COLLOQUIUM MATHEMATICUM

VOL. 125 2011 NO. 2

DIVERGENT SOLUTIONS TO THE 5D HARTREE EQUATIONS

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Abstract. We consider the Cauchy problem for the focusing Hartree equation $iu_t + \Delta u + (|\cdot|^{-3} * |u|^2)u = 0$ in $\mathbb{R}^5$ with initial data in H^1, and study the divergence property of infinite-variance and nonradial solutions. For the ground state solution of $-Q + \Delta Q + (|\cdot|^{-3} * |Q|^2)Q = 0$ in $\mathbb{R}^5$, we prove that if $u_0 \in H^1$ satisfies $M(u_0)E(u_0) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then the corresponding solution $u(t)$ either blows up in finite forward time, or exists globally for positive time and there exists a time sequence $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2 \to \infty$. A similar result holds for negative time.

1. Introduction. In this paper, we consider the following Cauchy problem for the 5D Hartree equation:

$$\begin{cases} iu_t + \Delta u + (V * |u|^2)u = 0, & (x,t) \in \mathbb{R}^5 \times \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^5), \end{cases} \tag{1.1}$$

where $V(x) = |x|^{-3}$, and $*$ denotes convolution in $\mathbb{R}^5$.

The Hartree type nonlinearity $(|\cdot|^{2-N} * |u|^2)u$ in $\mathbb{R}^N$ describes the dynamics of the mean-field limits of many-body quantum systems such as coherent states and condensates. Note that equation (1.1) is invariant under the scaling $u(x,t) \mapsto \lambda^2 u(\lambda x, \lambda^2 t)$ which also leaves the norm of the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^5)$ invariant. That is why we call equation (1.1) the $\dot{H}^{1/2}$-*Hartree equation*. Similarly, the case $N = 4$ gives the L^2-critical Hartree equation, the solution of which, by [25], scatters when the mass of the initial data is strictly less than that of the ground state. Moreover in [23], the dynamics of minimal mass blowing up solutions of the focusing L^2-critical Hartree equation is characterized. A large amount of work has been devoted to the scattering theory for the Hartree equation (see for example [24, 5, 6, 26, 3, 21, 22, 27]).

It is well known from [4] that (1.1) is locally well-posed in H^1. Namely, for $u_0 \in H^1$, there exist $0 < T \le \infty$ and a unique solution $u(\cdot) \in C([0,T); H^1)$ to (1.1). When $T < \infty$, we have $\lim_{t\uparrow T} \|\nabla u(t)\|_2 = \infty$, and say that solution u *blows up in finite positive time*. On the other hand, when $T = \infty$, the

2010 *Mathematics Subject Classification*: 35Q55, 35A15, 35B30.

Key words and phrases: Hartree equation, blow-up, profile decomposition, divergent solutions.

DOI: 10.4064/cm125-2-10

solution is called *positively global.* Note that the local theory gives nothing about the behavior of $\|\nabla u(t)\|_2$ as $t \uparrow \infty$. For solutions of (1.1) the following quantities are conserved:

$$L^2\text{-}norm: \quad M(u)(t) \equiv \int |u(x,t)|^2 dx = M(u_0);$$

$$Energy: \quad E(u)(t) \equiv \frac{1}{2}\int |\nabla u(x,t)|^2\,dx - \frac{1}{4}\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x,t)|^2|u(y,t)|^2}{|x-y|^3}\,dx\,dy = E(u_0);$$

$$Momentum: \quad P(u)(t) \equiv \operatorname{Im}\int \overline{u}(x,t)\nabla u(x,t)\,dx = P(u_0).$$

In [3], it is proved that if $u_0 \in H^1$, $M(u_0)E(u_0) < M(Q)E(Q)$ and $\|\nabla u_0\|_2\|u_0\|_2 > \|\nabla Q\|_2\|Q\|_2$, then the solution $u(t)$ to (1.1) blows up in finite time provided $\|xu_0\|_{L^2} < \infty$ or u_0 is radial. Note that this result is sharp in the sense that $u(t) = e^{it}Q(x)$ solves (1.1) and does not blow up in finite time.

In this paper, in the spirit of Holmer and Roudenko [9] dealing with the cubic 3D Schrödinger equation, without assuming finite variance and radiality we obtain the following result:

THEOREM 1.1. *Suppose that $u_0 \in H^1$, $M(u_0)E(u_0) < M(Q)E(Q)$ and $\|\nabla u_0\|_2\|u_0\|_2 > \|\nabla Q\|_2\|Q\|_2$. Then either $u(t)$ blows up in finite forward time, or $u(t)$ is forward global and there exists a time sequence $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2 \to \infty$. A similar statement holds for negative time.*

The difference between NLS and the Hartree equation is the nonlinearity. The former is a local interaction, while the latter is a nonlocal interaction. We cannot always use NLS's arguments to discuss the Hartree equation, especially when dealing with the nonlinearity. In fact, there are many essential differences between the NLS and Hartree cases.

REMARK 1.2. Using the same argument as in the introduction of [9] (see more details in Appendix B there), via the Galilean transformation, we will always assume that $P(u) = 0$. That is, we need only show Theorem 1.1 under the condition $P(u) = 0$. In fact, on the one hand, by [9], the dichotomy result of Propositions 2.1 and 2.2 below is preserved by the Galilean transformation; on the other hand, from the relationship between $u(t)$ with nonzero momentum and its Galilean transformation $\tilde{u}(t)$ satisfying

$$\tilde{u}(x,t) = e^{ix\xi}e^{-it|\xi|^2}u(x-2\xi t,t) \quad \text{with} \quad \xi = \frac{P(u)}{M(u)}$$

we get

$$P(\tilde{u}) = 0, \quad M(\tilde{u}) = M(u) = M(Q), \quad \|\nabla\tilde{u}\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \frac{P(u)^2}{2M(u)}.$$

Thus, Theorem 1.1 is also true by the Galilean transformation.

In this paper, H^1 denotes the usual Sobolev space $W^{1,2}(\mathbb{R}^5)$ and

$$\|u\|_{L^V} \equiv \Big(\iint_{\mathbb{R}^5\times\mathbb{R}^5} |u(x)|^2 V(x-y)|u(y)|^2\,dx\,dy\Big)^{1/4}.$$

As usual, we denote the L^p norm by $\|\cdot\|_p$, and use the convention that c always stands for absolute constants which may vary from line to line.

The rest of this paper is organized as follows. In Section 2 we recall the dichotomy and scattering results. In Section 3, we discuss blow-up of solutions based on the virial identity and its localized versions. Section 4 is devoted to the variational characterization of the ground state and can be considered as a preparation for Section 5, in which we set up the inductive argument that will be continued in Sections 7 and 8. In Section 6 we introduce the linear and nonlinear profile decomposition lemmas, which are needed in the argument of Sections 7 and 8, where we give the proof of Theorem 1.1.

2. Ground state and dichotomy. As argued in [30, 23], if C_{HLS} is the best constant in the Hardy–Littlewood–Sobolev inequality

$$\iint \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,dx\,dy \le C_{\mathrm{HLS}}\|u\|_2\|\nabla u\|_2^3, \tag{2.1}$$

then it is attained at Q that is the unique radial positive solution to

$$Q - \Delta Q = (V * Q^2)Q. \tag{2.2}$$

The uniqueness of the ground state of (2.2) can be obtained by the same method as in the cases of dimension three and four ([17] and [16]) by means of Newton's theorem [18]. In fact, it suffices to note that the convolution term in (1.1) is just the Newtonian potential in $\mathbb{R}^5$. It is worth pointing out that, generally, the uniqueness of the ground states of Hartree equations is an open problem. One can also find some results on this problem in [20].

From (2.2) we have

$$\int |Q|^2\,dx + \int |\nabla Q|^2\,dx - \|Q\|_{L^V}^4 = 0,$$

and the Pokhozhaev identity

$$\frac{5}{2}\int |Q|^2\,dx + \frac{3}{2}\int |\nabla Q|^2\,dx - \frac{7}{4}\|Q\|_{L^V}^4 = 0.$$

These two equalities imply that

$$\|Q\|_{L^V}^4 = \frac{4}{3}\|\nabla Q\|_2^2 = 4\|Q\|_2^2.$$

As a consequence,

$$C_{\mathrm{HLS}} = \frac{\|Q\|_{L^V}^4}{\|Q\|_2\|\nabla Q\|_2^3} = \frac{4}{3}\,\frac{1}{\|Q\|_2\|\nabla Q\|_2}, \tag{2.3}$$

and therefore

$$E(Q) = \frac{1}{6}\|\nabla Q\|_2^2. \tag{2.4}$$

Let

$$\eta(t) = \frac{\|\nabla u\|_2 \|u\|_2}{\|\nabla Q\|_2 \|Q\|_2}. \tag{2.5}$$

By (2.1), (2.3) and (2.4) we have

$$3\eta(t)^2 \geq \frac{E(u)M(u)}{E(Q)M(Q)} \geq 3\eta(t)^2 - 2\eta(t)^3. \tag{2.6}$$

Thus, it is not difficult to observe that if $0 \leq M(u)E(u)/M(Q)E(Q) < 1$, then there exist two solutions $0 \leq \lambda_- < 1 < \lambda$ of the following equation in λ:

$$\frac{E(u)M(u)}{E(Q)M(Q)} = 3\lambda^2 - 2\lambda^3. \tag{2.7}$$

On the other hand, if $E(u) < 0$, there exists exactly one solution $\lambda > 1$ to (2.7).

By the H^1 local theory [4], there exist $-\infty \leq T_- < 0 < T_+ \leq \infty$ such that (T_-, T_+) is the maximal time interval of existence for $u(t)$ solving (1.1), and if $T_+ < \infty$ then

$$\|\nabla u(t)\|_2 \to \infty \quad \text{as } t \uparrow T_+.$$

A similar conclusion holds if $T_- > -\infty$. Moreover, as a consequence of the continuity of the flow $u(t)$, we have the following dichotomy proposition:

PROPOSITION 2.1 (Global versus blow-up dichotomy). *Let $u_0 \in H^1$, and let $I = (T_-, T_+)$ be the maximal time interval of existence of $u(t)$ solving* (1.1). *Suppose that*

$$M(u)E(u) < M(Q)E(Q). \tag{2.8}$$

If (2.8) *holds and*

$$\|u_0\|_2 \|\nabla u_0\|_2 < \|Q\|_2 \|\nabla Q\|_2, \tag{2.9}$$

then $I = (-\infty, \infty)$, i.e., the solution exists globally in time, and for all $t \in \mathbb{R}$,

$$\|u(t)\|_2 \|\nabla u(t)\|_2 < \|Q\|_2 \|\nabla Q\|_2. \tag{2.10}$$

If (2.8) *holds and*

$$\|u_0\|_2 \|\nabla u_0\|_2 > \|Q\|_2 \|\nabla Q\|_2, \tag{2.11}$$

then for $t \in I$,

$$\|u(t)\|_2 \|\nabla u(t)\|_2 > \|Q\|_2 \|\nabla Q\|_2. \tag{2.12}$$

Proof. Multiplying the formula for the energy by $M(u)$ and using the Hardy–Littlewood–Sobolev inequality (2.1), we obtain

$$\begin{aligned} E(u)M(u) &= \frac{1}{2}\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^V}^4\|u\|_{L^2}^2 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2\|u\|_2^2 - \frac{1}{4}C_{\mathrm{HLS}}\|\nabla u\|_2^3\|u\|_2^3. \end{aligned}$$

Define $f(x) = \frac{1}{2}x^2 - \frac{1}{4}C_{\mathrm{HLS}}x^3$. Then $f'(x) = x(1-(3C_{\mathrm{HLS}}/4)x)$, and $f'(x) = 0$ when $x_0 = 0$ and $x_1 = \|\nabla Q\|_2\|Q\|_2 = 4/3C_{\mathrm{HLS}}$ by (2.3). Note that $f(0) = 0$ and $f(x_1) = \frac{1}{6}\|\nabla Q\|_2^2\|Q\|_2^2$. Thus f has two extrema: a local minimum at x_0 and a local maximum at x_1. (2.8) implies that $E(u_0)M(u_0) < f(x_1)$, which combined with energy conservation leads to

$$f(\|\nabla u\|_2\|u\|_2) \leq E(u)M(u_0) = E(u)M(u) < f(x_1). \tag{2.13}$$

If initially $\|\nabla u_0\|_2\|u_0\|_2 < x_1$, i.e., (2.9) holds, then by (2.13) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2 < x_1$ for all $t \in I$. In particular, the H^1 norm of the solution is bounded, which implies the global existence and (2.10) in this case.

If initially $\|\nabla u_0\|_2\|u_0\|_2 > x_1$, i.e., (2.11) holds, then by (2.13) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2 > x_1$ for all $t \in I$, which proves (2.12). ∎

The following is another statement of the Dichotomy Proposition in terms of λ and $\eta(t)$ defined by (2.7) and (2.5) respectively, which will be useful in what follows.

PROPOSITION 2.2. *Let $M(u)E(u) < M(Q)E(Q)$ and $0 \leq \lambda_- < 1 < \lambda$ be defined as* (2.7). *Then exactly one of the following statements holds:*

(1) *The solution $u(t)$ to* (1.1) *is global and*

$$\frac{1}{3}\frac{E(u)M(u)}{E(Q)M(Q)} \leq \eta(t)^2 \leq \lambda_-^2, \qquad \forall t \in (-\infty,\infty).$$

(2) $1 < \lambda \leq \eta(t)$ *for all* $t \in (T_-, T_+)$.

For a better understanding, one can refer to the figure in [9] describing the relationship between $M(u)E(u)/M(Q)E(Q)$ and $\eta(t)$. Whether the solution is of the first or second type in Proposition 2.2 is determined by the initial data. Note that the second case does not assert finite-time blow-up.

In the remainder of this section, we will review the Strichartz estimates and some facts about scattering. It is well-known that a pair of exponents (q,r) is *Strichartz admissible* if

$$\frac{2}{q} + \frac{5}{r} = \frac{5}{2}, \qquad 2 \leq q \leq \infty, \qquad 2 \leq r \leq \frac{10}{3}.$$

Similarly for $s > 0$, we say that (q, r) is $\dot{H}^s(\mathbb{R}^5)$ *admissible* and write $(q, r) \in \Lambda_s$ if

$$\frac{2}{q} + \frac{5}{r} = \frac{5}{2} - s, \qquad 4 < q \le \infty, \qquad \frac{10}{5-2s} \le r < \frac{10}{3}.$$

Correspondingly, the Hölder dual (q', r') to (q, r) is *dual* $\dot{H}^s(\mathbb{R}^5)$ *admissible*, written $(q', r') \in \Lambda'_s$, if $(q, r) \in \Lambda_{-s}$. We define the Strichartz norm

$$\|u\|_{S(\dot{H}^{1/2})} = \sup_{(q,r)\in\Lambda_{1/2}} \|u\|_{L_t^q L_x^r}$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-1/2})} = \inf_{(q',r')\in\Lambda'_{1/2}} \|u\|_{L_t^{q'} L_x^{r'}} = \inf_{(q,r)\in\Lambda_{-1/2}} \|u\|_{L_t^{q'} L_x^{r'}},$$

where (q', r') is the Hölder dual to (q, r).

So we have the following Strichartz estimates:

$$\|e^{it\Delta}\phi\|_{S(L^2)} \le c\|\phi\|_2 \quad \text{and} \quad \Big\|\int_0^t e^{i(t-t^1)\Delta} f(\cdot, t^1)\, dt^1\Big\|_{S(L^2)} \le c\|f\|_{S'(L^2)}.$$

Together with Sobolev embeddings, we obtain

$$\|e^{it\Delta}\phi\|_{S(\dot{H}^{1/2})} \le c\|\phi\|_{\dot{H}^{1/2}}$$

and

$$\Big\|\int_0^t e^{i(t-t^1)\Delta} f(\cdot, t^1)\, dt^1\Big\|_{S(\dot{H}^{1/2})} \le c\|D^{1/2} f\|_{S'(L^2)}.$$

In fact, we also have the following Kato inhomogeneous Strichartz estimate [10]:

$$\Big\|\int_0^t e^{i(t-t^1)\Delta} f(\cdot, t^1)\, dt^1\Big\|_{S(\dot{H}^{1/2})} \le c\|f\|_{S'(\dot{H}^{-1/2})}. \tag{2.14}$$

We will write $S(\dot{H}^{1/2}; I)$ to indicate the restriction to a time subinterval $I \subset (-\infty, \infty)$.

For the first case of the dichotomy proposition (Proposition 2.2), we have furthermore scattering results that will be used in the future discussion. We omit the proofs since they are similar to those in [3].

LEMMA 2.3 (Small data). *Let* $\|u_0\|_{\dot{H}^{1/2}} \le A$. *Then there exists* $\delta_{\mathrm{sd}} = \delta_{\mathrm{sd}}(A) > 0$ *such that* $\|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})} \le \delta_{\mathrm{sd}}$, *then* u *solving* (1.1) *is global and*

$$\|u\|_{S(\dot{H}^{1/2})} \le 2\|e^{it\Delta}u_0\|_{S(\dot{H}^{1/2})}, \tag{2.15}$$

$$\|D^{1/2}u\|_{S(L^2)} \le 2c\|u_0\|_{\dot H^{1/2}}. \tag{2.16}$$

Note that by Strichartz estimates, the hypotheses are satisfied if $\|u_0\|_{\dot H^{1/2}} \le c\delta_{\mathrm{sd}}$.

THEOREM 2.4 (Scattering). *Suppose that* $0 < M(u)E(u)/M(Q)E(Q) < 1$ *and the first case of Proposition* 2.2 *holds. Then* $u(t)$ *scatters as* $t \to \infty$ *or* $t \to -\infty$. *That is, there exist* $\phi_\pm \in H^1$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1} = 0. \tag{2.17}$$

Consequently,

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^V} = 0 \tag{2.18}$$

and

$$\lim_{t\to\pm\infty} \eta(t)^2 = \frac{1}{3}\,\frac{E(u)M(u)}{E(Q)M(Q)}. \tag{2.19}$$

LEMMA 2.5 (Existence of wave operators). *Suppose that* $\phi^+ \in H^1$ *and*

$$\frac{1}{2}\|\phi^+\|_2^2\|\nabla\phi^+\|_2^2 < E(Q)M(Q). \tag{2.20}$$

Then there exists $v_0 \in H^1$ *such that the corresponding solution* v *to* (1.1) *exists globally and satisfies*

$$\|\nabla v(t)\|_2\|v_0\|_2 \le \|\nabla Q\|_2\|Q\|_2, \quad M(v) = \|\phi^+\|_2^2, \quad E(v) = \frac{1}{2}\|\nabla\phi^+\|_2^2,$$

and

$$\lim_{t\to\infty} \|v(t) - e^{it\Delta}\phi^+\|_{H^1} = 0.$$

Moreover, if $\|e^{it\Delta}\phi^+\|_{S(\dot H^{1/2})} \le \delta_{\mathrm{sd}}$, *then*

$$\|v\|_{S(\dot H^{1/2})} \le 2\|e^{it\Delta}\phi^+\|_{S(\dot H^{1/2})}, \quad \|D^{1/2}v\|_{S(L^2)} \le 2c\|\phi^+\|_{\dot H^{1/2}}.$$

3. Virial identity and blow-up conditions. From now on we will focus on the second case of Proposition 2.2. Using the classical virial identity we first derive an upper bound of the finite blow-up time under the finite variance hypothesis.

PROPOSITION 3.1. *Suppose that* $\|xu_0\|_2 < \infty$. *Let* $M(u) = M(Q)$, $E(u) < E(Q)$ *and suppose that the second case of Proposition* 2.2 *holds with* $\lambda > 1$ *defined in* (2.7). *Let* $r(t)$ *be the scaled variance given by*

$$r(t) = \frac{\|xu\|_2^2}{48\lambda^2(\lambda-1)E(Q)}.$$

Then blow-up occurs in forward time before t_b, *where*

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}.$$

Proof. The virial identity gives

$$r''(t) = \frac{24E(u) - 4\|\nabla u\|_2^2}{48\lambda^2(\lambda-1)E(Q)}.$$

Using (2.3) we obtain

$$r''(t) = \frac{1}{2\lambda^2(\lambda-1)}\left(\frac{E(u)}{E(Q)} - \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2}\right).$$

By the definition of λ and η,

$$r''(t) = \frac{3\lambda^2 - 2\lambda^3 - \eta(t)^2}{2\lambda^2(\lambda-1)}.$$

Since $\eta(t) \geq \lambda > 1$, we have

$$r''(t) \leq -1,$$

which by twice integrating in time gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0).$$

Note that t_b is the positive root of the polynomial on the right hand side, which leads to $r(t) \leq t_b$. ∎

The next result is related to the local virial identity. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be radial such that $\varphi'' \leq 2$ and

$$\varphi(x) = \begin{cases} |x|^2, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

For $R > 0$ define

$$z_R(t) = \int R^2 \phi\left(\frac{x}{R}\right)|u(x,t)|^2\,dx. \tag{3.1}$$

Then by direct calculations we obtain the following local virial identity:

$$\begin{aligned} z_R''(t) = 4\sum_{j,k}\int \partial_j\partial_k\varphi\left(\frac{x}{R}\right)\partial_j\bar{u}\partial_k u\,dx - \frac{1}{R^2}\int \Delta^2\varphi\left(\frac{x}{R}\right)|u|^2\,dx \\ + R\iint\left(\nabla\varphi\left(\frac{x}{R}\right) - \nabla\varphi\left(\frac{y}{R}\right)\right)\nabla V(x-y)|u(x)|^2|u(y)|^2\,dx\,dy. \end{aligned} \tag{3.2}$$

Set

$$I = 3\sum_j\iint\left[\left(2x_j - R\partial_j\varphi\left(\frac{x}{R}\right)\right) - \left(2y_j - R\partial_j\varphi\left(\frac{y}{R}\right)\right)\right] \times \frac{x_j - y_j}{|x-y|^5}|u(x)|^2|u(y)|^2 dx dy,$$

and by the definition of φ, we have

$$z_R''(t) = 24E(u) - 4\|\nabla u\|_2^2 + A_R(u(t)),$$

where

$$A_R(u(t)) = 4\sum_{j\neq k}\int_{|x|>R} \partial_j\partial_k\varphi\Big(\frac{x}{R}\Big)\partial_j\bar{u}\partial_k u\,dx$$
$$+4\sum_j\int_{|x|\le R}\Big[\partial_j^2\varphi\Big(\frac{x}{R}\Big)-2\Big]|\nabla u|^2\,dx$$
$$-\frac{1}{R^2}\int_{|x|>R}\Delta^2\varphi\Big(\frac{x}{R}\Big)|u|^2\,dx + I.$$

Observe that I vanishes in the region $|x|,|y|\le R$, while in the region $|x|,|y|\ge R$, I becomes $6\int_{|x|\ge 2R}\int_{|y|\ge 2R}V(x-y)|u(x)|^2|u(y)|^2\,dy\,dx$. In other cases, since the integral is symmetric with respect to x and y, I is bounded by

$$6\sum_j\int\int_{|x|\ge R}\Big[\Big(2x_j - R\partial_j\varphi\Big(\frac{x}{R}\Big)\Big)-\Big(2y_j - R\partial_j\varphi\Big(\frac{y}{R}\Big)\Big)\Big]$$
$$\times\frac{x_j-y_j}{|x-y|^5}|u(x)|^2|u(y)|^2\,dx\,dy,$$

which is bounded by $c\int\int_{|x|\ge R}\frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,dx\,dy$. Thus, for a suitable radial function φ such that $\varphi''\le 2$, we have the following control:

$$A_R(u(t))\le c\Big(\frac{1}{R^2}\|u\|^2_{L^2(|x|>R)}+\|u\|^4_{L^V(|x|>R)}\Big). \tag{3.3}$$

The local virial identity will give another version of Proposition 3.1, for which, without the assumption of finite variance, we will assume that the solution is suitably localized in H^1 for all times.

PROPOSITION 3.2. *Let $M(u)=M(Q)$, $E(u)<E(Q)$ and suppose that the second case of Proposition* 2.2 *holds with $\lambda>1$ defined in* (2.7). *Select γ such that $0<\gamma<\min\{\lambda-1,1\}$. Suppose that there is a radius $R\ge\sqrt{c/6\gamma}$ such that for all t,*

$$\|u\|^4_{L^V(|x|\ge R)}<\frac{6\gamma E(Q)}{c}, \tag{3.4}$$

where the absolute constant c is determined in (3.3). *Let $\tilde{r}(t)$ be the scaled local variance given by*

$$\tilde{r}(t)=\frac{z_R(t)}{48\lambda^2(\lambda-1-\gamma)E(Q)}.$$

Then blow-up occurs in forward time before t_b, where

$$t_b=\tilde{r}'(0)+\sqrt{\tilde{r}'(0)^2+2\tilde{r}(0)}.$$

Proof. In view of the assumptions, by the local virial identity and the same steps as in the proof of Proposition 3.1,

$$\begin{aligned}\tilde{r}''(t) &= \frac{1}{48\lambda^2(\lambda-1-\gamma)E(Q)}\big(24E(u)-4\|\nabla u\|_2^2+A_R(u(t))\big)\\ &= \frac{1}{2\lambda^2(\lambda-1-\gamma)}\left(3\lambda^2-2\lambda^3-\eta(t)^2+\frac{A_R(u(t))}{24E(Q)}\right)\\ &\le \frac{3\lambda^2-2\lambda^3-\eta(t)^2}{2\lambda^2(\lambda-1-\gamma)}+\frac{\frac{c}{R^2}\|u\|^2_{L^2(|x|>R)}}{48E(Q)\lambda^2(\lambda-1-\gamma)}+\frac{c\|u\|^4_{L^V(|x|>R)}}{48E(Q)\lambda^2(\lambda-1-\gamma)}\\ &\le \frac{1}{2\lambda^2(\lambda-1-\gamma)}(3\lambda^2-2\lambda^3-\eta(t)^2+\gamma\eta(t)^2)\le 1.\end{aligned}$$

Finally, we complete our proof as in the proof of Proposition 3.1. ∎

Remark 3.3. Note that by the Hardy–Littlewood–Sobolev inequalities, Hölder estimates and Sobolev embeddings, the assumption (3.4) is satisfied by u which is H^1 bounded and H^1 localized, i.e. for any $\epsilon>0$ there exists $R>0$ large enough such that $\|u\|_{H^1(|x|\ge R)}\le\epsilon$.

We will finally give a quantified proof of finite-time blow-up for radial solutions, for which we need the following radial Sobolev embedding: If $u\in H^1(\mathbb{R}^d)$ is radially symmetric, then

$$\big\||x|^{(d-1)/2}u\big\|_\infty^2\le c\|u\|_2\|\nabla u\|_2. \tag{3.5}$$

Proposition 3.4. *Let $M(u)=M(Q)$, $E(u)<E(Q)$. Suppose u is radial and the second case of Proposition 2.2 holds with $\lambda>1$ defined in (2.7). Select γ such that $0<\gamma<\min\{\lambda-1,1\}$. Suppose that $R\ge\max\{\sqrt{c/6\gamma},(cE(Q)/12\gamma)^{5/4}\}$, where the absolute constant c is determined by those in (3.3) and (3.5). Let $\tilde{r}(t)$ be the scaled local variance given by*

$$\tilde{r}(t)=\frac{z_R(t)}{48\lambda^2(\lambda-1-\gamma)E(Q)}.$$

Then blow-up occurs in forward time before $t_b=\tilde{r}'(0)+\sqrt{\tilde{r}'(0)^2+2\tilde{r}(0)}$.

Proof. Again from the local virial identity,

$$\begin{aligned}\tilde{r}''(t) &= \frac{1}{48\lambda^2(\lambda-1-\gamma)E(Q)}(24E(u)-4\|\nabla u\|_2^2+A_R(u(t)))\\ &\le \frac{3\lambda^2-2\lambda^3-\eta(t)^2}{2\lambda^2(\lambda-1-\gamma)}+\frac{\frac{c}{R^2}\|u\|^2_{L^2(|x|>R)}}{48E(Q)\lambda^2(\lambda-1-\gamma)}\\ &\quad+\frac{c\|u\|^4_{L^V(|x|>R)}}{48E(Q)\lambda^2(\lambda-1-\gamma)}.\end{aligned}$$

The radial Sobolev embedding (3.5) implies that for any $p \geq 2$,

$$\|u\|^p_{L^p(|x|>R)} \leq \frac{c}{R^{2p-2}}\|u\|^{(p+2)/2}_{L^2(|x|>R)}\|\nabla u\|^{(p-2)/2}_{L^2(|x|>R)}.$$

This, combined with the Hardy–Littlewood–Sobolev inequalities and Hölder estimates, implies that

$$\begin{aligned}\|u\|^4_{L^V(|x|>R)} &\leq \|u\|^2_{L^{10/7}(\mathbb{R}^5)}\|u\|^2_{L^{10/7}(|x|>R)} \leq \frac{c}{R^{4/5}}\|u\|^{11/5}_{L^2(\mathbb{R}^5)}\|\nabla u\|^{9/5}_{L^2(\mathbb{R}^5)}\\ &\leq \frac{cE(Q)^2}{R^{(4)/5}}\eta(t)^2.\end{aligned}$$

Thus in view of the assumptions, we have

$$\tilde{r}''(t) \leq \frac{1}{2\lambda^2(\lambda-1-\gamma)}(3\lambda^2 - 2\lambda^3 - \eta(t)^2 + \gamma\eta(t)^2) \leq 1.$$

Arguing as in the proof of the preceding propositions we can complete our proof. ∎

4. Variational characterization of the ground state. In this section we deal with the variational characterization of Q defined in Section 2. This is an important preparation for the "near boundary case" in Section 5. Since time dependence plays no role in this section, we will write $u = u(x)$ from now on.

PROPOSITION 4.1. *There exists a function $\epsilon(\rho)$ with $\epsilon(\rho) \to 0$ as $\rho \to 0$ such that the following holds: Suppose there is $\lambda > 0$ satisfying*

$$\left|\frac{M(u)E(u)}{M(Q)E(Q)} - (3\lambda^2 - 2\lambda^3)\right| \leq \rho\lambda^3, \tag{4.1}$$

and

$$\left|\frac{\|u\|_2\|\nabla u\|_2}{\|Q\|_2\|\nabla Q\|_2} - \lambda\right| \leq \rho\begin{cases}\lambda, & \lambda \geq 1,\\ \lambda^2, & \lambda \leq 1.\end{cases} \tag{4.2}$$

Then there exist $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^5$ such that

$$\|u - e^{i\theta}\lambda^{5/2}\beta^{-2}Q(\lambda(\beta^{-1}\cdot - x_0))\|_2 \leq \beta^{1/2}\epsilon(\rho) \tag{4.3}$$

and

$$\|\nabla[u - e^{i\theta}\lambda^{5/2}\beta^{-2}Q(\lambda(\beta^{-1}\cdot - x_0))]\|_2 \leq \lambda\beta^{-1/2}\epsilon(\rho), \tag{4.4}$$

where $\beta = M(u)/M(Q)$.

REMARK 4.2. If we let $v(x) = \beta^2 u(\beta x)$, then $M(v) = \beta^{-1}M(u) = M(Q)$, and we can restate Proposition 4.1 as follows:

Suppose $\|v\|_2 = \|Q\|_2$ and there is $\lambda > 0$ such that

$$\left|\frac{E(v)}{E(Q)} - (3\lambda^2 - 2\lambda^3)\right| \leq \rho\lambda^3, \tag{4.5}$$

and

$$\left|\frac{\|\nabla v\|_2}{\|\nabla Q\|_2}-\lambda\right| \le \rho \begin{cases} \lambda, & \lambda \ge 1, \\ \lambda^2, & \lambda \le 1. \end{cases} \tag{4.6}$$

Then there exist $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^5$ such that

$$\|v - e^{i\theta}\lambda^{5/2}Q(\lambda(\cdot - x_0))\|_2 \le \epsilon(\rho) \tag{4.7}$$

and

$$\|\nabla[v - e^{i\theta}\lambda^{5/2}Q(\lambda(\cdot - x_0))]\|_2 \le \lambda\epsilon(\rho). \tag{4.8}$$

Thus it suffices to prove the scaled statement equivalent to Proposition 4.1. We will apply the following result of Lions [19].

LEMMA 4.3. *There exists a function $\epsilon(\rho)$, defined for small $\rho > 0$, such that* $\lim_{\rho\to 0}\epsilon(\rho) = 0$ *and for all $u \in H^1$ with*

$$\big|\|u\|_{L^V} - \|Q\|_{L^V}\big| + \big|\|u\|_2 - \|Q\|_2\big| + \big|\|\nabla u\|_2 - \|\nabla Q\|_2\big| \le \rho, \tag{4.9}$$

there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\|u - e^{i\theta_0}Q(\cdot - x_0)\|_{H^1} \le \epsilon(\rho). \tag{4.10}$$

Proof of Proposition 4.1. In view of Remark 4.2, we will prove the equivalent version rescaling the mass. Set $\tilde{u}(x) = \lambda^{-5/2}v(\lambda^{-1}x)$. Then (4.6) gives

$$\left|\frac{\|\nabla \tilde{u}\|_2}{\|\nabla Q\|_2} - 1\right| \le \rho. \tag{4.11}$$

On the other hand, (2.3), (4.5) and (4.6) imply

$$\begin{aligned} 2\left|\frac{\|v\|_{L^V}^4}{\|Q\|_{L^V}^4} - \lambda^3\right| &\le \left|\frac{E(v)}{E(Q)} - (2\lambda^3 - 3\lambda^2)\right| + 3\left|\frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \lambda^2\right| \\ &\le \rho\lambda^3 + 3\rho \begin{cases} \lambda^2, & \lambda \ge 1, \\ \lambda^4, & \lambda \le 1 \end{cases} \\ &\le 4\rho\lambda^3. \end{aligned}$$

Thus in terms of $\tilde{u}$, we obtain

$$\left|\frac{\|\tilde{u}\|_{L^V}^4}{\|Q\|_{L^V}^4} - 1\right| \le 2\rho. \tag{4.12}$$

Thus (2.20) and (4.12) imply that $\tilde{u}$ satisfies (4.9) (maybe with different ρ). By Lemma 4.3 and scaling back to v, we obtain (4.7) and (4.8). ∎

5. Near-boundary case. We know from Proposition 2.2 that if $M(u) = M(Q)$ and $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$ for some $\lambda > 1$ and $\|\nabla u_0\|_2/\|\nabla Q\|_2 \ge \lambda$, then $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \ge \lambda$ for all t. Now we claim that $\|\nabla u(t)\|_2/\|\nabla Q\|_2$ cannot remain near λ globally in time.

PROPOSITION 5.1. *Let $\lambda_0 > 1$. There exists $\rho_0 = \rho_0(\lambda_0) > 0$, with $\rho_0(\lambda_0) \to 0$ as $\lambda_0 \to 1$, such that for any $\lambda \geq \lambda_0$, the following holds: There does not exist a solution $u(t)$ of problem (1.1) with $P(u) = 0$ satisfying $M(u) = M(Q)$,*

$$\frac{E(u)}{E(Q)} = 3\lambda^2 - 2\lambda^3, \tag{5.1}$$

and for all $t \geq 0$,

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.2}$$

An equivalent statement is: For any solution $u(t)$ to (1.1) with $P(u) = 0$ satisfying $M(u) = M(Q)$, (5.1), and $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \geq \lambda$ for all $t \geq 0$, there exists a time $t_0 \geq 0$ such that $\|\nabla u(t_0)\|_2/\|\nabla Q\|_2 \geq \lambda(1 + \rho_0)$.

Before proving Proposition 5.1, following the idea of [2], we introduce a useful lemma.

LEMMA 5.2. *Suppose that $u(t)$ with $P(u) = 0$ solving (1.1) satisfies, for all t,*

$$\|u(t) - e^{i\theta(t)}Q(\cdot - x(t))\|_{H^1}^2 \leq \epsilon \tag{5.3}$$

for some continuous functions $\theta(t)$ and $x(t)$. Then if $\epsilon > 0$ is sufficiently small, we have

$$\frac{|x(t)|}{t} \leq c\epsilon \quad \text{as } t \to \infty.$$

Proof. If not, (5.3) holds for any small $\epsilon > 0$ while there exists a time sequence $t_n \to \infty$ such that $|x(t_n)|/t_n \geq \epsilon_0$ with some $\epsilon_0 > 0$. Without loss of generality we assume $x(0) = 0$. For $R > 0$ we define $t_0(R) = \inf\{t \geq 0 : |x(t)| \geq R\}$ and then by the continuity of $x(t)$ we find that 1) $t_0(R) > 0$; 2) $|x(t)| < R$ for $0 \leq t < t_0(R)$; and 3) $|x(t_0(R))| = R$. If we set $R_n = |x(t_n)|$ and $\tilde{t}_n = t_0(R_n)$, then $t_n \geq \tilde{t}_n$, which implies that $R_n/\tilde{t}_n \geq \epsilon_0$. From $|x(t_n)|/t_n \geq \epsilon_0$ and $t_n \to \infty$ we deduce that $R_n = |x(t_n)| \to \infty$. Thus, $\tilde{t}_n = t_0(R_n) \to \infty$. From now on, we will work on the time interval $[0, \tilde{t}_n]$ to get a contradiction.

For that purpose we need a uniform localization. That is, for any $\epsilon > 0$ there exists $R_0(\epsilon) \geq 0$ such that for all $t \geq 0$,

$$\int_{|x - x(t)| \geq R_0(\epsilon)} (|u|^2 + |\nabla u|^2)\, dx \leq 2\epsilon. \tag{5.4}$$

In fact, since the ground state Q is in H^1, there must exist $R(\epsilon) > 0$ such that

$$\int_{|x| \geq R(\epsilon)} (|Q|^2 + |\nabla Q|^2 + (V * |Q|^2)|Q|^2)\, dx \leq \epsilon. \tag{5.5}$$

Thus, taking $R_0(\epsilon) = R(\epsilon)$ we have

$$\begin{aligned}\int\limits_{|x-x(t)|\geq R_0(\epsilon)} &(|u|^2 + |\nabla u|^2)\,dx \\ &\leq \int \big(|u - e^{i\theta(t)}Q(\cdot - x(t))|^2 + |\nabla(u - e^{i\theta(t)}Q(\cdot - x(t)))|^2\big)\,dx \\ &\quad + \int\limits_{|x-x(t)|\geq R(\epsilon)} \big(|Q(\cdot - x(t))|^2 + |\nabla Q(\cdot - x(t))|^2\big)\,dx \leq 2\epsilon.\end{aligned}$$

Let $\theta \in C_c^\infty(\mathbb{R})$ be such that $\theta(x) = x$ for $-1 \leq x \leq 1$, $\theta(x) = 0$ for $|x| \geq 2^{1/3}$, $|\theta(x)| \leq |x|$, $\|\theta\|_\infty \leq 2$ and $\|\theta'\|_\infty \leq 4$. For $x \in \mathbb{R}^5$, let $\phi(x) = (\theta(x_1), \ldots, \theta(x_5))$; then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi\|_\infty \leq 2$. For $R > 0$, set $\phi_R(x) = R\phi(x/R)$. We consider the truncated center of mass: $z_R(t) = \int \phi_R(x)|u(x,t)|^2\,dx$ and $[z_R'(t)]_j = 2\operatorname{Im}\int \theta'(x_j/R)\partial_j u\bar{u}\,dx$.

By the zero momentum property we obtain $|z_R'(t)| \leq 5\int_{|x|\geq R}(|u|^2 + |\nabla u|^2)\,dx$. Setting $\tilde{R}_n = R_n + R_0(\epsilon)$, we have $|x - x(t)| \geq R_0(\epsilon)$ for $0 \leq t \leq \tilde{t}_n$ and $|x| > \tilde{R}_n$. Then by the uniform localization (5.4), we obtain

$$|z_{\tilde{R}_n}'(t)| \leq 5\epsilon. \tag{5.6}$$

Now we claim that

$$|z_{\tilde{R}_n}(0)| \leq R_0(\epsilon)M(u) + 2\tilde{R}_n\epsilon \tag{5.7}$$

and

$$|z_{\tilde{R}_n}(\tilde{t}_n)| \geq \tilde{R}_n(M(u) - 3\epsilon) - 2R_0(\epsilon)M(u). \tag{5.8}$$

In fact, first, the upper bound for $z_{\tilde{R}_n}(0)$ can be obtained from

$$z_{\tilde{R}_n}(0) = \int\limits_{|x|<R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u_0(x)|^2\,dx + \int\limits_{|x|\geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u_0(x)|^2\,dx$$

and (5.4) immediately. Next, we show the lower bound for $z_{\tilde{R}_n}(t)$ as follows. For $0 \leq t \leq \tilde{t}_n$, we split $z_{\tilde{R}_n}(t)$ into

$$\begin{aligned}z_{\tilde{R}_n}(t) &= \int\limits_{|x-x(t)|<R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u(x,t)|^2\,dx + \int\limits_{|x-x(t)|\geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u(x,t)|^2\,dx \\ &\equiv I + II.\end{aligned}$$

Again from (5.4), we obtain $|II| \leq 2\tilde{R}_n\epsilon$. For I, since $|x| \leq |x - x(t)| + |x(t)| \leq R_0(\epsilon) + R_n = \tilde{R}_n\epsilon$, we can write

$$\begin{aligned} I &= \int\limits_{|x-x(t)|<R_0(\epsilon)} (x-x(t))|u(x,t)|^2\,dx + x(t)\int\limits_{|x-x(t)|<R_0(\epsilon)} |u(x,t)|^2\,dx \\ &= \int\limits_{|x-x(t)|<R_0(\epsilon)} (x-x(t))|u(x,t)|^2\,dx + x(t)M(u) \\ &\quad - x(t)\int\limits_{|x-x(t)|\geq R_0(\epsilon)} |u(x,t)|^2\,dx \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Since $|I_1| \leq R_0(\epsilon)M(u)$, and since $|I_3| \leq |x(t)|\epsilon$ by (5.4), we have

$$|z_{\tilde{R}_n}(t)| \geq |I_2| - |I_1| - |I_3| - |II| \geq |x(t)|M(u) - R_0(\epsilon)M(u) - 3\tilde{R}_n\epsilon,$$

which gives (5.8) since $|x(\tilde{t}_n)| = R_n$.

Combining (5.6)–(5.8), we obtain

$$5\epsilon\tilde{t}_n \geq \left|\int_0^{\tilde{t}_n} z'_{\tilde{R}_n}(t)\,dt\right| \geq |z_{\tilde{R}_n}(\tilde{t}_n) - z_{\tilde{R}_n}(0)| \geq \tilde{R}_n(M(u)-5\epsilon) - 3R_0(\epsilon)M(u).$$

Thus assuming $\epsilon \leq M(u)/5$, since $\tilde{R}_n \geq R_n$ and $R_n/\tilde{t}_n \geq \epsilon_0$, we finally obtain

$$5\epsilon \geq \epsilon_0(M(u)-5\epsilon) - \frac{3R_0(\epsilon)M(u)}{\tilde{t}_n}.$$

If we take $\epsilon < M(u)\epsilon_0/20$ and let $n \to \infty$ (hence $\tilde{t}_n \to \infty$), we get a contradiction. ■

We shall prove Proposition 5.1 using the above lemma; our arguments, unlike those for the Schrödinger equation, will not use any exponential decay property of the ground state Q.

Proof of Proposition 5.1. Towards a contradiction, suppose that there exists a solution $u(t)$ satisfying $M(u) = M(Q)$, $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$ and

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1+\rho_0). \tag{5.9}$$

Since $\|\nabla u(t)\|_2^2 \geq \lambda^2\|\nabla Q\|_2^2 = 6\lambda^2 E(Q)$, we have

$$24E(u) - 4\|\nabla u(t)\|_2^2 \leq -48E(Q)\lambda^2(\lambda-1).$$

By Proposition 4.1, there exist functions $\theta(t)$ and $x(t)$ such that for $\rho = \rho_0$,

$$\|u(t) - e^{i\theta(t)}\lambda^{5/2}Q(\lambda(\cdot - x(t)))\|_2 \leq \epsilon(\rho) \tag{5.10}$$

and

$$\|\nabla[u(t) - e^{i\theta(t)}\lambda^{5/2}Q(\lambda(\cdot - x(t)))]\|_2 \leq \lambda\epsilon(\rho). \tag{5.11}$$

By the continuity of the $u(t)$ flow, we may assume $\theta(t)$ and $x(t)$ are continuous. Let

$$R(T) = \max\Big\{ \max_{0\le t\le T} |x(t)|, R(\epsilon(\rho))\Big\},$$

where $R(\epsilon(\rho))$ is given by (5.5) with $R(\epsilon(\rho)) \to \infty$ as $\rho \to 0$. For fixed T, take $R = 2R(T)$ in the local virial identity (3.2). Then we claim

$$|A_R(u(t))| \le c\lambda^3 \epsilon(\rho)^2.$$

In fact,

$$\begin{aligned}\|u\|_{L^V(|x|\ge R)} \le \|u - e^{i\theta(t)}\lambda^{5/2} Q(\lambda(\cdot - x(t)))\|_{L^V} \\ + \|e^{i\theta(t)}\lambda^{5/2} Q(\lambda(\cdot - x(t)))\|_{L^V(|x|\ge R)}.\end{aligned}$$

By the Hardy–Littlewood–Sobolev inequality (2.1), (5.10) and (5.11) imply

$$\|u - e^{i\theta(t)}\lambda^{5/2} Q(\lambda(\cdot - x(t)))\|_{L^V}^4 \le \lambda^3 \epsilon(\rho)^4.$$

On the other hand, by (5.5), we have

$$\begin{aligned}\|e^{i\theta(t)}\lambda^{5/2} Q(\lambda(\cdot - x(t)))\|_{L^V(|x|\ge R)}^4 &\le \|\lambda^{5/2} Q(\lambda(\cdot))\|_{L^V(|x|\ge R - \max_{0\le t\le T}|x(t)|)}^4 \\ &\le \|\lambda^{5/2} Q(\lambda(\cdot))\|_{L^V(|x|\ge R(T))}^4 \le \|\lambda^{5/2} Q(\lambda(\cdot))\|_{L^V(|x|\ge R(\epsilon(\rho)))}^4 \le \lambda^3\epsilon(\rho)^4.\end{aligned}$$

Similarly, $\|u\|_{L^2(|x|>R)}^2 \le c\epsilon(\rho)^2$. Thus (3.3) implies the claim.

Taking ρ_0 small enough to make $\epsilon(\rho)$ small we obtain, for all $0 \le t \le T$,

$$z_R''(t) \le -24E(Q)\lambda^2(\lambda - 1),$$

and so

$$\frac{z_R(T)}{T^2} \le \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - 12E(Q)\lambda^2(\lambda-1).$$

By definition of $z_R(t)$ we have

$$|z_R(0)| \le cR^2\|u_0\|_2^2 = c\|Q\|_2^2 R^2$$

and

$$|z_R'(0)| \le cR\|u_0\|_2\|\nabla u_0\|_2 \le c\|Q\|_2\|\nabla Q\|_2 R(1+\rho_0)\lambda.$$

Consequently,

$$\frac{z_{2R(T)}(T)}{T^2} \le c\bigg(\frac{R(T)^2}{T^2} + \frac{\lambda R(T)}{T}\bigg) - 12E(Q)\lambda^2(\lambda-1).$$

Taking T sufficiently large, from Lemma 5.2 we have

$$0 \le \frac{z_{2R(T)}(T)}{T^2} \le c(\lambda\epsilon(\rho)^2 - \lambda^2(\lambda-1)) < 0$$

provided ρ_0 is small enough.

Note that ρ_0 is independent of T. Thus, we get a contradiction. ∎

6. Profile decomposition. The following Keraani-type profile decomposition will play an important role in our future discussion.

LEMMA 6.1 (Profile expansion). *Let ϕ_n be a uniformly bounded sequence in H^1. Then for each M there exists a subsequence of ϕ_n, also denoted by ϕ_n, and*

(1) *for each $1 \le j \le M$, there exists a (fixed in n) profile $\tilde{\psi}^j(x)$ in H^1,*
(2) *for each $1 \le j \le M$, there exists a sequence (in n) of time shifts t_n^j,*
(3) *for each $1 \le j \le M$, there exists a sequence (in n) of space shifts x_n^j,*
(4) *there exists a sequence (in n) of remainders $\tilde{W}_n^M(x)$ in H^1,*

such that

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j \Delta} \tilde{\psi}^j(x - x_n^j) + \tilde{W}_n^M(x).$$

The time and space sequences have the pairwise divergence property, i.e., for $1 \le j \ne k \le M$, we have

$$\lim_{n\to\infty} (|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty. \tag{6.1}$$

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to\infty} \lim_{n\to\infty} \|e^{it\Delta} \tilde{W}_n^M\|_{S(\dot{H}^{1/2})} = 0. \tag{6.2}$$

For fixed M and any $0 \le s \le 1$, we have the asymptotic Pythagorean expansion:

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{M} \|\tilde{\psi}^j\|_{\dot{H}^s}^2 + \|\tilde{W}_n^M\|_{\dot{H}^s}^2 + o_n(1). \tag{6.3}$$

REMARK 6.2. By refining the subsequence for each j and using a standard diagonalization argument, we may assume that for each j the sequence t_n^j is convergent to some time in the compactified time interval $[-\infty, \infty]$. If t_n^j converges to some finite time $t^j \in (-\infty, \infty)$, we may shift $\tilde{\psi}^j$ by the linear propagator $e^{-it^j \Delta}$ to assume without loss of generality that t_n^j converges to either $-\infty$, 0, or ∞. If t_n^j converges to 0, we may absorb the error $e^{-it_n^j \Delta}\tilde{\psi}^j - \tilde{\psi}^j$ in the remainder $\tilde{W}_n^M$ without affecting significantly the scattering size of the linear evolution of $\tilde{W}_n^M$ and so assume, without loss of generality, that in this case $t_n^j \equiv 0$.

Since the profile decomposition corresponds to the linear equation and there is no difference in the linear terms between the Hartree equation and the Schrödinger equation, there is no essential difference in the proof from that in [2] for the 3D cubic Schrödinger equation. Furthermore, we also have the following energy expansion.

LEMMA 6.3 (Energy Pythagorean expansion). *Under the assumptions of Lemma* 6.1,

$$E(\phi_n) = \sum_{j=1}^{M} E(e^{-it_n^j\Delta}\tilde{\psi}^j) + E(\tilde{W}_n^M) + o_n(1). \tag{6.4}$$

Similar to [14] and [12], we give the following definition of a nonlinear profile:

DEFINITION 6.4. Let V be a solution to the linear Schrödinger equation. We say that U is a *nonlinear profile* associated to $(V, \{t_n\})$ if U is a solution to (1.1) satisfying

$$\|(U-V)(-t_n)\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Note that, similar to the arguments in [12], by the local theory and Lemma 2.5, there always exists a nonlinear profile associated to a given $(V, \{t_n\})$. In fact, it is obtained by solving (1.1) with $U(-t_0, x) = V(-t_0, x)$, where $t_0 = \lim_n t_n$. $V(-t_0, x)$ is an initial data if t_0 is finite, and an asymptotic state otherwise. Thus for every j, there exists a solution v^j to (1.1) associated to $(\tilde{\psi}^j, \{t_n^j\})$ such that

$$\|v^j(\cdot - x_n^j, -t_n^j) - e^{-it_n^j\Delta}\tilde{\psi}^j(\cdot - x_n^j)\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

If we denote the solution to (1.1) with the initial data ψ by $\mathrm{NLH}(t)\psi$, by shifting the linear profile $\tilde{\psi}^j$ when necessary, we may denote $v^j(-t_n^j)$ as $\mathrm{NLH}(-t_n^j)\psi^j$ with some $\psi^j \in H^1$. Thus using the same method of replacing linear flows by nonlinear flows as in [8] we can get the following proposition:

PROPOSITION 6.5. *Let ϕ_n be a uniformly bounded sequence in H^1. There exists a subsequence, also denoted by ϕ_n, profiles ψ^j in H^1, and parameters x_n^j, t_n^j such that for each M,*

$$\phi_n(x) = \sum_{j=1}^{M} \mathrm{NLH}(-t_n^j)\psi^j(x - x_n^j) + W_n^M(x), \tag{6.5}$$

where

- *For each j, either $t_n^j = 0$, $t_n^j \to \infty$ or $t_n^j \to -\infty$ as $n \to \infty$.*
- *If $t_n^j \to \infty$ as $n \to \infty$, then $\|\mathrm{NLH}(-t)\psi^j\|_{S(\dot{H}^{1/2};[0,\infty))} < \infty$; if $t_n^j \to -\infty$, then $\|\mathrm{NLH}(-t)\psi^j\|_{S(\dot{H}^{1/2};(-\infty,0])} < \infty$* [1].
- *For $j \neq k$,*

$$\lim_{n\to\infty} (|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty.$$

[1] This property is obtained by solving an asymptotic problem similar to that in the proof of the existence of the wave operator. In fact, we deduce further that $\|D^{1/2}\,\mathrm{NLH}(-t)\psi^j\|_{S(L^2;[0,\infty))} < \infty$ in the case of $t_n^j \to \infty$, and a similar result for $t_n^j \to -\infty$.

- $\mathrm{NLH}(t)W_n^M$ *is global for* M *large enough with*

$$\lim_{M\to\infty}\lim_{n\to\infty}\|\mathrm{NLH}(t)W_n^M\|_{S(\dot H^{1/2})}=0.$$

We also have the H^s*-Pythagorean decomposition: for fixed* M *and* $0\le s\le 1$,

$$\|\phi_n\|_{\dot H^s}^2=\sum_{j=1}^M\|\mathrm{NLH}(-t_n^j)\psi^j\|_{\dot H^s}^2+\|W_n^M\|_{\dot H^s}^2+o_n(1), \tag{6.6}$$

and, by energy conservation $E(\mathrm{NLH}(-t_n^j)\psi^j)=E(\psi^j)$, *the energy Pythagorean decomposition*

$$E(\phi_n)=\sum_{j=1}^M E(\psi^j)+E(W_n^M)+o_n(1). \tag{6.7}$$

REMARK 6.6. As is stated in [9], (6.7) was proven by first establishing the following orthogonal decomposition:

$$\|\phi_n\|_{L^V}^4=\sum_{j=1}^M\|\mathrm{NLH}(-t_n^j)\psi^j\|_{L^V}^4+\|W_n^M\|_{L^V}^4+o_n(1), \tag{6.8}$$

and we will find a similar one in the proof of Lemma 6.8.

The next perturbation lemma is essential to get our main theorem.

LEMMA 6.7 (Long time perturbation theory). *For any given* $A\gg 1$, *there exist* $\epsilon_0=\epsilon_0(A)\ll 1$ *and* $c=c(A)$ *such that the following statement holds: For fixed* $T>0$, *let* $u=u(x,t)\in L^\infty([0,T];H^1)$ *solve*

$$iu_t+\Delta u+(V*|u|^2)u=0$$

on $[0,T]$. *Let* $\tilde u=\tilde u(x,t)\in L^\infty([0,T];H^1)$ *and set*

$$e\equiv i\tilde u_t+\Delta\tilde u+(V*|\tilde u|^2)\tilde u.$$

For each $\epsilon\le\epsilon_0$, *if*

$\|\tilde u\|_{S(\dot H^{1/2};[0,T])}\le A$, $\|e\|_{S'(\dot H^{-1/2};[0,T])}\le\epsilon$, $\|e^{it\Delta}(u(0)-\tilde u(0)\|_{S(\dot H^{1/2};[0,T])}\le\epsilon$,

then

$$\|u-\tilde u\|_{S(\dot H^{1/2};[0,T])}\le c(A)\epsilon.$$

Proof. Define $w=u-\tilde u$. Then w solves the equation

$$iw_t+\Delta w+(V*|w+\tilde u|^2)w+(V*|w+\tilde u|^2)\tilde u-(V*|\tilde u|^2)\tilde u+e=0.$$

That is,

$$\begin{aligned}iw_t+\Delta w&+(V*|w|^2)w+(V*(\bar w\tilde u))w+(V*(w\bar{\tilde u}))w\\&+(V*|w|^2)\tilde u+(V*|\tilde u|^2)w+(V*(\bar w\tilde u))\tilde u+(V*(w\bar{\tilde u}))\tilde u+e=0.\end{aligned} \tag{6.9}$$

Since $\|\tilde u\|_{S(\dot H^{1/2};[0,T])}\le A$, we can split $[0,T]$ into $N=N(A)$ intervals $I_j=[t_j,t_{j+1})$ such that, for each $0\le j\le N-1$, $\|\tilde u\|_{S(\dot H^{1/2};I_j)}<\delta$ with a sufficiently small δ to be specified later. The integral equation of (6.9) with the initial time t_j is

$$w(t) = e^{i(t-t_j)\Delta} w(t_j) + i \int_{t_j}^{t} e^{i(t-s)\Delta} W(\cdot, s)\, ds, \tag{6.10}$$

where

$$W = (V * |w|^2)w + (V * (\bar{w}\tilde{u}))w + (V * (w\bar{\tilde{u}}))w + (V * |w|^2)\tilde{u} + (V * |\tilde{u}|^2)w + (V * (\bar{w}\tilde{u}))\tilde{u} + (V * (w\bar{\tilde{u}}))\tilde{u} + e.$$

Applying the Kato–Strichartz estimate (2.14) on I_j we have

$$\begin{aligned} \|w\|_{S(\dot{H}^{1/2};I_j)} \le{}& \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^{1/2};I_j)} + c\|(V * |w|^2)w\|_{L_{I_j}^{24/13} L_x^{12/7}} \\ &+ c\|(V * (\bar{w}\tilde{u}))w\|_{L_{I_j}^{24/13} L_x^{12/7}} + c\|(V * (w\bar{\tilde{u}}))w\|_{L_{I_j}^{24/13} L_x^{12/7}} \\ &+ c\|(V * |w|^2)\tilde{u}\|_{L_{I_j}^{24/13} L_x^{12/7}} + c\|(V * |\tilde{u}|^2)w\|_{L_{I_j}^{24/13} L_x^{12/7}} \\ &+ c\|(V * (\bar{w}\tilde{u}))\tilde{u}\|_{L_{I_j}^{24/13} L_x^{12/7}} + c\|(V * (w\bar{\tilde{u}}))\tilde{u}\|_{L_{I_j}^{24/13} L_x^{12/7}}. \end{aligned} \tag{6.11}$$

In fact, we can easily check that $(24/13, 12/7) \in \Lambda'_{1/2}$ and $(24/5, 60/19)$, $(8, 20/7) \in \Lambda_{1/2}$. And by the Hardy–Littlewood–Sobolev inequalities and Hölder estimates we have

$$\begin{aligned} \|(V * |\tilde{u}|^2)w\|_{L_{I_j}^{24/13} L_x^{12/7}} &\le \|\tilde{u}\|^2_{L_{I_j}^{24/5} L_x^{60/19}} \|w\|_{L_{I_j}^{8} L_x^{20/7}} \\ &\le \|\tilde{u}\|^2_{S(\dot{H}^{1/2};I_j)} \|w\|_{S(\dot{H}^{1/2};I_j)} \le \delta^2 \|w\|_{S(\dot{H}^{1/2};I_j)}, \\ \|(V * |w|^2)\tilde{u}\|_{L_{I_j}^{24/13} L_x^{12/7}} &\le \|w\|^2_{L_{I_j}^{24/5} L_x^{60/19}} \|\tilde{u}\|_{L_{I_j}^{8} L_x^{20/7}} \le \delta \|w\|^2_{S(\dot{H}^{1/2};I_j)}. \end{aligned}$$

Similarly, we can estimate other terms in (2.17) and get

$$\begin{aligned} \|w\|_{S(\dot{H}^{1/2};I_j)} &\le \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^{1/2};I_j)} + c\delta^2 \|w\|_{S(\dot{H}^{1/2};I_j)} \\ &\quad + c\delta \|w\|^2_{S(\dot{H}^{1/2};I_j)} + c\|w\|^3_{S(\dot{H}^{1/2};I_j)} + c\|e\|_{S'(\dot{H}^{-1/2};I_j)} \\ &\le \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^{1/2};I_j)} + c\delta^2 \|w\|_{S(\dot{H}^{1/2};I_j)} \\ &\quad + c\delta \|w\|^2_{S(\dot{H}^{1/2};I_j)} + c\|w\|^3_{S(\dot{H}^{1/2};I_j)} + c\epsilon. \end{aligned} \tag{6.12}$$

Now if $\delta \le \min(1, 1/6c)$ and

$$\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^{1/2};I_j)} + c\epsilon \le \min\left\{1, \frac{1}{2\sqrt{6c}}\right\}, \tag{6.13}$$

we obtain

$$\|w\|_{S(\dot{H}^{1/2};I_j)} \le 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^{1/2};I_j)} + 2c\epsilon. \tag{6.14}$$

Next, taking $t = t_j$ in (6.10) and applying $e^{i(t-t_{j+1})\Delta}$ to both sides, we obtain

$$e^{i(t-t_{j+1})\Delta}w(t_{j+1}) = e^{i(t-t_j)\Delta}w(t_j) + i\int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta}W(\cdot,s)\,ds. \tag{6.15}$$

Noting that the Duhamel integral is confined to I_j, similarly to (6.12) we have the estimate

$$\begin{aligned}\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{S(\dot H^{1/2};[0,T])} &\\ \le \|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\dot H^{1/2};[0,T])} &+ c\delta^2\|w\|_{S(\dot H^{1/2};I_j)}\\ + c\delta\|w\|_{S(\dot H^{1/2};I_j)} &+ c\|w\|^3_{S(\dot H^{1/2};I_j)} + c\epsilon.\end{aligned}$$

Then (2.19) and (6.14) imply

$$\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{S(\dot H^{1/2};[0,T])} \le 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\dot H^{1/2};[0,T])} + 2c\epsilon.$$

Now, beginning with $j = 0$ we get by iteration

$$\begin{aligned}\|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\dot H^{1/2};[0,T])} &\le 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{S(\dot H^{1/2};[0,T])} + (2^j-1)2c\epsilon\\ &\le 2^{j+2}c\epsilon.\end{aligned}$$

Since the second part of (2.19) is needed for each I_j, $0 \le j \le N-1$, we require

$$2^{N+2}c\epsilon_0 \le \min\left\{1, \frac{1}{2\sqrt{6c}}\right\}. \tag{6.16}$$

Recall that δ is an absolute constant satisfying (2.19); the number of intervals N is determined by the given A; and then by (6.16), ϵ_0 is determined by $N = N(A)$. Thus, the iteration completes our proof. ∎

Note that from the proof above the parameters in Lemma 6.7 are independent of T. As is stated in [9], besides the H^1 asymptotic orthogonality (6.6) at $t = 0$, this property can be extended to the nonlinear flow for $0 \le t \le T$ as an application of Lemma 6.7 with a constant $A = A(T)$ depending on T (but only through A). As for the Hartree equation (1.1), we will show a similar result:

LEMMA 6.8. *Let ϕ_n be a uniformly bounded sequence in H^1. Fix any time $0 < T < \infty$. Suppose that $u_n(t) \equiv \mathrm{NLH}(t)\phi_n$ exists up to time T for all n and*

$$\lim_{n\to\infty}\|\nabla u_n(t)\|_{L^\infty([0,T];L^2)} < \infty.$$

Let $W_n^M(t) \equiv \mathrm{NLH}(t)W_n^M$. Then, for all j, $v^j(t) \equiv \mathrm{NLH}(t)\psi^j$ exists up to *time* T *and for all* $t \in [0,T]$,

$$(6.17)\qquad \|\nabla u_n\|_2^2 = \sum_{j=1}^{M} \|\nabla v^j(t-t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2 + o_n(1),$$

where $o_n(1) \to 0$ *uniformly for* $0 \le t \le T$.

Proof. Let M_0 be such that for $M \ge M_0$ and for $\delta_{\rm sd}$ in Lemma 2.3, we have

$$\|\mathrm{NLH}(t)W_n^M\|_{S(\dot H^{1/2})} \le \delta_{\rm sd}/2$$

and $\|v^j\|_{S(\dot H^{1/2})} \le \delta_{\rm sd}$ for $j > M_0$. Reorder the first M_0 profiles and introduce an index M_2, $0 \le M_2 \le M_0$, such that:

- For each $0 \le j \le M_2$ we have $t_n^j = 0$ (there is no j in this case if $M_2 = 0$).
- For each $M_2 + 1 \le j \le M_0$ we have $|t_n^j| \to \infty$ (there is no j in this case if $M_2 = M_0$).

By the definition of M_0, $v^j(t)$ for $j > M_0$ scatters in both time directions. We claim that for fixed T and $M_2+1 \le j \le M_0$, $\|v^j(t-t_n^j)\|_{S(\dot H^{1/2};[0,T])} \to 0$ as $n \to \infty$. Indeed, take the case $t_n^j \to \infty$ for example. By Proposition 6.5, $\|v^j(-t)\|_{S(\dot H^{1/2};[0,\infty))} < \infty$. Then for $q < \infty$, $\|v^j(-t)\|_{L^q([0,\infty);L^r)} < \infty$ implies $\|v^j(t-t_n^j)\|_{L^q([0,T];L^r)} \to 0$. On the other hand, since $v^j(t)$ in Proposition 6.5 is constructed from the existence of wave operators which converge in H^1 to a linear flow at $-\infty$, the $L^{5/2}$ decay of the linear flow implies immediately that $\|v^j(t-t_n^j)\|_{L^\infty([0,T];L^{5/2})} \to 0$. Similarly, we can deduce that for $M_2+1 \le j \le M_0$, $\|D^{1/2}v^j(t-t_n^j)\|_{S(L^2;[0,T])} \to 0$ as $n \to \infty$.

Let $B = \max\{1, \lim_n \|\nabla u_n\|_{L^\infty([0,T];L^2)}\}$. For each $1 \le j \le M_2$, define $T^j \le T$ to be the maximal forward time for which $\|\nabla v^j\|_{L^\infty([0,T^j];L^2)} \le 2B$. Let $\widetilde T = \min_{1\le j\le M_2} T^j$, and if $M_2 = 0$, just take $\widetilde T = T$. Note that if we prove (6.17) for $T = \widetilde T$, then by the definition of T^j, using continuity arguments, it follows from (6.17) that for each $1 \le j \le M_2$, we have $T^j = T$. Hence $\widetilde T = T$. Thus, for the remainder of the proof, we just work on $[0,\widetilde T]$.

For each $1 \le j \le M_2$, $\|v^j\|_{L^\infty([0,\widetilde T];L^2)} = \|\psi^j\|_2 \le \lim_n \|\phi_n\|_2$ by (6.6), thus we have

$$\begin{aligned}(6.18)\qquad &\|v^j(t)\|_{S(\dot H^{1/2};[0,\widetilde T])}\\ &\le c(\|v^j\|_{L^\infty([0,\widetilde T];L^{5/2})} + \|v^j\|_{L^4([0,\widetilde T];L^{10/3})})\\ &\le c(\|v^j\|^{1/2}_{L^\infty([0,\widetilde T];L^2)}\|\nabla v^j\|^{1/2}_{L^\infty([0,\widetilde T];L^2)} + \widetilde T^{1/4}\|\nabla v^j\|_{L^\infty([0,\widetilde T];L^2)})\\ &\le c(1+\widetilde T^{1/4})B.\end{aligned}$$

In fact, from the local theory (see Chapter 4 in [1]), we further deduce that for each $1 \le j \le M_2$,

$$\|D^{1/2}v^j(t)\|_{S(L^2;[0,\widetilde{T}])} \le C(\widetilde{T}, B). \tag{6.19}$$

For a fixed M, let

$$\tilde{u}_n(x,t) = \sum_{j=1}^{M} v^j(x - x_n^j, t - t_n^j),$$
$$e_n = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + (V * |\tilde{u}_n|^2)\tilde{u}_n.$$

Claim 1. *There exists $A = A(\widetilde{T})$ (independent of M) such that for all $M > M_0$, there exists $n_0 = n_0(M)$ such that for all $n > n_0$,*

$$\|\tilde{u}_n\|_{S(\dot{H}^{1/2};[0,\widetilde{T}])} \le A.$$

Claim 2. *For each $M > M_0$ and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that for $n > n_1$ and for some $\dot{H}^{-1/2}$ admissible (q, r),*

$$\|e_n\|_{L^{q'}([0,\widetilde{T}];L^{r'})} \le \epsilon.$$

We postpone the proof of those claims and just suppose they both hold. Since $u_n(0) - \tilde{u}_n(0) = W_n^M$, there exists $M' = M'(\epsilon)$ large enough such that for each $M > M'$ there exists $n_2 = n_2(M')$ such that for $n > n_2$,

$$\|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{S(\dot{H}^{1/2};[0,\widetilde{T}])} \le \epsilon.$$

For $A = A(\widetilde{T})$ in the first claim, Lemma 6.7 gives $\epsilon_0 = \epsilon_0(A) \ll 1$. We select an arbitrary $\epsilon \le \epsilon_0$ and obtain from the above arguments an index $M' = M'(\epsilon)$. Now select an arbitrary $M > M'$, and set $n' = \max(n_0, n_1, n_2)$. Then by Lemma 6.7 and the above arguments, for $n > n'$, we have

$$\|u_n - \tilde{u}_n\|_{S(\dot{H}^{1/2};[0,\widetilde{T}])} \le c(\widetilde{T})\epsilon. \tag{6.20}$$

To obtain the $\|\nabla \tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^2)}$ bound, we also have to discuss $j \ge M_2 + 1$. As was noted in the first paragraph of the proof, $\|v^j(t - t_n^j)\|_{S(\dot{H}^{1/2};[0,\widetilde{T}])} \to 0$ as $n \to \infty$. By the Strichartz estimate we can get $\|\nabla v^j(t - t_n^j)\|_{L^\infty([0,\widetilde{T}];L^2)} \le c\|\nabla v^j(-t_n^j)\|_2$. By the pairwise divergence of parameters,

$$\begin{aligned}
&\|\nabla \tilde{u}_n\|^2_{L^\infty([0,\widetilde{T}];L^2)} \\
&\quad = \sum_{j=1}^{M_2} \|\nabla v^j(t)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + \sum_{M_2+1}^{M} \|\nabla v^j(t - t_n^j)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + o_n(1) \\
&\quad \le c\Big(M_2 B^2 + \sum_{M_2+1}^{M} \|\nabla \, \mathrm{NLH}(-t_n^j)\psi^j\|_2^2 + o_n(1)\Big) \\
&\quad \le c(M_2 B^2 + \|\nabla \phi_n\|_2^2 + o_n(1)) \le c(M_2 B^2 + B^2 + o_n(1)).
\end{aligned}$$

Note that for $5/2 < p < 10/3$, from (6.20) we have, for some $0 < \theta < 1$,

$$\|u_n - \tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^p)} \le c(\|u_n - \tilde{u}_n\|^{\theta}_{L^\infty([0,\widetilde{T}];L^{5/2})} \|\nabla(u_n - \tilde{u}_n)\|^{1-\theta}_{L^\infty([0,\widetilde{T}];L^2)}) \\ \le c(\widetilde{T})^\theta (M_2 B^2 + B^2 + o_n(1))^{(1-\theta)/2} \epsilon^\theta.$$

Thus, by the Hardy–Littlewood–Sobolev inequalities and Hölder estimates we obtain

$$\sup_{t\in[0,\widetilde{T}]} \|u_n - \tilde{u}_n\|^4_{L^V} \le c(\widetilde{T})^2 (M_2 B^2 + B^2 + o_n(1))\epsilon^2. \tag{6.21}$$

Now we first replace the large parameter M in the notation $\tilde{u}_n$ and all other arguments above by M_1. Then for any fixed M, we will prove (6.17) on $[0,\widetilde{T}]$. In fact, we only need to establish that, for each $t \in [0,\widetilde{T}]$,

$$\|u_n\|^4_{L^V} = \sum_{j=1}^{M} \|v^j(t - t^j_n)\|^4_{L^V} + \|W^M_n(t)\|^4_{L^V} + o_n(1). \tag{6.22}$$

By (6.7) and the energy conservation we have

$$E(u_n(t)) = \sum_{j=1}^{M} E(v^j(t - t^j_n)) + E(W^M_n(t)) + o_n(1). \tag{6.23}$$

Thus (6.22) combined with (6.23) gives (6.17), which completes our proof. So now we have to establish (6.22)

First, we apply the perturbation Lemma 6.7 to $u_n(t) = W^M_n(t)$ and $\tilde{u}_n = \sum_{j=M+1}^{M_1} v^j(t-t^j_n)$. For any fixed $M < M_1$, since $u_n(0) - \tilde{u}_n(0) = W^{M_1}_n$, similar to the above two claims and the arguments above, we obtain

$$\Big\| W^M_n(t) - \sum_{j=M+1}^{M_1} v^j(t - t^j_n) \Big\|^4_{L^V} \to 0 \quad \text{as } n \to \infty.$$

From all arguments above and by the pairwise divergence of parameters,

$$\begin{aligned} \|u_n\|^4_{L^V} = \|\tilde{u}_n\|^4_{L^V} + o_n(1) &= \Big\| \sum_{j=1}^{M_1} v^j(t - t^j_n) \Big\|^4_{L^V} + o_n(1) \\ &= \sum_{j=1}^{M} \|v^j(t - t^j_n)\|^4_{L^V} + \Big\| \sum_{j=M+1}^{M_1} v^j(t - t^j_n) \Big\|^4_{L^V} + o_n(1) \\ &= \sum_{j=1}^{M} \|v^j(t - t^j_n)\|^4_{L^V} + \|W^M_n(t)\|^4_{L^V} + o_n(1). \end{aligned}$$

If on the other hand $M \ge M_1$, we easily deduce from the selection of M_1 (see the above analysis) that $\|W^M_n(t)\|_{L^V} = o_n(1)$ and (6.21) implies (6.22).

Now it remains to establish the two claims. Recall that M_0 is sufficiently large such that $\|e^{it\Delta}W_n^{M_0}\|_{S(\dot H^{1/2})} \le \delta_{\rm sd}/2$ and for each $j > M_0$, we have $\|e^{it\Delta}v^j(-t_n^j)\|_{S(\dot H^{1/2})} \le \delta_{\rm sd}$. Similar to the small data scattering and Proposition 2.5, we obtain

$$\|v^j(t-t_n^j)\|_{S(\dot H^{1/2})} \le 2\|e^{it\Delta}v^j(-t_n^j)\|_{S(\dot H^{1/2})} \le 2\delta_{\rm sd}, \tag{6.24}$$

and

$$\|D^{1/2}v^j(t-t_n^j)\|_{S(L^2)} \le c\|v^j(-t_n^j)\|_{\dot H^{1/2}} \quad \text{for } j > M_0. \tag{6.25}$$

Thus by the elementary inequality, for $a_j > 0$,

$$\Big|\Big|\sum_{j=1}^M a_j\Big|^{7/2} - \sum_{j=1}^M |a_j|^{7/2}\Big| \le C_M \sum_{j\ne k} |a_j|\,|a_k|^{5/2},$$

we have

$$\begin{aligned}
\|\tilde u_n\|^{7/2}_{L^{7/2}([0,\widetilde T];L^{7/2})} \tag{6.26}
&\le \sum_{j=1}^{M_2} \|v^j\|^{7/2}_{L^{7/2}([0,\widetilde T];L^{7/2})} + \sum_{j=M_2+1}^{M_0} \|v^j(t-t_n^j)\|^{7/2}_{L^{7/2}([0,\widetilde T];L^{7/2})}\\
&\quad + \sum_{j=M_0+1}^{M} \|v^j(t-t_n^j)\|^{7/2}_{L^{7/2}([0,\widetilde T];L^{7/2})} + \text{crossterms}\\
&\le \sum_{j=1}^{M_2} \|D^{1/2}v^j\|^{7/2}_{S(L^2;[0,\widetilde T])} + \sum_{j=M_2+1}^{M_0} \|D^{1/2}v^j(t-t_n^j)\|^{7/2}_{S(L^2;[0,\widetilde T])}\\
&\quad + \sum_{j=M_0+1}^{M} \|D^{1/2}v^j(t-t_n^j)\|^{7/2}_{S(L^2;[0,\widetilde T])} + \text{crossterms}\\
&\le M_0 C(\widetilde T, B) + M_0\epsilon^{7/2} + c\sum_{j=M_0+1}^{M} \|v^j(-t_n^j)\|^{7/2}_{\dot H^{1/2}} + \text{crossterms},
\end{aligned}$$

where we have used (6.19) and the fact that

$$\sum_{j=M_2+1}^{M_0} \|D^{1/2}v^j(t-t_n^j)\|^{7/2}_{S(L^2;[0,\widetilde T])} \le M_0\epsilon^{7/2}$$

for M_2 large enough. Now by (6.6),

$$\begin{aligned}
\|u_{n,0}\|^2_{\dot H^{1/2}} &= \sum_{j=1}^{M_0} \|v^j(-t_n^j)\|^2_{\dot H^{1/2}} + \sum_{j=M_0+1}^{M} \|v^j(-t_n^j)\|^2_{\dot H^{1/2}} \tag{6.27}\\
&\quad + \|W_n^M\|^2_{\dot H^{1/2}} + o_n(1),
\end{aligned}$$

we know that the quantity

$$\sum_{j=M_0+1}^{M} \|v^j(-t_n^j)\|_{\dot H^{1/2}}^2 \quad \text{and so} \quad \sum_{j=M_0+1}^{M} \|v^j(-t_n^j)\|_{\dot H^{1/2}}^{7/2}$$

is bounded independently of M provided $n > n_0$ is sufficiently large. On the other hand, the "crossterms" can also be made bounded by taking n_0 large enough owing to the pairwise divergence of parameters. Above all, we have shown that $\|\tilde u_n\|_{L^{7/2}([0,\widetilde T];L^{7/2})}$ is bounded independently of M for $n > n_0$. A similar argument shows that $\|\tilde u_n\|_{L^\infty([0,\widetilde T];L^{5/2})}$ is also bounded independently of M for $n > n_0$, and the first claim holds true since the Strichartz norm $\|\tilde u_n\|_{S(\dot H^{1/2};[0,\widetilde T])}$ can be bounded by interpolation between the time-space norms with the above two exponents.

Now we prove the second claim. We easily find the following expansion of e_n which consists of $O(M^3)$ terms involving $V * |v^j(t-t_n^j)|^2 v^k(t-t_n^k)$, $k \neq j$ (we will call such terms *crossterms*):

$$\begin{aligned} e_n &= \Big(V * \Big|\sum_{j=1}^{M} v^j(t-t_n^j)\Big|^2\Big) \sum_{j=1}^{M} v^j(t-t_n^j) \\ &\quad - \sum_{j=1}^{M} (V * |v^j(t-t_n^j)|^2) v^j(t-t_n^j) \\ &= \Big(V * \Big(\Big|\sum_{j=1}^{M} v^j(t-t_n^j)\Big|^2 - \sum_{j=1}^{M} |v^j(t-t_n^j)|^2\Big)\Big) \sum_{j=1}^{M} v^j(t-t_n^j) \\ &\quad + \sum_{j=1}^{M} (V * |v^j(t-t_n^j)|^2) \sum_{k\neq j} v^k(t-t_n^k). \end{aligned}$$

The point is to estimate the crossterms. Assume first that $j \neq k$ and $|t_n^j - t_n^k| \to \infty$; then at least one index is $\geq M_2 + 1$. Take the Strichartz estimate of one of the crossterms:

$$\begin{aligned} &\|(V * |v^j|^2)(t-t_n^j) v^k(t-t_n^k)\|_{L^{24/13}([0,\widetilde T];L^{12/7})} \\ &\qquad = \|(V * |v^j|^2)(t) v^k(t+t_n^j-t_n^k)\|_{L^{24/13}([0,\widetilde T];L^{12/7})}. \end{aligned}$$

Similar to the analysis in the second paragraph, this term goes to zero since $v^j, v^k \in L_t^{24/5} L_x^{60/19} \cap L_t^8 L_x^{20/7}$ and

$$\begin{aligned} &\|(V * |v^j|^2)(t) v^k(t+t_n^j-t_n^k)\|_{L^{24/13}([0,\widetilde T];L^{12/7})} \\ &\qquad \leq \|v^j\|_{L^{24/5}([0,\widetilde T];L^{60/19})}^2 \|v^k(t+t_n^j-t_n^k)\|_{L^8([0,\widetilde T];L^{20/7})}. \end{aligned}$$

If $j \neq k$ and $t_n^j = t_n^k$, then by (6.1), $|x_n^j - x_n^k| \to \infty$. Taking the same crossterm, we have

$$\begin{aligned}\left\| \int \frac{|v^j(y - x_n^j)|^2 v^k(x - x_n^k)}{|x-y|^3}\, dy \right\|_{L^{24/13}([0,\widetilde{T}];L^{12/7})} &= \left\| \int \frac{|v^j(y')|^2 v^k(x - x_n^k)}{|x - x_n^j - y'|^3}\, dy \right\|_{L^{24/13}([0,\widetilde{T}];L^{12/7})} \\ &= \left\| \int \frac{|v^j(y')|^2 v^k(x' + x_n^j - x_n^k)}{|x'-y'|^3}\, dy \right\|_{L^{24/13}([0,\widetilde{T}];L^{12/7})} \\ &= \|(V * |v^j|^2) v^k(\cdot + x_n^j - x_n^k)\|_{L^{24/13}([0,\widetilde{T}];L^{12/7})}.\end{aligned}$$

In the same way, we infer that it must go to zero again. Observe that all other crossterms will have the same property through similar estimates, and in fact we have proved the second claim. ∎

LEMMA 6.9 (Profile reordering). *Let ϕ_n be a bounded sequence in H^1 and let $\lambda_0 > 1$. Suppose that $M(\phi_n) = M(Q)$, $E(\phi_n)/E(Q) = 3\lambda_n^2 - 2\lambda_n^3$ with $\lambda_n \geq \lambda_0 > 1$ and $\|\nabla \phi_n\|_2 / \|\nabla Q\|_2 \geq \lambda_n$ for each n. Then, for a given M, the profiles can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ satisfying:*

(1) *For each $1 \leq j \leq M_1$, we have $t_n^j = 0$ and $v^j(t) \equiv \mathrm{NLH}(t)\psi^j$ does not scatter as $t \to \infty$. (We in fact assert that at least one j belongs to this category.)*
(2) *For each $M_1 + 1 \leq j \leq M_2$, we have $t_n^j = 0$ and $v^j(t)$ scatters as $t \to \infty$. (There is no j in this category if $M_2 = M_1$.)*
(3) *For each $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. (There is no j in this category if $M_2 = M$.)*

Proof. Firstly, we claim that there exists at least one j such that t_n^j converges as $n \to \infty$. In fact,

$$\begin{aligned}\frac{\|\phi_n\|_{L^V}^4}{\|Q\|_{L^V}^4} &= -\frac{1}{2}\frac{E(\phi_n)}{E(Q)} + \frac{3}{2}\frac{\|\nabla \phi_n\|_2^2}{\|\nabla Q\|_2^2} \\ &\geq -\frac{1}{2}(3\lambda_n^2 - 2\lambda_n^3) + \frac{3}{2}\lambda_n^2 = \lambda_n^3 \geq \lambda_0^3 > 1.\end{aligned} \tag{6.28}$$

If $|t_n^j| \to \infty$, then $\|\mathrm{NLH}(-t_n^j)\psi^j\|_{L^V} \to 0$ and (6.8) implies our conclusion. Now if j is such that t_n^j converges as $n \to \infty$, then we might as well assume $t_n^j = 0$.

Reorder the profiles ψ^j so that for $1 \leq j \leq M_2$, we have $t_n^j = 0$, and for $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. It remains to show that there exists one j, $1 \leq j \leq M_2$, such that $v^j(t)$ does not scatter as $t \to \infty$. If, on the contrary, $v^j(t)$ scatters for all $1 \leq j \leq M_2$, then $\lim_{t\to\infty} \|v^j(t)\|_{L^V} = 0$. Let t_0 be sufficiently large so that for all $1 \leq j \leq M_2$, we have $\|v^j(t_0)\|_{L^V}^4 \leq \epsilon / M_2$.

The L^V orthogonality (6.22) along the NLH flow and an argument as in (6.28) imply

$$\begin{aligned}\lambda_0^3\|Q\|_{L^V}^4 &\le \|u_n(t_0)\|_{L^V}^4 \\ &= \sum_{j=1}^{M_2}\|v^j(t_0)\|_{L^V}^4 + \sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{L^V}^4 + \|W_n^M(t_0)\|_{L^V}^4 + o_n(1).\end{aligned}$$

We know from Proposition 6.5 that $\sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{L^V}^4 \to 0$ as $n\to\infty$, and thus

$$\lambda_0^3\|Q\|_{L^V}^4 \le \epsilon + \|W_n^M(t_0)\|_{L^V}^4 + o_n(1).$$

This gives a contradiction since $W_n^M(t)$ is a scattering solution. ∎

7. Recurrence argument and existence of a critical solution. We now begin to prove Theorem 1.1. By Remark 1.2 we only need to deal with the case $P(u)=0$. We will use the notation from [9] and give some definitions first.

Definition 7.1. Let $\lambda>1$. We say that $\exists\,\mathrm{GB}(\lambda,\sigma)$ holds if there exists a solution $u(t)$ to
(1.1) such that

$$P(u)=0,\qquad M(u)=M(Q),\qquad \frac{E(u)}{E(Q)}=3\lambda^2-2\lambda^3$$

and

$$\lambda\le\frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2}\le\sigma\qquad\text{for all } t\ge 0.$$

$\exists\,\mathrm{GB}(\lambda,\sigma)$ means that there exist solutions with energy $3\lambda^2-2\lambda^3$ globally bounded by σ. Thus by Proposition 5.1, $\exists\,\mathrm{GB}(\lambda,\lambda(1+\rho_0(\lambda_0)))$ is false for all $\lambda\ge\lambda_0>1$.

$\exists\,\mathrm{GB}(\lambda,\sigma)$ false is equivalent to saying that for every solution $u(t)$ of (1.1) with $M(u)=M(Q)$ and $E(u)/E(Q)=3\lambda^2-2\lambda^3$ such that $\|\nabla u(t)\|_2/\|\nabla Q\|_2\ge\lambda$ for all t, there exists a time $t_0\ge 0$ such that $\|\nabla u(t_0)\|_2/\|\nabla Q\|_2\ge\sigma$. By resetting the initial time, we can find a sequence $t_n\to\infty$ such that $\|\nabla u(t_n)\|_2/\|\nabla Q\|_2\ge\sigma$ for all n.

Note that if $\lambda\le\sigma_1\le\sigma_2$, then $\exists\,\mathrm{GB}(\lambda,\sigma_2)$ false implies $\exists\,\mathrm{GB}(\lambda,\sigma_1)$ false. We will induct on the statement and define a threshold.

Definition 7.2 (The critical threshold). Fix $\lambda_0>1$. Let $\sigma_c=\sigma_c(\lambda_0)$ be the supremum of all $\sigma>\lambda_0$ such that $\exists\,\mathrm{GB}(\lambda,\sigma)$ is false for all λ such that $\lambda_0\le\lambda\le\sigma$.

Proposition 5.1 implies that $\sigma_c(\lambda_0)>\lambda_0$. Let $u(t)$ be any solution to (1.1) with $P(u)=0$, $M(u)=M(Q)$, $E(u)/E(Q)\le 3\lambda_0^2-2\lambda_0^3$ and $\|\nabla u(0)\|_2/\|\nabla Q\|_2>1$. If $\lambda_0>1$ and $\sigma_c=\infty$, we claim that there exists a sequence of times t_n such that $\|\nabla u(t_n)\|_2 \to \infty$. In fact, if not, let $\lambda \geq \lambda_0$ be such that $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$. Since there is no sequence t_n such that $\|\nabla u(t_n)\|_2 \to \infty$, there exists $\sigma < \infty$ such that $\lambda \leq \|\nabla u(t)\|_2/\|\nabla Q\|_2 \leq \sigma$ for all $t \geq 0$, which means that $\exists\,\mathrm{GB}(\lambda,\sigma)$ holds true. Thus $\sigma_c \leq \sigma < \infty$ and we get a contradiction.

In view of the above results, if we can prove that $\sigma_c(\lambda_0) = \infty$ for every $\lambda_0 > 1$, we will have in fact proved Theorem 1.1. For a contradiction, fix $\lambda_0 > 1$ and assume $\sigma_c < \infty$. Of course, it suffices to consider λ_0 close to 1, so we might as well assume that $\lambda_0 < 3/2$. We first need to obtain the existence of a critical solution:

Lemma 7.3. *If $\sigma_c(\lambda_0) < \infty$, then there exist initial data $u_{c,0}$ and $\lambda_c \in [\lambda_0, \sigma_c(\lambda_0)]$ such that $u_c(t) \equiv \mathrm{NLH}(t)u_{c,0}$ is global, $P(u_c) = 0$, $M(u_c) = M(Q)$, $E(u_c)/E(Q) = 3\lambda_c^2 - 2\lambda_c^3$, and*

$$\lambda_c \leq \frac{\|\nabla u_c(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_c \quad \textit{for all } t \geq 0.$$

Proof. By the definition of σ_c, there exist sequences λ_n and σ_n such that $\lambda_0 \leq \lambda_n \leq \sigma_n$ and $\sigma_n \downarrow \sigma_c$ for which $\exists\,\mathrm{GB}(\lambda_n,\sigma_n)$ holds. This means that there exists $u_{n,0}$ such that $u_n(t) \equiv \mathrm{NLH}(t)u_{n,0}$ is global with $P(u_n) = 0$, $M(u_n) = M(Q)$, $E(u_n)/E(Q) = 3\lambda_n^2 - 2\lambda_n^3$, and

$$\lambda_n \leq \frac{\|\nabla u_n(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_n \quad \text{for all } t \geq 0.$$

The boundedness of λ_n enables us to pass to a subsequence that converges to a $\lambda' \in [\lambda_0, \sigma_c]$.

According to Lemma 6.9, where we take $\phi_n = u_{n,0}$, for $M_1+1 \leq j \leq M_2$, $v^j(t) \equiv \mathrm{NLH}(t)\psi^j$ scatters as $t \to \infty$ and combined with Proposition 6.5, for $M_2+1 \leq j \leq M$, v^j also scatters in one or the other time direction. Thus by the scattering theory, for $M_1+1 \leq j \leq M$, we have $E(v_j) = E(\psi_j) \geq 0$, and then by (6.7),

$$\sum_{j=1}^{M_1} E(\psi^j) \leq E(\phi_n) + o_n(1).$$

Thus, there exists at least one $1 \leq j \leq M_1$ with

$$E(\psi^j) \leq \max\{\lim_n E(\phi_n), 0\},$$

Without loss of generality, we might take $j = 1$. Since, by the profile composition, also $M(\psi^1) \leq \lim_n M(\phi_n) = M(Q)$, we have

$$\frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} \leq \max\left\{\lim_n \frac{E(\phi_n)}{E(Q)}, 0\right\}.$$

Thus, there exists $\tilde{\lambda} \geq \lambda_0$ ([2]) such that

$$\frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} = 3\tilde{\lambda}^2 - 2\tilde{\lambda}^3.$$

Note that by Lemma 6.9, v^1 does not scatter, so it follows from Theorem 2.4 that $\|\psi^1\|_2\|\nabla\psi^1\|_2 < \|Q\|_2\|\nabla Q\|_2$ cannot hold. Then, by the dichotomy Proposition 2.2, we have $\|\psi^1\|_2\|\nabla\psi^1\|_2 \geq \tilde{\lambda}\|Q\|_2\|\nabla Q\|_2$.

Now if $\tilde{\lambda} > \sigma_c$, recalling that $t_n^1 = 0$, for all t we know that

$$\tilde{\lambda}^2 \leq \frac{\|v^1(t)\|_2^2\|\nabla v^1(t)\|_2^2}{\|Q\|_2^2\|\nabla Q\|_2^2} \leq \frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\sum_{j=1}^M \|\nabla v^j(t-t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2}{\|\nabla Q\|_2^2}. \tag{7.1}$$

Taking $t = 0$, for example, by Lemma 6.8 we have

$$\tilde{\lambda}^2 \leq \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1),$$

which contradicts the assumption $\tilde{\lambda} > \sigma_c$. Hence we must have $\tilde{\lambda} \leq \sigma_c$.

Now if $\tilde{\lambda} < \sigma_c$, we know from the definition of σ_c that $\exists\,\mathrm{GB}(\tilde{\lambda}, \sigma_c - \delta)$ is false for any $\delta > 0$ sufficiently small, and so there exists a nondecreasing sequence t_k of times such that

$$\lim_k \frac{\|v^1(t_k)\|_2\|\nabla v^1(t_k)\|_2}{\|Q\|_2\|\nabla Q\|_2} \geq \sigma_c.$$

Noting that $t_n^1 = 0$, we have

$$\begin{aligned} \sigma_c^2 - o_k(1) &\leq \frac{\|v^1(t_k)\|_2^2\|\nabla v^1(t_k)\|_2^2}{\|Q\|_2^2\|\nabla Q\|_2^2} \leq \frac{\|\nabla v^1(t_k)\|_2^2}{\|\nabla Q\|_2^2} \\ &\leq \frac{\sum_{j=1}^M \|\nabla v^j(t_k-t_n^j)\|_2^2 + \|\nabla W_n^M(t_k)\|_2^2}{\|\nabla Q\|_2^2} \\ &\leq \frac{\|\nabla u_n(t_k)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1), \end{aligned} \tag{7.2}$$

where by Lemma 6.8 we take $n = n(k)$ large. Taking $k \to \infty$ and hence $n(k) \to \infty$, we conclude that all inequalities must be equalities. Thus $W_n^M(t_k) \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \geq 2$. Hence easily $P(v^1) = P(u_n) = 0$. On the other hand, if $\tilde{\lambda} = \sigma_c$, we do not need the inductive hypothesis but, similar to (7.1), we obtain

([2]) If $\lim_n E(\phi_n) \geq 0$, we have $\tilde{\lambda} \geq \lambda' \geq \lambda_0$; while in the case $\lim_n E(\phi_n) < 0$, we will have $\tilde{\lambda} \geq 3/2 > \lambda_0$ though we might not have $\tilde{\lambda} \geq \lambda'$.

$$\sigma_c^2 \le \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \le \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \le \sigma_c^2 + o_n(1),$$

and then again $W_n^M \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \ge 2$. Moreover, by Lemma 6.8, for all t,

$$\frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \le \lim_n \frac{\|\nabla u_n(t)\|_2^2}{\|\nabla Q\|_2^2} \le \sigma_c^2.$$

Hence, we take $u_{c,0} = v^1(0) = \psi^1$ and $\lambda_c = \tilde\lambda$ to complete our proof. ∎

8. Concentration of critical solutions and proof of Theorem 1.1. In this section, we will complete our proof of Theorem 1.1 by virtue of the precompactness of the flow of the critical solution. For convenience, we take $u(t) = u_c(t)$.

LEMMA 8.1. *There exists a path $x(t)$ in $\mathbb{R}^N$ such that*

$$K \equiv \{u(t, \cdot - x(t)) \mid t \ge 0\} \subset H^1$$

is precompact in H^1.

Proof. As was shown in [2], it suffices to prove that for each sequence of times $t_n \to \infty$, there exists a sequence x_n such that, by passing to a subsequence, $u(t_n, \cdot - x_n)$ converges in H^1.

Taking $\phi_n = u(t_n)$ in Lemma 6.9 and by the definition of $u(t) = u_c(t)$, similarly to the proof of Lemma 7.3, we find that there exists at least one $1 \le j \le M_1$ with

$$E(\psi^j) \le \max\{\lim_n E(\phi_n), 0\}.$$

Without loss of generality, we can take $j = 1$. Since also $M(\psi^1) \le \lim_n M(\phi_n) = M(Q)$, there exists $\tilde\lambda \ge \lambda_0$ such that

$$\frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} = 3\tilde\lambda^2 - 2\tilde\lambda^3.$$

Note that by Lemma 6.9, v^1 does not scatter, so we must have $\|\psi^1\|_2\|\nabla\psi^1\|_2 \ge \tilde\lambda\|Q\|_2\|\nabla Q\|_2$. Then, in the same way as in the proof of Lemma 7.3, we deduce that $W_n^M(t_k) \to 0$ in H^1 and $v^j \equiv 0$ for all $j \ge 2$. Since we know that $W_n^M(t)$ is a scattering solution, this implies that

$$W_n^M(0) = W_n^M \to 0 \quad \text{in } H^1. \tag{8.1}$$

Consequently, we have

$$u(t_n) = \mathrm{NLH}(-t_n^1)\psi^1(x - x_n^1) + W_n^M(x).$$

Note that by Lemma 6.9, $t_n^1 = 0$, and thus

$$u(t_n, x + x_n^1) = \psi^1(x) + W_n^M(x + x_n^1).$$

This equality and (8.1) imply our conclusion. ∎

Using the uniform-in-time H^1 concentration of $u(t) = u_c(t)$ and changing variables, we can easily get

COROLLARY 8.2. *For each $\epsilon > 0$, there exists $R > 0$ such that for all t,*

$$\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon.$$

With the localization property of u_c, we show, similar to [9], that u_c must blow up in finite time using the same method as in the proof of Proposition 3.2 and Remark 3.3. However, this contradicts the boundedness of u_c in H^1. Hence, u_c cannot exist and $\sigma_c = \infty$. As is argued in Section 7, this completes the proof of Theorem 1.1.

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Received 6 June 2011;
revised 2 December 2011 (5514)