COLLOQUIUM MATHEMATICUM

VOL. 125 2011 NO. 2

BANACH SPACES WHICH EMBED INTO THEIR DUAL

BY

VALERIO CAPRARO (Neuchâtel) and STEFANO ROSSI (Roma)

Abstract. We use Birkhoff–James' orthogonality in Banach spaces to provide new conditions for the converse of the classical Riesz representation theorem.

1. Introduction. It is well-known that the two most basic properties of a complex Hilbert space $\mathcal{H}$ are

- If X is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = X \oplus X^{\perp}$.
- (Riesz's representation theorem). There is a conjugate-linear isometry from $\mathcal{H}$ onto $\mathcal{H}^*$.

It was shown by Lindenstrauss and Tzafriri in [Li-Tz71] that the first property essentially characterizes Hilbert spaces among the Banach spaces. A longstanding question asks instead whether Riesz's representation theorem also characterizes Hilbert spaces; namely, let X be a Banach space and $F : X \to X^*$ an isometric isomorphism; is it true that X is a Hilbert space? In general, the answer is clearly negative. Indeed, if Y is a reflexive Banach space which is not a Hilbert space, one can easily check that $X = Y \oplus Y^*$ is isometrically isomorphic to its dual, but X is not a Hilbert space. So, over the years, there have been many attempts to add some condition on F in order to guarantee that X turns to be a Hilbert space (see, for instance, [Dr-Ya05], [Li70], [Pa86a], [Sz-Za81]). In this paper we contribute to this problem proposing some different conditions, by making use of the so-called Birkhoff–James orthogonality (see Theorems 2 and 4). We also propose some weaker statement, as in Theorem 5 and its corollary.

2. Some converses of the Riesz representation theorem. Throughout this note $(X, \|\cdot\|)$ will denote a complex normed Banach space (the real case is analogous). We start by recalling Birkhoff–James' definition of orthogonality in Banach spaces (cf. [Bi35] and [Ja47]).

2010 *Mathematics Subject Classification*: Primary 52A01; Secondary 46L36.
Key words and phrases: orthogonality in Banach spaces, Riesz representation theorem.

DOI: 10.4064/cm125-2-11

DEFINITION 1. $x \in X$ is said to be *orthogonal* to $y \in X$ if for each scalar λ one has $\|x\| \le \|x + \lambda y\|$.

It is clear that if X is a Hilbert space, then this definition reduces to the usual one. In this general context, where there is no inner product, it describes the following geometric property: a vector x is orthogonal to y if each triangle with one side equal to x and another side constructed along y has the third side longer than x. By the way, this is not the unique definition of orthogonality in Banach spaces, but it is surely the oldest and the most intuitive one (see [Al-Be97], [Di83], [Ja45] and [Pa86b] for other notions of orthogonality).

A simple but important remark is that the classical Riesz representation $\mathcal{H} \ni x \mapsto f_x \in \mathcal{H}^*$ has the property $x \in \mathrm{Ker}(f_x)^{\perp}$, which we can require in our context of normed spaces by using the Birkhoff–James orthogonality (by the way, it would be interesting to know if the following result holds true also using other notions of orthogonality).

THEOREM 2. *Let X be a complex normed* (*resp. Banach*) *space and $F : x \in X \mapsto f_x \in X^*$ an isometry such that for all $x, y \in X$ one has*

$$f_x(y) = \overline{f_y(x)}, \tag{1}$$

$$x \in \mathrm{Ker}(f_x)^{\perp} \text{ in the sense of Birkhoff and James.} \tag{2}$$

Then X is a pre-Hilbert (*resp. Hilbert*) *space with respect to the inner product given by $(x, y) = f_x(y)$ and $(x, x) = \|x\|^2$.*

Proof. Clearly $(x, y) \doteq f_x(y)$ defines a sesquilinear hermitian form on X (thanks to (1)). We will prove that this form is also positive definite. Let $x \in X$ be such that $(x, x) = 0$. Then $x \in \mathrm{Ker}(f_x)$ and we can apply Definition 1 with $\lambda y = -x$: $\|x\| \le 0$, i.e. $x = 0$. Now we observe that the real-valued function $\Phi : X \ni x \mapsto f_x(x) \in \mathbb{R}$ is continuous (by the triangle inequality), $X \setminus \{0\}$ is connected (unless $\dim X = 1$ and X is real, which is a trivial case) and thus $\Phi(X \setminus \{0\})$ is an interval $I \subseteq \mathbb{R}$ not containing 0. Hence $I \subseteq (-\infty, 0)$ or $I \subseteq (0, \infty)$ and we can assume that $f_x(x) > 0$ for all $x \neq 0$ (otherwise take $-f_x(x)$). It remains to prove that $f_x(x) = \|x\|^2$. Clearly $f_x(x) \le \|f_x\|\,\|x\| = \|x\|^2$. Conversely, let $p(x)$ be such that $f_x(x) = p(x)\|x\|$. We have to prove that $p(x) \ge \|x\|$. Let $y \in \mathrm{Ker}(f_x)$ and $\lambda \in \mathbb{C}$. By Definition 1, we have

$$|f_x(\lambda x + y)| = |\lambda| f_x(x) = |\lambda| p(x)\|x\| = p(x)\|\lambda x\| \le p(x)\|\lambda x + y\|.$$

Now, remember that when y runs over $\mathrm{Ker}(f_x)$ and $\lambda \in \mathbb{C}$, $\lambda x + y$ describes the whole X (indeed $\mathrm{Ker}(f_x)$ has codimension 1 and does not contain x), whence $\|x\| = \|f_x\| \le p(x)$. ∎

REMARK 3. In [Pa86a] the author proved that the existence of an orthogonality relation on a Banach space which satisfies certain properties is sufficient to guarantee that the Banach space is actually a Hilbert space. Unfortunately, the Birkhoff–James orthogonality does not have those properties. In [Dr-Ya05], the authors proved a result similar to our Theorem 2. Indeed they obtained the same conclusion under the two conditions $f_x(x) \geq 0$ for all $x \in X$ and $x \mapsto f_x$ surjective. The first condition is weaker than ours, while the second one is stronger. Indeed, we have not required that $F : X \to X^*$ is surjective, which is a consequence of the other hypothesis, at least when X is norm-complete. In fact, requiring surjectivity a priori, we are able to relax our second hypothesis.

THEOREM 4. *Let $F : x \in X \mapsto f_x \in X^*$ be an isometric isomorphism that satisfies*

$$f_x(y) = \overline{f_y(x)}, \qquad x \in \operatorname{Ker}(f_x) \Rightarrow x = 0.$$

Then X is a Hilbert space with respect to the inner product given by $(x, y) = f_x(y)$ and $(x, x) = \|x\|^2$.

Proof. The argument of the previous proof shows that $(\cdot,\cdot)$ is positive definite. Setting $|x| = (x, x)^{1/2}$, it remains to prove that $|x| = \|x\|$ for each $x \in X$. Clearly $|x|^2 \leq \|f_x\| \, \|x\| = \|x\|^2$. Conversely, by the Hahn–Banach theorem, there exists $f \in X^*$, with $\|f\| = 1$, such that $\|x\| = f(x)$. By the surjectivity of the embedding we have $f = f_y$ for some $y \in X$ with $\|y\| = \|f_y\| = \|f\| = 1$. So

$$\|x\| = f_y(x) = (y, x) \leq |y| \, |x| \leq \|y\| \cdot |x| = |x|$$

in which the first inequality is just the Cauchy–Schwarz inequality applied to $(\cdot,\cdot)$. ■

Now we propose a minor refinement of the previous results. Indeed, if X is reflexive and $\operatorname{Ran} F$ is closed we get the same conclusion, up to norm-equivalence. More precisely:

THEOREM 5. *Let $F : x \in X \mapsto f_x \in X^*$ be a continuous map from the reflexive Banach space X into its dual with closed range and such that*

$$f_x(y) = \overline{f_y(x)}, \tag{3}$$

$$x \in \operatorname{Ker}(f_x) \;\Rightarrow\; x = 0. \tag{4}$$

Then the norm of X is equivalent to the Hilbert norm given by $|x| = f_x(x)^{1/2}$.

Proof. We start by observing that F is injective (by (4)), so it is an isomorphism between X and $\operatorname{Ran} F$. Then, by the Banach inverse operator theorem we get $\|f_x\| \geq \delta \|x\|$ for each $x \in X$ (for some $\delta > 0$). As in the previous proofs we set $(x, y) = f_x(y)$ and we see easily that it is a positive definite sesquilinear form. Now

$$|x|^2 = f_x(x) \leq \|f_x\| \, \|x\| \leq \|F\| \, \|x\|^2.$$

To prove the reverse inequality, we need to show first the surjectivity of F. It is a straightforward consequence of the reflexivity of X : $\operatorname{Ran} F$ is a dense (and closed) subspace of X^* because the polar space of $\operatorname{Ran} F$ is the null space, as one can easily check. Now, let $x \in X$ and $f \in X^*$ with $\|f\| = 1$ and $\|x\| = f(x)$. By the surjectivity of F, we have $f = f_y$ for a unique $y \in X$. So (using the Cauchy–Schwarz inequality on the positive definite form $(\cdot, \cdot)$)

$$\|x\| = f_y(x) = (y, x) \le |y|\,|x| \le \|F\|^{1/2}\|y\|\,|x| \le \delta^{-1}\|F\|^{1/2}|x|.$$

This ends the proof. ∎

REMARK 6. The assumption about the reflexivity of X is, in some sense, necessary. Indeed, a straightforward application of James' characterization of reflexivity shows that if X is a real Banach space which is isometrically isomorphic to its dual via $x \mapsto f_x$ and this isomorphism is such that $\overline{f_x(y)} = f_y(x)$, then X is reflexive (see for instance [Li70]).

3. Contraction of a Banach space into a Hilbert space. Theorem 5 suggests an observation that might be of interest. Indeed, we have used the fact that $\operatorname{Ran} F$ is closed and the reflexivity of X only to prove that $C\|x\| \le |x|$. Thus we have the following

COROLLARY 7. *Let X be a normed space and $F : x \in X \mapsto f_x \in X^*$ be continuous and such that*

$$f_x(y) = \overline{f_y(x)}, \qquad x \in \operatorname{Ker}(f_x) \Rightarrow x = 0.$$

Then $(x, y) = f_x(y)$ defines a pre-Hilbertian structure on X, and the topology induced by $(\cdot, \cdot)$ is weaker than the norm-topology.

Let $\widetilde{X}$ denote the completion of X with respect to the inner product $f_x(y)$. We can compute $\widetilde{X}$ in some simple cases.

- We consider the contraction of l^1 into its dual l^∞ given by the "identity". It is easy to check that $\widetilde{l^1} = l^2$.
- Let $L^1(B(H))$ and $L^2(B(H))$ be respectively the trace class and the Hilbert–Schmidt operators on a Hilbert space H. $L^1(B(H))$ is canonically embedded into its dual $B(H)$ through the conjugate-linear map $T \mapsto \operatorname{tr}(T^* \cdot)$. Thus $\widetilde{L^1(B(H))} = L^2(B(H))$.

In both these examples, a Banach space turns out to be contracted into a Hilbert space. The contraction of a Banach space into a Hilbert space is nothing special, at least when the space is separable. Indeed the classical Banach–Mazur representation theorem provides an isometry from every separable Banach space into $C[0, 1]$, which is obviously contracted into $L^2[0, 1]$. On the other hand, it is not clear what happens when the space is not separable: one can still apply the Banach–Mazur theorem to obtain an isometry from X onto a closed subspace of $C(X_1^*)$, where X_1^* stands for the weak* closed unit ball in the dual space of X. When does $C(X_1^*)$ embed into $L^2(X_1^*)$? To obtain the canonical embedding we need a positive Borel measure whose support is the whole X_1^*, but it is clear that such a measure might not exist. For instance, let H be a non-separable Hilbert space, $\{e_a : a \in A\}$ an orthonormal basis for H, and $X = \{x \in H : \|x\| \leq 1\}$ with the weak topology. We set $U_a = \{x \in X : |(x, e_a)|^2 > 1/2\}$. This is an uncountable family of non-empty ($e_a \in U_a$!) disjoint (by Parseval!) open (because the functionals $x \mapsto |(x, e_a)|^2$ are continuous with respect the weak topology) sets. Thus, if μ is a Borel measure on X, there exists $a \in A$ such that $\mu(U_a) = 0$ and thus $\mathrm{supp}(\mu) \subseteq U_a^c$.

Acknowledgments. The first author was partially supported by Swiss SNF Sinergia project CRSI22-130435.

REFERENCES

[Al-Be97] J. Alonso and C. Benitez, *Area orthogonality in linear normed space*, Arch. Math. (Basel) 68 (1997), 70–76.

[Bi35] G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. 1 (1935), 169–172.

[Di83] C. R. Diminnie, *A new orthogonality relation for normed linear spaces*, Math. Nachr. 114 (1983), 192–203.

[Dr-Ya05] D. Drivaliaris and N. Yannakakis, *Hilbert space structure and positive operators*, J. Math. Anal. Appl. 305 (2005), 560–565.

[Ja47] R. C. James, *Orthogonality and linear functionals on normed linear spaces*, Trans. Amer. Math. Soc. 61 (1947), 265–292.

[Ja45] —, *Orthogonality in normed linear spaces*, Duke Math. J. 12 (1945), 291–302.

[Le78] K. Leichtweiss, *Zur expliziten Bestimmung der Norm der selbstadjungierten Minkowski-Räume*, Res. Math. 1 (1978), 61–87.

[Li70] B. L. Lin, *On Banach spaces isomorphic to its conjugate*, in: Studies and Essays presented to Yu-why Chen on his 60th Birthday, Math. Res. Center, Nat. Taiwan Univ., Taipei, 1970, 151–156.

[Li-Tz71] J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem*, Israel J. Math. 9 (2) (1971), 263–269.

[Pa86a] J. R. Partington, *Self-conjugate polyhedral Banach spaces*, Bull. London Math. Soc.

[Pa86b] —, *Orthogonality in normed spaces*, Bull. Austral. Amer. Soc. 33 (1986), 449–455.

[Sz-Za81] R. Sztencel and P. Zaremba, *On self-conjugate Banach spaces*, Colloq. Math. 44 (1981), 111–115.

Valerio Capraro
Université de Neuchâtel
Neuchâtel, Switzerland
E-mail: valerio.capraro@unine.ch

Stefano Rossi
Università di Roma “La Sapienza”
Roma, Italy
E-mail: s-rossi@mat.uniroma1.it

Received 17 October 2011;
revised 5 December 2011 (5559)