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# ON THE LEBESGUE–NAGELL EQUATION

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**Abstract.** We completely solve the Diophantine equations $x^2 + 2^a q^b = y^n$ (for $q = 17, 29, 41$). We also determine all $C = p_1^{a_1} \cdots p_k^{a_k}$ and $C = 2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are fixed primes satisfying certain conditions. The corresponding Diophantine equations $x^2 + C = y^n$ may be studied by the method used by Abu Muriefah et al. (2008) and Luca and Togbé (2009).

**1. Introduction.** The Diophantine equation $x^2+C = y^n$ ($x \geq 1$, $y \geq 1$, $n \geq 3$) has a rich history. Lebesgue [9] proved that this equation has no solution when $C = 1$. Cohn [7] solved the equation for several values of $1 \leq C \leq 100$. The remaining values of $C$ in the above range were covered by Mignotte and de Weger [13] and by Bugeaud, Mignotte and Siksek [5]. Barros in his recent PhD thesis considered the range $-100 \leq C \leq -1$. Also, several authors (Abu Muriefah, Arif, Le, Luca, Pink, Togbé,...) became interested in the case where only the prime factors of $C$ are specified (see, for instance, introductions to [2], [11] and [12]). Abu Muriefah, Luca, Siksek and Tengely [1] studied the more general equation $x^2 + C = 2y^n$.

Consider the Diophantine equation $x^2+C = y^n$, where $C = p_1^{a_1} \cdots p_k^{a_k}$ or $2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$, and $p_1, \ldots, p_k$ are fixed primes satisfying the following three conditions:

(I) $p_i \equiv 1 \pmod 4$ for all $i = 1, \ldots, k$.

Write $C = dz^2$ with $d$ squarefree. Let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let $\mathrm{rad}(n)$ denote the radical of the positive integer $n$ (product of all prime divisors of $n$).

(II) $\mathrm{rad}(h(-d)) \mid 6$ for any decomposition $C = dz^2$ as above.
(III) $\mathrm{rad}(p_i \pm 1) \mid 2 \cdot 3 \cdot 5 \cdot 7$ for all $i = 1, \ldots, k$.

In such cases we can apply the method used in [2] and [12]. If we are able to determine all $S$-integral points (with $S$ an explicit set of rational primes) on some associated elliptic curves, then we can completely solve

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such Diophantine equations. Conditions (I)–(III) above were suggested by Section 5 in [2].

In this paper, we determine all values of $C$ satisfying conditions (I)–(III) (Lemma 2). Radicals of $C$ take exactly 41 values. Some of the equations $x^2 + C = y^n$ with $C$ listed in Lemma 2 were studied in the literature; these include $\mathrm{rad}(C) \in \{5, 13, 17, 29, 41, 97, 2\cdot 5, 2\cdot 13, 5\cdot 13, 2\cdot 5\cdot 13\}$.

Consider $C$ listed in Lemma 2, with $\mathrm{rad}(C) = 2q$. The cases $\mathrm{rad}(C) = 10, 26$ were studied in [11] and [12]. We consider the three remaining cases. We solve completely the Diophantine equations $x^2 + 2^a q^b = y^n$ for $q = 17$, 29, and 41. We apply the method used in [2] and [12]. For $n = 3$ and $n = 4$, the problem is reduced to finding all $\{2, q\}$-integral points on some elliptic curves. For $n \geq 5$ we use the theory of primitive divisors for Lucas sequences [3] to deduce that, at most, the cases $n = 5$, $n = 7$ are possible. In these cases, we reduce again the problem to computation of all $\{2, q\}$-integral points on some elliptic curves. The calculations were done using Magma [4].

Theorem 1. *The only solutions of the equation*

$$x^2 + 2^a 17^b = y^n, \quad x, y \geq 1,\ \gcd(x, y) = 1,\ n \geq 3,\ a, b \geq 0, \tag{1}$$

*are:*

$n = 3$, $(x, y, a, b) \in \{(5, 3, 1, 0), (11, 5, 2, 0)\}$;
$n = 4$, $(x, y, a, b) \in \{(47, 9, 8, 1), (8, 3, 0, 1), (1087, 33, 8, 1), (7, 3, 5, 0), (9, 5, 5, 1), (4785, 71, 9, 3), (15, 7, 7, 1), (495, 23, 11, 1)\}$;
$n = 8$, $(x, y, a, b) = (47, 3, 8, 1)$.

Theorem 2. *The only solutions of the equation*

$$x^2 + 2^a 29^b = y^n, \quad x, y \geq 1,\ \gcd(x, y) = 1,\ n \geq 3,\ a, b \geq 0, \tag{2}$$

*are:*

$n = 3$, $(x, y, a, b) \in \{(5, 3, 1, 0), (11, 5, 2, 0), (3, 5, 2, 1), (26661, 905, 20, 1), (14149, 585, 8, 1), (79, 33, 10, 1), (1465, 129, 4, 1), (95, 33, 5, 2), (73052815, 174753, 17, 2)\}$;
$n = 4$, $(x, y, a, b) = (7, 3, 5, 0)$;
$n = 7$, $(x, y, a, b) = (278, 5, 0, 2)$.

Theorem 3. *The only solutions of the equation*

$$x^2 + 2^a 41^b = y^n, \quad x, y \geq 1,\ \gcd(x, y) = 1,\ n \geq 3,\ a, b \geq 0, \tag{3}$$

*are:*

$n = 3$, $(x, y, a, b) \in \{(5, 3, 1, 0), (11, 5, 2, 0)\}$;
$n = 4$, $(x, y, a, b) \in \{(840, 29, 0, 2), (7, 3, 5, 0), (87, 13, 9, 1), (33, 7, 5, 1)\}$;
$n = 5$, $(x, y, a, b) = (38, 5, 0, 2)$.

**2. Some useful results.** First, let us determine all the primes $p \equiv 1 \pmod 4$ satisfying condition (III).

LEMMA 1. *There are exactly eight primes* $p \equiv 1 \pmod 4$ *satisfying condition* (III)*:* 5, 13, 17, 29, 41, 97, 449, 4801.

*Proof.* We have to find all primes $p \equiv 1 \pmod 4$ satisfying $p+1 = 2\cdot 3^b 5^c 7^d$ and $p-1 = 2^\alpha 3^\beta 5^\gamma 7^\delta$. We consider two cases.

CASE (i): $b+\beta > 0$, $c+\gamma > 0$ and $d+\delta > 0$. Using [8] (or [10, Theorem 4]), we find that the equation $p^2-1 = 2^{\alpha+1}3^{b+\beta}5^{c+\gamma}7^{d+\delta}$ has exactly six solutions:

$$5^2-1 = 2^3\cdot 3,\quad 17^2-1 = 2^5\cdot 3^2,\quad 29^2-1 = 2^3\cdot 3\cdot 5\cdot 7,\quad 41^2-1 = 2^4\cdot 3\cdot 5\cdot 7,$$
$$449^2-1 = 2^7\cdot 3^2\cdot 5^2\cdot 7,\qquad 4801^2-1 = 2^7\cdot 3\cdot 5^2\cdot 7^4.$$

CASE (ii): $b+\beta = 0$ or $c+\gamma = 0$ or $d+\delta = 0$. In this case, we obtain two additional primes 13 and 97. To check this statement, one can use, for instance, [6, Theorems 1 and 2]. We omit the details. ∎

Now we are ready to determine all values of $C$ satisfying (I)–(III).

LEMMA 2.

(i) *The prime power* $p^a$ *satisfies conditions* (I)–(III) *iff* $p \in \{5, 13, 17, 29, 41, 97\}$.
(ii) *The number* $C = 2^{a_0}p^a$ *satisfies* (I)–(III) *iff* $p \in \{5, 13, 17, 29, 41\}$.
(iii) *The odd number* $C = p^a q^b$ *(*$p$*,* $q$ *different odd primes) satisfies* (I)–(III) *iff* $pq \in \{5\cdot 13, 5\cdot 17, 5\cdot 29, 5\cdot 41, 13\cdot 17, 13\cdot 29, 13\cdot 41, 17\cdot 29, 17\cdot 41, 17\cdot 97, 29\cdot 41\}$.
(iv) *The number* $C = 2^{a_0}p^a q^b$ *(*$p$*,* $q$ *different odd primes) satisfies* (I)–(III) *iff* $pq \in \{5\cdot 13, 5\cdot 17, 5\cdot 41, 13\cdot 17, 17\cdot 41\}$.
(v) *The odd number* $C = p_1^{a_1}p_2^{a_2}p_3^{a_3}$ *(*$p_1$*,* $p_2$*,* $p_3$ *different odd primes) satisfies* (I)–(III) *iff* $p_1p_2p_3 \in \{5\cdot 13\cdot 17, 5\cdot 13\cdot 29, 5\cdot 13\cdot 41, 5\cdot 17\cdot 29, 5\cdot 17\cdot 41, 5\cdot 29\cdot 41, 13\cdot 17\cdot 29, 13\cdot 17\cdot 41, 13\cdot 29\cdot 41\}$.
(vi) *The number* $C = 2^{a_0}p_1^{a_1}p_2^{a_2}p_3^{a_3}$ *(*$p_1$*,* $p_2$*,* $p_3$ *different odd primes) satisfies* (I)–(III) *iff* $p_1p_2p_3 \in \{5\cdot 13\cdot 29, 5\cdot 17\cdot 29, 13\cdot 17\cdot 29, 13\cdot 29\cdot 41\}$.
(vii) *The number* $C$ *with* $\geq 4$ *different odd prime factors satisfies* (I)–(III) *iff* $C = 5^a 13^b 17^c 41^d$.

*Proof.* Class number calculations, using Pari. For instance, (i) and (ii) follow from the following data:

$$h(-5) = h(-10) = 2,\quad h(-13) = 2,\quad h(-26) = 6,$$
$$h(-17) = h(-34) = 4,\quad h(-29) = 6,\quad h(-58) = 2,$$
$$h(-41) = 8,\quad h(-82) = 4,\quad h(-97) = 4,\quad h(-194) = 20,$$
$$h(-449) = 20,\quad h(-898) = 12,\quad h(-4801) = 56,\quad h(-9602) = 88.$$ ∎

## 3. The case $n = 3$

LEMMA 3. *Let $n = 3$.*

(i) *The only solutions to equation* (1) *are*

$$(x, y, a, b) \in \{(5, 3, 1, 0), (11, 5, 2, 0)\}.$$

(ii) *The only solutions to equation* (2) *are*

$$\begin{aligned}(x, y, a, b) \in \{&(5, 3, 1, 0), (11, 5, 2, 0), (26661, 905, 20, 1),\\ &(14149, 585, 8, 1), (79, 33, 10, 1), (1465, 129, 4, 1),\\ &(95, 33, 5, 2), (73052815, 174753, 17, 2)\}.\end{aligned}$$

(iii) *The only solutions to equation* (3) *are*

$$(x, y, a, b) \in \{(5, 3, 1, 0), (11, 5, 2, 0)\}.$$

*Proof.* Let $q \in \{17, 29, 41\}$. Write the equation $x^2 + 2^a q^b = y^3$ as $(x/z^3)^2 + A = (y/z^2)^3$, where $A$ is a 6th power free positive integer, defined by $2^a q^b = Az^6$ with some integer $z$. Of course, $A = 2^\alpha q^\beta$ with $\alpha, \beta \in \{0, 1, 2, 3, 4, 5\}$, and we obtain the equations

$$V^2 = U^3 - 2^\alpha q^\beta$$

with $U = y/z^2$, $V = x/z^3$. We have to determine $\{2, q\}$-integral points on these 36 elliptic curves; this can be done using Magma. Note that we only need to consider "admissible" points $(U, V)$ (see [12, p. 141]), i.e.

- we discard the solutions with $U \leq 0$ or $V = 0$;
- we do not consider the solutions having the numerators of $U$ and $V$ not coprime;
- if $U, V \in \mathbb{Z}$, then $z = 1$;
- if $U$ and $V$ are rationals which are not integers, then their numerators give $x$ and $y$, and $z$ is determined by their denominators. Therefore, $a$ and $b$ are determined from the formula $2^a q^b = Az^6$.

Here are the results of our Magma calculations.

(i) The only "admissible" $\{2, 17\}$-integral points on $V^2 = U^3 - 2^\alpha 17^\beta$ are

$$(U, V, \alpha, \beta) \in \{(3, 5, 1, 0), (5, 11, 2, 0)\}.$$

(ii) The only "admissible" $\{2, 29\}$-integral points on $V^2 = U^3 - 2^\alpha 29^\beta$ are

$$\begin{aligned}(U, V, \alpha, \beta) \in \{&(3, 5, 1, 0), (5, 11, 2, 0), (5, 3, 2, 1), (905, 26661, 20, 1),\\ &(585, 14149, 8, 1), (33, 79, 10, 1), (129, 1465, 4, 1),\\ &(33, 95, 5, 2), (174753, 73052815, 17, 2)\}.\end{aligned}$$

(iii) The only "admissible" $\{2, 41\}$-integral points on $V^2 = U^3 - 2^\alpha 41^\beta$ are

$$(U, V, \alpha, \beta) \in \{(3, 5, 1, 0), (5, 11, 2, 0)\}. \blacksquare$$

**4. The case** $n = 4$

LEMMA 4. *Let* $n = 4$.

(i) *The only solutions to equation* (1) *are*

$$(x, y, a, b) \in \{(47, 9, 8, 1), (8, 3, 0, 1), (1087, 33, 8, 1), (7, 3, 5, 0), (9, 5, 5, 1), (4785, 71, 9, 3), (15, 7, 7, 1), (495, 23, 11, 1)\}.$$

(ii) *The only solution to equation* (2) *is* $(x, y, a, b) = (7, 3, 5, 0)$.

(iii) *The only solutions to equation* (3) *are*

$$(x, y, a, b) \in \{(840, 29, 0, 2), (7, 3, 5, 0), (87, 13, 9, 1), (33, 7, 5, 1)\}.$$

*Proof.* Let $q \in \{17, 29, 41\}$. Write the equation $x^2 + 2^a q^b = y^4$ as $(x/z^2)^2 + A = (y/z)^4$, where $A$ is a 4th power free positive integer, defined by $2^a q^b = Az^4$ with some integer $z$. Of course, $A = 2^\alpha q^\beta$ with $\alpha, \beta \in \{0, 1, 2, 3\}$, and we obtain the equations

$$V^2 = U^4 - 2^\alpha q^\beta$$

with $U = y/z$, $V = x/z^2$. We have to determine $\{2, q\}$-integral points on these 16 elliptic curves. As in the case $n = 3$, we only need to consider "admissible" points $(U, V)$.

Here are the results of our Magma calculations.

(i) The only "admissible" $\{2, 17\}$-integral points on $V^2 = U^4 - 2^\alpha 17^\beta$ are

$$(U, V, \alpha, \beta) \in \{(9, 47, 8, 1), (3, 8, 0, 1), (33, 1087, 8, 1), (3, 7, 5, 0), (5, 9, 5, 1), (71, 4785, 9, 3), (7, 15, 7, 1), (23, 495, 11, 1)\}.$$

(ii) The only "admissible" $\{2, 29\}$-integral point on $V^2 = U^4 - 2^\alpha 29^\beta$ is

$$(U, V, \alpha, \beta) = (3, 7, 5, 0).$$

(iii) The only "admissible" $\{2, 41\}$-integral points on $V^2 = U^4 - 2^\alpha 41^\beta$ are

$$(U, V, \alpha, \beta) \in \{(29, 840, 0, 2), (3, 7, 5, 0), (13, 87, 9, 1), (7, 33, 5, 1)\}. \blacksquare$$

**5. The case** $n \geq 5$**.** Let $q \in \{17, 29, 41\}$. We rewrite the Diophantine equation $x^2 + 2^a q^b = y^n$ as $x^2 + dz^2 = y^n$, where $d = 1, 2, q, 2q$ according to the parities of the exponents of $a$ and $b$. Factoring the last equation in $\mathbb{Q}(\sqrt{-d})$ we get $(x + z\sqrt{-d})(x - z\sqrt{-d}) = y^n$. Here $z = 2^\alpha q^\beta$ for some nonnegative integers $\alpha$ and $\beta$. Conditions (I) and (II) allow us to assume that $x + z\sqrt{-d} = \gamma^n$ with some algebraic integer $\gamma = u + v\sqrt{-d} \in \mathbb{Z}[\sqrt{-d}]$. As a consequence,

$$2^{\alpha+1} q^\beta \sqrt{-d} = \gamma^n - \overline{\gamma}^n. \tag{4}$$

Let $n \geq 5$ be a prime. The Lucas number $L_n := (\gamma^n - \overline{\gamma}^n)/(\gamma - \overline{\gamma})$ has a primitive prime factor (it cannot be defective, see Table 1 in [3]). A primitive prime factor $r$ of $L_n$ satisfies the congruence $r \equiv e \pmod{n}$, where $e = \left(\frac{-4d}{r}\right)$.

**5.1. The Diophantine equation $x^2 + 2^a 17^b = y^n$.** In this case $r = 17$, hence $n \mid 16$ or $n \mid 18$. Therefore (1) has no solution with prime $n \geq 5$. Note, using Lemma 4(i), that (1) has a solution $(x, y, a, b) = (47, 3, 8, 1)$ for $n = 8$.

**5.2. The Diophantine equation $x^2 + 2^a 29^b = y^n$.** In this case $r = 29$, hence $n \mid 28$ or $n \mid 30$. Therefore, $n = 7$ and $d = 1$ or $n = 5$ and $d = 2$.

Case $n = 7$. Using (4) with $n = 7$, $d = 1$, we obtain

$$v(7u^6 - 35u^4v^2 + 21u^2v^4 - v^6) = 2^\alpha 29^\beta. \tag{5}$$

Since $u$ and $v$ are coprime, we have the following possibilities.

$$\text{(a)}\, v = \pm 2^\alpha 29^\beta, \quad \text{(b)}\, v = \pm 29^\beta, \quad \text{(c)}\, v = \pm 2^\alpha, \quad \text{(d)}\, v = \pm 1.$$

We only need to look at the last two possibilities.

In case (c), $v = \pm 2^\alpha$, and the Diophantine equation (5) is

$$7u^6 - 35u^4v^2 + 21u^2v^4 - v^6 = \pm 29^\beta.$$

Dividing both sides by $v^6$, we obtain

$$7X^3 - 35X^2 + 21X - 1 = D_1 Y^2, \tag{6}$$

where $X = u^2/v^2$, $Y = 29^{\beta_1}/v^3$, $\beta_1 = \lfloor \beta/2 \rfloor$, $D_1 = \pm 1, \pm 29$.

In the case $D_1 = \pm 1$, we have to find $\{2\}$-integral points on the elliptic curves

$$7X^3 - 35\eta X^2 + 21X - \eta = D_1 Y^2, \quad \eta = \pm 1. \tag{7}$$

We mutiply both sides of (7) by $7^2$ to obtain

$$U^3 - 35\eta U^2 + 147U - 49\eta = V^2, \tag{8}$$

where $(U, V) = (7\eta X, 7Y)$ are $\{2\}$-integral points on the above elliptic curves.

Using Magma, we find $(U, V) \in \{(1, 8), (58, -293)\}$ (hence, $(X, Y) \in \{(1/7, 8/7), (58/7, -293/7)\}$) for $\eta = 1$. These do not lead to solutions of (2).

If $\eta = -1$, we find $(U, V) \in \{(-21, 56), (-5, 8), (0, 7), (7, -56), (39, 344), (301/4, -6377/8)\}$ (and hence $(X, Y) \in \{(3, 8), (5/7, 8/7), (0, 1), (-1, -8), (-39/7, 344/7), (-43/4, -911/8)\}$). These do not lead to solutions of (2) either.

Consider the case $D_1 = \pm 29$. The unique $\{2\}$-integral point $(2349, -87464)$ on the elliptic curve $U^3 - 35 \cdot 29U^2 + 21 \cdot 7 \cdot 29^2 U - 7^2 \cdot 29^3 = V^2$ does not lead to a solution of (2). Magma finds the $\{2\}$-integral points $(-812, 5887)$, $(-377, 6728)$, $(-5, -776)$, $(91, 4648)$, $(1015, 47096)$, $(8365/4, -941297/8)$ on the elliptic curve $U^3 + 35 \cdot 29U^2 + 21 \cdot 7 \cdot 29^2 U + 7^2 \cdot 29^3 = V^2$. The point $(-812, 5887)$ leads to the solution $(x, y, a, b) = (278, 5, 0, 2)$ of (2).

Consider case (d), $v = \pm 1$. We have to find integral points on

$$7X^3 - 35X^2 + 21X - 1 = D_1 Y^2, \tag{9}$$

where $D_1 = \pm 1, \pm 2, \pm 29, \pm 58$.

The cases $D_1 = \pm 1, \pm 29$ were treated above.

Consider the case $D_1 = \pm 2$. There exists no integral point on the curve $U^3 - 35 \cdot 2U^2 + 21 \cdot 7 \cdot 2^2 U - 7^2 \cdot 2^3 = V^2$, and there are two integral points $(-14, 56)$, $(7, 91)$ on the curve $U^3 + 35 \cdot 2U^2 + 21 \cdot 7 \cdot 2^2 U + 7^2 \cdot 2^3 = V^2$. These do not lead to solutions of (2).

Consider the case $D_1 = \pm 58$.There exists no integral point on the curve $U^3 - 35 \cdot 2 \cdot 29U^2 + 21 \cdot 7 \cdot 2^2 \cdot 29^2 U - 7^2 2^3 \cdot 29^3 = V^2$ and there are two integral points $(58, 6728), (879, -51883)$ on the curve $U^3 + 35 \cdot 2 \cdot 29U^2 + 21 \cdot 7 \cdot 2^2 \cdot 29^2 U + 7^2 2^3 \cdot 29^3 = V^2$. These do not lead to solutions of (2).

Case $n = 5$. Using (4) with $n = 5$, $d = 2$, we obtain

$$v(5u^4 - 20u^2v^2 + 4v^4) = 2^\alpha 29^\beta. \tag{10}$$

As in the case $n = 7$, we only need to check $v = \pm 2^\alpha$, $v = \pm 1$.

In the first case, the Diophantine equation (10) is $5u^4 - 20u^2v^2 + 4v^4 = \pm 29^\beta$. Dividing both sides by $v^4$, we obtain

$$5X^4 - 20X^2 + 4 = D_1 Y^2, \tag{11}$$

where $X = u/v$, $Y = 29^{\beta_1}/v^2$, $\beta_1 = \lfloor \beta/2 \rfloor$, and $D_1 = \pm 1, \pm 29$. Using Magma we find three $\{2\}$-integral points $(0, 2)$, $(2, 2)$, $(-2, 2)$ on (11) with $D_1 = 1$, and none in the remaining cases. These points do not lead to solutions of (2).

In the second case, the Diophantine equation (10) is $5u^4 - 20u^2 + 4 = \pm 2^\alpha 29^\beta$. We need to find integral points on the curves $5X^4 - 20X^2 + 4 = D_1 Y^2$, $D_1 = \pm 1, \pm 2, \pm 29, \pm 58$. Magma finds no solution.

**5.3. The Diophantine equation $x^2 + 2^a 41^b = y^n$.** We have $\left(\frac{-4}{41}\right) = \left(\frac{-8}{41}\right) = 1$, hence in this case $n = 5$, $d = 1$ or $n = 5$, $d = 2$.

Using (4) with $n = 5$, $d = 2$, we obtain

$$v(5u^4 - 20u^2v^2 + 4v^4) = 2^\alpha 41^\beta. \tag{12}$$

We only need to check $v = \pm 2^\alpha$, $v = \pm 1$.

In the first case, the Diophantine equation (12) is $5u^4 - 20u^2v^2 + 4v^4 = \pm 41^\beta$. Dividing both sides by $v^4$, we obtain

$$5X^4 - 20X^2 + 4 = D_1 Y^2, \tag{13}$$

where $X = u/v$, $Y = 41^{\beta_1}/v^2$, $\beta_1 = \lfloor \beta/2 \rfloor$, and $D_1 = \pm 1, \pm 41$. Using Magma we find three $\{2\}$-integral points $(0, 2)$, $(2, 2)$, $(-2, 2)$ on (13) with $D_1 = 1$, and none in the remaining cases. These points do not lead to solutions of (3).

In the second case, the Diophantine equation (12) is $5u^4 - 20u^2 + 4 = \pm 2^\alpha 41^\beta$. We need to find integral points on the curves $5X^4 - 20X^2 + 4 = D_1 Y^2$, $D_1 = \pm 1, \pm 2, \pm 41, \pm 82$. Magma finds no solution.

Using (4) with $n = 5$, $d = 1$, we obtain $v(5u^4 - 10u^2v^2 + v^4) = 2^\alpha 41^\beta$. In the case $v = \pm 2^a$ we obtain $5u^4 - 10u^2v^2 + v^4 = \pm 41^\beta$. Magma finds $\{2\}$-integral points on

$$5X^4 - 10X^2 + 1 = \pm D_1 Y^2, \quad D_1 = \pm 1, \pm 41,$$

namely, $(1, 2)$ if $D_1 = -1$, and $(2, 1)$ if $D_1 = 41$. The point $(2, 1)$ gives the new solution $(x, y) = (38, 5)$ of (3).

In the case $v = \pm 1$, we obtain $5u^4 - 10u^2v^2 + v^4 = \pm 2^\alpha 41^\beta$. Magma finds no integral points on the curves

$$5X^4 - 10X^2 + 1 = \pm D_1 Y^2, \quad D_1 = \pm 1, \pm 41 \pm 2, \pm 82.$$

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