus (B_\alpha \cup \{x_\beta : \beta < \alpha\})$ such that $x_\alpha - y_\alpha = u_\alpha$. Such a choice is always possible as $U \subset (K \setminus B) - (K \setminus B)$ for any Borel subset $B \subset K$ of G. Lemma 1 guarantees that the sets $\{x_\alpha\}_{\alpha<\omega_1}$ and $\{y_\alpha\}_{\alpha<\omega_1}$ are universally small in G and so is the set $A = \{c + x_\alpha, y_\alpha : c \in C,\ \alpha < \omega_1\}$. It remains to prove that $\mathrm{Pack}(A) = 1$. This will follow as soon as we check that $A \cap (z + A)$ has cardinality of continuum for every $z \in G$. Since $C + U = G$, we can find $c \in C$ and $u \in U$ such that $z = c + u$. The choice of the enumeration $\{u_\alpha\}_{\alpha<\omega_1}$ guarantees that the set $\Omega_u = \{\alpha < \omega_1 : u_\alpha = u\}$ has cardinality continuum. Now observe that for every $\alpha \in \Omega_u$ we get $z = c + u = c + u_\alpha = c + x_\alpha - y_\alpha$ and hence $c + x_\alpha = z + y_\alpha \in A \cap (z + A)$, which implies that $A \cap (z + A) \supset \{c + x_\alpha\}_{\alpha \in \Omega_u}$ has cardinality continuum.

4. Proof of Lemma 2. Fix an invariant metric $d \le 1$ generating the topology of G. This metric is complete because the group G is Polish. The metric d induces a norm $\|\cdot\| : G \to [0, 1]$ on G defined by $\|x\| = d(x, 0)$. For an $\varepsilon > 0$ we denote by $B(\varepsilon) = \{x \in G : \|x\| < \varepsilon\}$ and $\bar{B}(\varepsilon) = \{x \in G : \|x\| \le \varepsilon\}$ the open and closed ε-balls centered at zero.

We define a subset D of G to be *ε-separated* if $d(x, y) \geq \varepsilon$ for any distinct $x, y \in D$. By Zorn's Lemma, each ε-separated subset S of any subset $A \subset G$ can be enlarged to a maximal ε-separated subset $\tilde{S}$ of A. This set $\tilde{S}$ is an *ε-net* for A in the sense that for each $a \in A$ there is an $s \in \tilde{S}$ with $d(a, s) < \varepsilon$.

Fix any non-zero element $a_{-1} \in G$ and let $\varepsilon_{-1} = \frac{1}{12}\|a_{-1}\|$. By induction we can define a sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers and a sequence $(a_n)_{n\in\omega}$ of points of G such that

- $16\varepsilon_n \leq \|a_n\| < \varepsilon_{n-1}$ for every $n \in \omega$.

For every $n \in \omega$, fix a maximal $2\varepsilon_n$-separated subset $X_n \ni 0$ in $B(2\varepsilon_{n-1})$.

The choice of (ε_n) guarantees that the series $\sum_{n\in\omega} \varepsilon_n$ is convergent and thus for any $(x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n$ the series $\sum_{n\in\omega} x_n$ is convergent in G (because $\|x_n\| < 2\varepsilon_{n-1}$ for all $n \in \mathbb{N}$). Therefore the following subsets of G are well-defined:

$$\Sigma_0 = \Big\{ \sum_{n\in\omega} x_{2n} : (x_{2n})_{n\in\omega} \in \prod_{n\in\omega} X_{2n} \Big\},$$
$$\Sigma_1 = \Big\{ \sum_{n\in\omega} x_{2n+1} : (x_{2n+1})_{n\in\omega} \in \prod_{n\in\omega} X_{2n+1} \Big\}.$$

These sets have the following properties:

Claim 1.

(1) $\Sigma_0 \cup \Sigma_1 \subset B(4\varepsilon_{-1})$.
(2) $B(2\varepsilon_{-1}) \subset \Sigma_1 + \Sigma_0$.
(3) *For every $i \in \{0, 1\}$ the closure $\overline{\Sigma_i - \Sigma_i}$ of $\Sigma_i - \Sigma_i$ in G is not a neighborhood of zero.*

Proof. (1) For every $x \in \Sigma_0 \cup \Sigma_1$ we can find $(x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n$ with $x = \sum_{n=0}^{\infty} x_n$ and observe that

$$\|x\| \leq \sum_{n=0}^{\infty} \|x_n\| \leq \sum_{n=0}^{\infty} 2\varepsilon_{n-1} < \sum_{n\in\omega} \frac{2\varepsilon_{-1}}{16^n} < 4\varepsilon_{-1}.$$

(2) Given any $x \in B(2\varepsilon_{-1})$, find $x_0 \in X_0$ such that $\|x - x_0\| < 2\varepsilon_0$ (use the fact that X_0 is a $2\varepsilon_0$-net in $B(2\varepsilon_{-1})$). Continuing by induction, for every $n \in \omega$ find $x_n \in X_n$ such that $\|x - \sum_{i=0}^{n} x_i\| < 2\varepsilon_n$. After completing the inductive construction, we obtain a sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n$ such that

$$x = \sum_{n\in\omega} x_n = \sum_{n\in\omega} x_{2n} + \sum_{n\in\omega} x_{2n+1} \in \Sigma_0 + \Sigma_1.$$

(3) We shall give a detailed proof of the third statement for $i = 0$ (for $i = 1$ the proof is analogous). Since the sequence $(a_{2k+1})_{k\in\omega}$ converges to zero, it suffices to show that $d(a_{2k+1}, \Sigma_0 - \Sigma_0) > 0$ for all $k \in \omega$.

Given $x, y \in \Sigma_0$, we shall prove that $d(a_{2k+1}, x - y) \geq \varepsilon_{2k+1}$. If $x = y$, then $d(a_{2k+1}, x - y) = d(a_{2k+1}, 0) = \|a_{2k+1}\| > \varepsilon_{2k+1}$ by the choice of a_{2k+1}.

So, assume that $x \neq y$ and find infinite sequences $(x_{2n})_{n\in\omega}, (y_{2n})_{n\in\omega} \in \prod_{n\in\omega} X_{2n}$ with $x = \sum_{n\in\omega} x_{2n}$ and $y = \sum_{n\in\omega} y_{2n}$.

Let $m = \min\{n \in \omega : x_{2n} \neq y_{2n}\}$. If $m \geq k+1$, then

$$\begin{aligned}\|x-y\| &= \Big\|\sum_{n\geq m} x_{2n} - y_{2n}\Big\| \leq \sum_{n\geq m} (\|x_{2n}\| + \|y_{2n}\|) \\ &\leq 2\sum_{n\geq m} 2\varepsilon_{2n-1} \leq 8\,\varepsilon_{2m-1} \leq 8\,\varepsilon_{2k+1} < \|a_{2k+1}\| - \varepsilon_{2k+1}\end{aligned}$$

and hence $d(x-y, a_{2k+1}) \geq \varepsilon_{2k+1}$.

If $m \leq k$, then

$$\begin{aligned}\|x-y\| &= \Big\|(x_{2m} - y_{2m}) + \sum_{n>m} (x_{2n} - y_{2n})\Big\| \\ &\geq \|x_{2m} - y_{2m}\| - \sum_{n>m} (\|x_{2n}\| + \|y_{2n}\|) \\ &\geq 2\varepsilon_{2m} - 2\sum_{n>m} 2\varepsilon_{2n-1} \geq 2\varepsilon_{2m} - 8\varepsilon_{2m+1} \\ &\geq \frac{3}{2}\varepsilon_{2m} \geq \frac{3}{2}\varepsilon_{2k} > \|a_{2k+1}\| + \frac{1}{2}\varepsilon_{2k}\end{aligned}$$

according to the choice of the point a_{2k+1}. Consequently,

$$d(x-y, a_{2k+1}) \geq \frac{1}{2}\varepsilon_{2k} \geq \varepsilon_{2k+1}. \ \blacksquare$$

A subset C of G will be called a *Cantor set* in G if C is homeomorphic to the Cantor cube $\{0,1\}^\omega$. By the classical Brouwer Theorem [5, 7.4], this happens if and only if C is compact, zero-dimensional and has no isolated points.

Claim 2. *For every $i \in \{0,1\}$ there is a Cantor set $C_i \subset B(\varepsilon_0)$ such that the map $h_i : C_i \times \overline{\Sigma}_i \to G$, $(x,y) \mapsto x+y$, is a closed topological embedding.*

Proof. Taking into account that $\overline{\overline{\Sigma}_i - \overline{\Sigma}_i} = \overline{\Sigma_i - \Sigma_i}$ is not a neighborhood of zero in G, and repeating the proof of Lemma 2.1 of [3], we can construct a Cantor set $C_i \subset B(\varepsilon_0)$ such that for any distinct points $x, y \in C_i$ the shifts $x + \overline{\Sigma}_i$ and $y + \overline{\Sigma}_i$ are disjoint. This implies that the map $h_i : C_i \times \overline{\Sigma}_i \to G$, $(x,y) \mapsto x+y$, is injective. Since C_i is compact and $\overline{\Sigma}_i$ is closed in G, the map h_i is closed and hence a closed topological embedding. $\blacksquare$

Observe that for every $i \in \{0,1\}$, $h_i(C_i \times \overline{\Sigma}_i) = C_i + \overline{\Sigma}_i \subset B(\varepsilon_0) + \bar{B}(4\varepsilon_{-1}) \subset B(5\varepsilon_{-1})$. Now we modify the closed embeddings h_0 and h_1 to closed embeddings

$$\tilde{h}_0 : C_0 \times \overline{\Sigma}_0 \to G, \quad (x,y) \mapsto a_{-1} + x + y,$$

and

$$\tilde{h}_1 : C_1 \times \overline{\Sigma}_1 \to G, \qquad (x, y) \mapsto -x - y.$$

These have images $\tilde{h}_0(C_0 \times \overline{\Sigma}_0) \subset a_{-1} + B(5\varepsilon_{-1})$ and $\tilde{h}_1(C_1 \times \overline{\Sigma}_1) \subset -B(5\varepsilon_{-1}) = B(5\varepsilon_{-1})$. Since $\|a_{-1}\| = 12\varepsilon_{-1}$, we conclude that the closed subsets $\tilde{h}_i(C_i \times \overline{\Sigma}_i)$, $i \in \{0,1\}$, of G are disjoint.

For every $i \in \{0,1\}$ fix a coanalytic non-analytic subset K_i in the Cantor set C_i. It follows that the disjoint union $K = \tilde{h}_0(K_0 + \overline{\Sigma}_0) \cup \tilde{h}_1(K_1 + \overline{\Sigma}_1)$ is a coanalytic subset of G.

The following claim completes the proof of the lemma and shows that the coanalytic set K and the open set $U = a_{-1} + B(\varepsilon_{-1})$ have the required property.

CLAIM 3. *$U \subset (K \setminus A) - (K \setminus A)$ for any analytic subspace $A \subset K$.*

Proof. Given an analytic subspace $A \subset K$, for every $i \in \{0,1\}$, consider its preimage $A_i = \tilde{h}_1^{-1}(A) \subset C_i \times \overline{\Sigma}_i$ and its projection $\mathrm{pr}_i(A_i)$ onto C_i. It follows from $A \subset K$ and $\tilde{h}_0(C_0 \times \overline{\Sigma}_0) \cap \tilde{h}_1(C_1 \times \overline{\Sigma}_1) = \emptyset$ that each A_i is an analytic subspace of the coanalytic set K_i. Since the space K_i is not analytic, there is a point $c_i \in K_i \setminus \mathrm{pr}_i(A_i)$. It follows that

$$\tilde{h}_0(\{c_0\} \times \Sigma_0) \cup \tilde{h}_1(\{c_1\} \times \Sigma_1) = (a_{-1} + c_0 + \Sigma_0) \cup (-c_1 - \Sigma_1) \subset K \setminus A$$

and hence

$$\begin{aligned}(K \setminus A) - (K \setminus A) &\supset a_{-1} + c_0 + \Sigma_0 + c_1 + \Sigma_1 \\ &\supset a_{-1} + c_0 + c_1 + B(2\varepsilon_{-1}) \supset a_{-1} + B(\varepsilon_{-1}) = U\end{aligned}$$

according to Claim 1(2). The inclusion $B(\varepsilon_{-1}) \subset c_0 + c_1 + B(2\varepsilon_{-1})$ follows from $c_0 + c_1 \in C_0 + C_1 \subset B(\varepsilon_0) + B(\varepsilon_0) \subset B(2\varepsilon_0) \subset B(\varepsilon_{-1})$. ∎

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Taras Banakh
Institute of Mathematics
Jan Kochanowski University
Kielce, Poland
and
Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
79000 Lviv, Ukraine
E-mail: t.o.banakh@gmail.com

Nadya Lyaskovska
Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
Universytetska 1
79000 Lviv, Ukraine
E-mail: lyaskovska@yahoo.com

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