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FINITE-FINITARY, POLYCYCLIC-FINITARY AND CHERNIKOV-FINITARY AUTOMORPHISM GROUPS

BY

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Abstract. If **X** is a property or a class of groups, an automorphism ϕ of a group G is **X**-finitary if there is a normal subgroup N of G centralized by ϕ such that G/N is an **X**-group. Groups of such automorphisms for G a module over some ring have been very extensively studied over many years. However, for groups in general almost nothing seems to have been done. In 2009 V. V. Belyaev and D. A. Shved considered the general case for **X** the class of finite groups. Here we look further at the finite case but our main results concern the cases where **X** is either the class of polycyclic-by-finite groups or the class of Chernikov groups. The latter presents a new perspective on some work of Ya. D. Polovitskiĭ in the 1960s, which seems to have been at least partially overlooked in recent years. Our polycyclic cases present a different view of work of S. Franciosi, F. de Giovanni and M. J. Tomkinson from 1990. We describe the polycyclic cases in terms of matrix groups over the integers, and the Chernikov case in terms of matrix groups over the complex numbers.

1. Introduction. Let **X** be a class of groups. Say that an automorphism γ of a group G is **X**-*finitary* if there is a normal subgroup N of G such that $[N,\gamma] = \langle 1\rangle$ and G/N is an **X**-group. We are only interested here in the cases of **X** being either the class **F** of all finite groups, or the class **P** of all polycyclic groups, or the class **PF** of all polycyclic-by-finite groups, or the class **Ch** of all Chernikov groups. In each of these cases the set $\mathrm{F}_{\mathbf{X}}\mathrm{Aut}\,G$ of all **X**-finitary automorphisms of G is a (normal) subgroup of $\mathrm{Aut}\,G$ and N can be taken to be $C_G(\gamma)_G = \bigcap_{g\in G} C_G(\gamma)^g$. We consider here the structure of subgroups Γ of these $\mathrm{F}_{\mathbf{X}}\mathrm{Aut}\,G$. (Unless otherwise indicated, **X** below denotes one of **F**, **P**, **PF** or **Ch**. Also we sometimes use the expanded terms finite-finitary, polycyclic-finitary etc. instead of **F**-finitary, **P**-finitary etc.)

First we consider the finite-finitary case. Note that here an alternative definition is: $\gamma \in \mathrm{Aut}\,G$ is finite-finitary if and only if the index $(G : C_G(\gamma))$ is finite. According to Mathematical Reviews Belyaev and Shved [1] prove the following (I have been unable to obtain a copy of this paper and hence have been unable to check its contents for myself): $\Gamma \le \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\,G$ is abelian-by-(locally finite), (locally finite)-by-abelian, locally centre-by-finite and (pe-

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riodic abelian)-by-(centre-by-(locally finite)). (These four theorems all follow from our results below.) In the earlier paper [13] the current author considered the case where G is abelian. In this case we showed in particular that Γ is locally finite with a normal subgroup N lying in the Fitting subgroup of Γ such that Γ/N embeds into a direct product of finitary linear groups over fields of prime order. Further we showed that any periodic abelian group A always embeds into $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ for some abelian group G, and embeds into $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ for some periodic abelian group G if and only if A has a residually finite subgroup B such that A/B is a direct product of cyclic groups. [13] contains further results of this type.

Let $\mathrm{F}_{\mathbf{F}}$ denote the class of all groups Γ for which there exists an embedding of Γ into $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ for some group G. If V is a vector space over a finite field F, then $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, V$ contains the finitary general linear group $\mathrm{FGL}(V) = \mathrm{FAut}_F\, V$ over V with equality if F has prime order, so all finitary linear groups over finite fields lie in $\mathrm{F}_{\mathbf{F}}$. All such groups are locally finite. Also the following holds (see below):

STATEMENT 1.1. *Every abelian group lies in* $\mathrm{F}_{\mathbf{F}}$*; more generally, so does each* FC*-group.*

Obviously such groups are not necessarily locally finite. Also there are finitary linear groups over finite fields that are not FC-groups, for example infinite simple such groups. It is easy to see that the class $\mathrm{F}_{\mathbf{F}}$ is closed under the subgroup operator S and the direct product operator D. (The infinite dihedral group is not in $\mathrm{F}_{\mathbf{F}}$, being finitely generated but not centre-by-finite; it is also residually finite. Thus $\mathrm{F}_{\mathbf{F}}$ is not closed under the poly, residual and cartesian operators P, R and C. However, it is R_0-closed since $\mathrm{R}_0 \leq \mathrm{SD}$.)

Every group that can be constructed from the above examples of $\mathrm{F}_{\mathbf{F}}$-groups and repeated use of the S and D operators is centre-by-(locally finite). Moreover every centre-by-(locally finite) group satisfies the conclusions of the four theorems of Belyaev and Shved quoted above. Frequently $\mathrm{F}_{\mathbf{F}}$-groups are centre-by-(locally finite). Below, $\mathrm{Fitt}\, G$ denotes the Fitting subgroup of a group G, $\tau(G)$ its unique maximal locally-finite normal subgroup, $\Delta(G)$ its FC-centre and $\zeta_1(G)$ its centre. The *rank* of G is the supremum, over the finitely generated subgroups X of G, of the minimum number of generators of X, so $\mathrm{rank}\, G$ is a non-negative integer or infinity.

PROPOSITION 1.2. *Let G be a group and Γ a subgroup of* $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$*. Under any one of the following five conditions Γ is at least centre-by-*(*locally finite*)*:*

(a) *G is an* FC*-group; here Γ is locally finite.*
(b) $\mathrm{Fitt}\, G$ *is periodic; here Γ is locally finite.*
(c) $[G, \Gamma] \leq \tau(G) \cap \Delta(G)$*; here Γ is locally finite.*
(d) $\tau(G) \cap \Delta(G) = \langle 1 \rangle$*; here Γ is abelian.*

(e) *G is a finite extension of a torsion-free soluble group of finite rank; here Γ is centre-by-finite.*

Notice that (e) here covers all polycyclic groups. However:

STATEMENT 1.3. *Not every* $\mathrm{F}_{\mathbf{F}}$*-group is centre-by-*(*locally finite*).

Below we present a range of groups G for which $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\,G$ is not centre-by-periodic. The simplest to state and prove is the wreath product of A by a cyclic group of order 2, where A is the direct product of an infinite cyclic group and a Prüfer p^{∞}-group for some prime p. Requiring only a little more proof is the split extension $\langle x\rangle A$, where A is as above and x has order 2 and inverts A (meaning that $a^x = a^{-1}$ for all a in A).

Conversely:

STATEMENT 1.4. *Not every centre-by-*(*locally finite*) *group is in* $\mathrm{F}_{\mathbf{F}}$*; indeed, nor is every locally finite group.*

For example, P. Hall's countable universal locally finite group U (see [5, Chapter 6, especially 6.4 and 6.1]) is not, the basic reason being the following proposition and the simplicity of U.

PROPOSITION 1.5. *Any simple group Σ in* $\mathrm{F}_{\mathbf{F}}$ *is isomorphic to a finitary linear group over a field of prime order.*

Obviously the case where Σ is finite is of no interest in this context. The infinite locally-finite simple finitary linear groups have been completely classified by J. I. Hall [4]. Thus if Σ is infinite in Proposition 1.5, then Σ is either a finitary analogue of the alternating groups, the symplectic groups, the special unitary groups or the orthogonal groups, or one of the finitary analogues of the (projective) special linear groups, clearly over a locally finite field and in fact over a finite field, though this is far from obvious.

We now turn to the polycyclic cases. It is not as straightforward as just replacing $\mathbf{F}$ in the Belyaev–Shved theorem by $\mathbf{P}$ or $\mathbf{PF}$ and in some ways the conclusions here are more elegant.

THEOREM 1.6. *Let Γ be a locally* (*soluble-by-finite*) *subgroup of* $\mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$ *for some group G. Then Γ is locally* (*polycyclic-by-finite*).

We abbreviate the conclusion here to $\Gamma \in \mathrm{L}(\mathbf{PF})$. The first hypothesis on Γ we also shorten to $\Gamma \in \mathrm{L}(\mathbf{SF})$. If $\Gamma \in \mathrm{L}(\mathbf{PF})$ then clearly Γ satisfies all four of the Belyaev–Shved conclusions with $\mathbf{PF}$ replacing the word finite. (In the finite-finitary case the conclusion corresponding to that of 1.6 would be that $\Gamma \leq \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\,G$ and $\Gamma \in \mathrm{L}(\mathbf{SF})$ together imply that Γ is locally finite, a conclusion that is clearly false, since every abelian group is in $\mathrm{F}_{\mathbf{F}}$.)

In 1.6 we do need some restriction on Γ beyond $\Gamma \leq \mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$, for clearly $\mathrm{Aut}\,G = \mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$ whenever $G \in \mathbf{PF}$. For example $\mathrm{GL}(2,\mathbb{Z}) =$

$\mathrm{F}_{\mathbf{P}}\mathrm{Aut}\,\mathbb{Z}^{(2)}$ is not in $\mathrm{L}(\mathbf{PF})$; it contains non-abelian free subgroups. ($\mathbb{Z}$ denotes the integers.) However, we can draw positive conclusions about arbitrary subgroups Γ of $\mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$, weaker, of course, than $\Gamma \in \mathrm{L}(\mathbf{PF})$. Denote by $\mathbf{L}_{\mathbb{Z}}$ the class of all groups that can be embedded into some $\mathrm{GL}(n,\mathbb{Z})$, where n is a positive integer. As usual $\mathbf{A}$ denotes the class of all abelian groups and $\mathbf{G}\cap\mathbf{A}$ the class of all finitely generated abelian groups.

THEOREM 1.7. *Let $\Gamma \le \mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$ for some group G. Then Γ is locally in the class $(G\cap A)\mathbf{L}_{\mathbb{Z}}$. Also Γ is locally residually finite.*

Thus if Γ is finitely generated in 1.7, then Γ has a free abelian normal subgroup of finite rank with $\Gamma/A \in \mathbf{L}_{\mathbb{Z}}$. Analogously to the class $\mathrm{F}_{\mathbf{F}}$ we define the classes $\mathrm{F}_{\mathbf{P}}$ and $\mathrm{F}_{\mathbf{PF}}$. Trivially $\mathrm{F}_{\mathbf{PF}} \supseteq \mathrm{F}_{\mathbf{F}} \cup \mathrm{F}_{\mathbf{P}}$. This is not an equality. For example, let G denote the wreath product of an infinite cyclic group by a finite non-abelian simple group S. Then $G/\zeta_1(G) \cong \Gamma \le \mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\,G$ in the obvious way. Now S embeds into Γ and finite $\mathrm{F}_{\mathbf{P}}$-groups are easily seen to be soluble, so $\Gamma \notin \mathrm{F}_{\mathbf{P}}$. Further Γ is finitely generated but not centre-by-finite, so $\Gamma \notin \mathrm{F}_{\mathbf{F}}$. (With different terminology, the class $\mathbf{F}_{\mathbf{G}\cap\mathbf{A}}$ is extensively discussed in [14], at least where G is abelian.)

In our final two sections we indicate how a similar analysis can be applied to the class $\mathbf{Ch}$ of Chernikov groups. This involves replacing the maximal condition on subgroups by the minimal condition. The results here are necessarily weaker; for a start the automorphism group of a Chernikov group need not be countable, unlike that of a $\mathbf{PF}$-group. Also it need not have a faithful representation of finite degree over any field. However, its outer automorphism group does at least have a faithful representation over the complex numbers $\mathbb{C}$ (e.g. [5, 3.38]) and some use can be made of this. Let $\mathbf{L}_{\mathbb{C}}$ denote the class of all groups that can be embedded into $\mathrm{GL}(n,\mathbb{C})$ for some positive integer n. We can at least prove the following (compare 1.7 above).

THEOREM 1.8. *Let $\Gamma \le \mathrm{F}_{\mathbf{Ch}}\mathrm{Aut}\,G$ for some group G. Then Γ is locally in the class $(\mathbf{F}^{-\mathrm{S}}\cap\mathbf{A})\mathrm{L}_{\mathbb{C}}$. Also Γ is locally residually finite.*

Here $\mathbf{F}^{-\mathrm{S}}\cap A$ denotes the class of torsion-free abelian groups. Theorem 1.8 and for that matter Theorem 1.7 are really results about finitely generated such subgroups Γ and more information about these Γ are given by Lemmas 5.2 and 7.5 below. Actually all finitely generated $\mathbf{L}_{\mathbb{C}}$ groups lie in $\mathrm{F}_{\mathbf{Ch}}$, the latter denoting the $\mathbf{Ch}$ analogue of $\mathrm{F}_{\mathbf{F}}$ and $\mathrm{F}_{\mathbf{PF}}$ (and trivially all $\mathbf{L}_{\mathbb{Z}}$ groups are in $\mathrm{F}_{\mathbf{P}}$). More precisely, we prove the following.

THEOREM 1.9. *Let Γ be a finitely generated group. The following are equivalent:*

(a) $\Gamma \in \mathbf{L}_{\mathbb{C}}$.
(b) Γ *embeds into* $\mathrm{Aut}\,G$ *for some* $\mathbf{Ch}$ *group* G.

(c) *There exists a positive integer r such that for almost all primes p the group Γ embeds into the automorphism group of the divisible abelian p-group of rank r.*

However, not every finitely generated $(\mathbf{F}^{-\mathrm{S}} \cap \mathbf{A})\mathbf{L}_{\mathbb{C}}$ group lies in $\mathbf{L}_{\mathbb{C}}$, the free (soluble group of derived length at most 3) of rank 2 being such an example, since free metabelian groups of finite rank lie in $\mathbf{L}_{\mathbb{C}}$ by a theorem of Magnus (see [9, 2.11] and use [9, 3.6]). A similar example is the wreath product of a cyclic group of infinite order by the free metabelian group of rank 2 (see [9, 10.21]). Of course Theorems 1.8 and 1.9 leave unanswered questions.

2. Belyaev and Shved type results

LEMMA 2.1. *Let N be a normal subgroup of a group G. If $\Gamma = \langle \gamma \in \operatorname{Aut} G : [N, \gamma] = \langle 1 \rangle \rangle$, then $[G, \Gamma, N] = \langle 1 \rangle$.*

Proof. Clearly $[N, G] \le N$ and $[N, \Gamma] = \langle 1 \rangle$, so $[N, G, \Gamma] = \langle 1 \rangle = [\Gamma, N, G]$. Therefore $[G, \Gamma, N] = \langle 1 \rangle$.

LEMMA 2.2. *Let A be an abelian normal subgroup of a group G. If $A/C_A(g)$ is periodic for all $g \in G$, then $[A, G]$ is periodic.*

In particular this shows that the fourth Belaev and Shved property follows from the first and third (actually it also follows from the first and the second).

Proof. We may factor by the torsion subgroup of A and assume that A is torsion-free. Let $a \in A$ and $g \in G$. There is a positive integer n with $[a^n, g] = 1$. Then $(a^g)^n = (a^n)^g = a^n$ and a^g and a both lie in the torsion-free abelian group A. Hence $a^g = a$ and consequently $[A, G] = \langle 1 \rangle$.

From now on in this section we consider an arbitrary group G and some subgroup Γ of $\mathrm{F}_{\mathbf{F}}\operatorname{Aut} G$. We work throughout inside the holomorph of G and more particularly in its subgroup ΓG. Thus Γ^G denotes the normal subgroup $\langle g^{-1}\Gamma_g : g \in G \rangle$ of ΓG.

LEMMA 2.3. *Suppose Γ is finitely generated. Then Γ' is finite, Γ is centre-by-finite, $[G, \Gamma]$ is centre-by-finite and $[G, \Gamma, \Gamma^G]$ is finite.*

Proof. Now $\Gamma = \langle \gamma_1, \dots, \gamma_m \rangle$ is finitely generated and $C_G(\Gamma) = \bigcap_i C_G(\gamma_i)$. Thus G has a subgroup N of finite index centralized by Γ and we can choose N normal in G. Then $C_\Gamma(G/N)$ has finite index in Γ and by stability theory embeds into the abelian group $\operatorname{Der}(G, \zeta_1(N))$ of derivations. Therefore Γ is at least abelian-by-finite. Set $C = C_G(N)$ and $Z = C \cap N = \zeta_1(N)$. By Lemma 2.1 we have $[G, \Gamma] \le C$.

Now $(C : Z)$ is finite, so C is centre-by-finite and hence C' is finite (Schur's Theorem, e.g. [15, 1.18] or use [9, 4.21]). If T denotes the maximal periodic normal subgroup of C, then $C' \leq T$ and C/T is torsion-free abelian with $\mathit{\Gamma}$ centralizing its subgroup ZT/T of finite index. Therefore $[C, \mathit{\Gamma}] \leq T$ by Lemma 2.2. Also $[G, \mathit{\Gamma}] \leq C$, so $[G, \mathit{\Gamma}]$ is centre-by-finite.

Certainly G, N, C and C' are all normal in $\mathit{\Gamma} G$, and Z is central in $\mathit{\Gamma} C$ and of finite index in C. Then $C_{\mathit{\Gamma}}(C/Z)$ has finite index in $\mathit{\Gamma}$ and is finitely generated. Further $C_{\mathit{\Gamma}}(C/Z)/C_{\mathit{\Gamma}}(C)$ embeds into the abelian group $\mathrm{Hom}(C/Z, Z)$, which has finite exponent dividing the order of C/Z. Hence $C_{\mathit{\Gamma}}(C)$ has finite index in $\mathit{\Gamma}$ and therefore the conjugacy class $c^{\mathit{\Gamma} C}$ is finite for every $c \in C$. But $C = XZ$ and $\mathit{\Gamma} = \langle Y \rangle$ for some finite sets X and Y. Thus

$$[C, \mathit{\Gamma}] = \langle [x, y]^{\mathit{\Gamma} C} : x \in X \ \&\ y \in Y \rangle$$

is finitely generated. But $[C, \mathit{\Gamma}] \leq T$; consequently, $[C, \mathit{\Gamma}]$ is finite.

Now $[C, \mathit{\Gamma}^G] = [C, [G, \mathit{\Gamma}]\mathit{\Gamma}] \leq [C, \mathit{\Gamma}]C'$, since $[G, \mathit{\Gamma}] \leq C$. Therefore $[C, \mathit{\Gamma}^G]$ is also finite and consequently so is $[G, \mathit{\Gamma}, \mathit{\Gamma}^G]$. Clearly $\mathit{\Gamma}$ stabilizes the series $G \geq C \geq [C, \mathit{\Gamma}^G]$, so $\mathit{\Gamma}' \leq C_{\mathit{\Gamma}}(G/[C, \mathit{\Gamma}^G])$. Also $\mathit{\Gamma}/C_{\mathit{\Gamma}}([C, \mathit{\Gamma}^G])$ is finite and

$$\mathit{\Sigma} = C_{\mathit{\Gamma}}(G/[C, \mathit{\Gamma}^G]) \cap C_{\mathit{\Gamma}}([C, \mathit{\Gamma}^G])$$

embeds into $\mathrm{Der}(G, \zeta_1([C, \mathit{\Gamma}^G]))$, which is abelian of finite exponent (dividing the order of $[C, \mathit{\Gamma}^G]$). But $\mathit{\Gamma}$ is finitely generated and abelian-by-finite, therefore $\mathit{\Sigma}$, $C_{\mathit{\Gamma}}(G/[C, \mathit{\Gamma}^G])$ and hence $\mathit{\Gamma}'$ are all finite. By the 'converse' of Schur's Theorem (e.g. [9, 4.24]), $\mathit{\Gamma}$ is centre-by-finite.

Remark. In Lemma 2.3 let d denote the minimal number of generators of $\mathit{\Gamma}$ and n the index of N in G. Following through the proof above shows that the orders of $\mathit{\Gamma}'$, $\mathit{\Gamma}/\zeta_1(\mathit{\Gamma})$ and $[G, \mathit{\Gamma}, \mathit{\Gamma}^G]$ are all bounded by integer-valued functions of d and n only.

Lemma 2.4. *Suppose $[G, \mathit{\Gamma}]$ is periodic and lies in the* FC*-centre of G. Then $\mathit{\Gamma}$ is locally finite.*

Proof. We may assume that $\mathit{\Gamma}$ is finitely generated. Let N and C be as in the proof of Lemma 2.3, so $[G, \mathit{\Gamma}] \leq C$. Let $c \in [G, \mathit{\Gamma}]$. Then c lies in the FC-centre of G, so c^G is finite. As in the proof of Lemma 2.3 we have $c^{\mathit{\Gamma} C}$ finite. It follows that $c^{\mathit{\Gamma} G}$ is finite for every $c \in [G, \mathit{\Gamma}]$. Now $G = XN$ and $\mathit{\Gamma} = \langle Y \rangle$ for some finite sets X and Y, so

$$[G, \mathit{\Gamma}] = \langle [x, y]^{\mathit{\Gamma} G} : x \in X \ \&\ y \in Y \rangle$$

is finitely generated. By hypothesis $[G, \mathit{\Gamma}]$ is periodic, FC and hence locally finite. Therefore $[G, \mathit{\Gamma}]$ is finite. If $\mathit{\Sigma} = C_{\mathit{\Gamma}}([G, \mathit{\Gamma}])$, then $\mathit{\Sigma}$ is normal in $\mathit{\Gamma}$, $\mathit{\Gamma}/\mathit{\Sigma}$ is finite, $\mathit{\Sigma}$ stabilizes the series $G \geq [G, \mathit{\Gamma}] \geq \langle 1 \rangle$ and $\mathit{\Sigma}$ embeds into the abelian group $\mathrm{Der}(G, \zeta_1([G, \mathit{\Gamma}]))$ of finite exponent dividing the order of $[G, \mathit{\Gamma}]$. Also $\mathit{\Gamma}$ is finitely generated. Therefore $\mathit{\Gamma}$ is finite.

Theorem 2.5 (Belyaev and Shved).

(a) Γ' *is locally finite.*
(b) Γ *is abelian-by-(locally finite).*
(c) Γ *is locally centre-by-finite and* $[G,\Gamma]$ *is locally (centre-by-finite and normal in* G*).*
(d) *If* H *is any subgroup of* Γ*, then* $[H,\Gamma]$ *is periodic and normal in* Γ*.*
(e) Γ *is (periodic abelian)-by-(centre-by-(locally finite)).*

Proof. Let Δ be a finitely generated subgroup of Γ. Then Δ' is finite by Lemma 2.3 and these Δ' form a local system for Γ'. Therefore Γ' is locally finite. Further by Lemma 2.3 each $[G,\Delta,\Delta^G]$ is finite and normal in G and these subgroups form a local system for $[G,\Gamma,\Gamma^G]$. Thus $[G,\Gamma,\Gamma^G]$ is periodic and lies in the FC-centre of G. Set $\Sigma = C_\Gamma([G,\Gamma])$. Then Γ/Σ is locally finite by Lemma 2.4. Clearly Σ stabilizes the series $G \geq [G,\Gamma] \geq \langle 1\rangle$, so Σ is abelian and Γ is abelian-by-(locally finite). This proves (a) and (b). Now (c) follows from Lemma 2.3 and the simple fact that $[G,\Delta]$ is normal in G for all Δ, and (d) follows from (a) since clearly $[H,\Gamma] \leq \Gamma'$. Finally if Σ is as in the proof of (b), then $[\Sigma,\Gamma]$ is periodic abelian by (d). Clearly $\Sigma/[\Sigma,\Gamma]$ is central in Γ and Γ/Σ is locally finite. The proof is complete.

Lemma 2.6. *Suppose* G *is an* FC*-group. Then* Γ *is locally finite.*

Proof. If G is periodic the claim is immediate from Lemma 2.4. We may assume that Γ is finitely generated. Suppose first that G is abelian. With N and Z as in the proof of Lemma 2.3 we see that $C_\Gamma(G/N)$ has finite index in Γ and embeds into $\mathrm{Hom}(G/N,Z)$. The latter is abelian with exponent dividing $(G:N)$, and Γ is finitely generated. Therefore Γ is finite. (Actually this is part of [13, Proposition 1.1].)

Now consider the general case and set $A = \zeta_1(G)$, so G/A is locally finite (e.g. see [9, 4.32]). There exists $A_1 \leq A$ with A_1 torsion-free and A/A_1 periodic. Set $B = \bigcap_{\gamma\in\Gamma}(A_1)^\gamma$. Since $\Gamma/C_\Gamma(A)$ is finite by the abelian case, it follows that A/B is periodic and B is torsion-free, central in G and normal in ΓG. Clearly $\Gamma/C_\Gamma(B)$ is finite since $\Gamma/C_\Gamma(A)$ is finite, and $\Gamma/C_\Gamma(G/B)$ is finite by the periodic case (G/B is periodic). Finally $C_\Gamma(B)\cap C_\Gamma(G/B)$ embeds into $\mathrm{Hom}(G/B,B)$. The latter is trivial since G/B is periodic and B is torsion-free. Therefore Γ is finite.

Lemma 2.7. *If* $\mathrm{Fitt}\, G$ *is periodic, then* Γ *is locally finite.*

Proof. Let $\gamma \in \Gamma$. There exists N normal in G of finite index and centralized by γ. For some $n > 0$ we have γ^n centralizing G/N. If Z denotes the centre of N then $\langle\gamma^n\rangle$ embeds into the direct product $Z^{(G:N)}$ of $(G:N)$ copies of Z (e.g. see [5, 1.C.3]) and $Z \leq \mathrm{Fitt}\, G$, so Z is periodic. Therefore $Z^{(G:N)}$ is periodic, γ has finite order, Γ is periodic and consequently Γ is locally finite.

Lemma 2.8. *Suppose $H \le \mathrm{GL}(n, F)$ is a linear group of finite degree n over the field F. If H is locally centre-by-finite, then $H/\zeta_1(H)$ is locally finite.*

Proof. Let $X \le Y \le H$, where X and Y are finitely generated and hence centre-by-finite. Then $X^0 \le Y^0 \le \zeta_1(Y)$ by [10, 5.4], where X^0 denotes the connected component of X containing 1 (ditto Y^0 for Y). Thus $X^0 \le \bigcap_{Y \ge X} \zeta_1(Y) \le \zeta_1(H)$. But $(X : X^0)$ is always finite. Therefore $H/\zeta_1(H)$ is locally finite.

Lemma 2.9. *If G is a finite extension of a torsion-free soluble group of finite rank, then $\Gamma = \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ is centre-by-finite.*

Proof. Since $\Gamma \le \mathrm{Aut}\, G$, Γ embeds into $\mathrm{GL}(n, \mathbb{Q})$ for some integer n and $\mathbb{Q}$ the field of rational numbers (see [11, 1.2]). Then $\Gamma/\zeta_1(\Gamma)$ is locally finite by 2.5 and 2.8. But $\Gamma/\zeta_1(\Gamma)$ is also isomorphic to a linear group of finite degree over $\mathbb{Q}$ by [10, 6.4 and 5.4]. Consequently, $\Gamma/\zeta_1(\Gamma)$ is finite by [10, 9.33].

Proposition 1.2 now follows, for part (a) is given by Lemma 2.6, part (b) by Lemma 2.7, part (c) by Lemma 2.4, part (d) by Lemma 2.3 since here by hypothesis $[G, \Gamma, \Gamma] = \langle 1 \rangle$, and part (e) by Lemma 2.9.

3. Examples: non-(centre-by-periodic) groups. Consider an additive abelian group $A = T \oplus E$, where T is periodic and E is torsion-free. Let $H = \{\theta \in \mathrm{Hom}(E, T) : (E : \ker\theta) < \infty\}$. Clearly H is a subgroup of $\mathrm{Hom}(E, T)$ and $H.\mathrm{Aut}\, T \le H$. For $\theta \in H$ let $[\theta]$ denote the map $t{+}e \mapsto (t{+}e\theta){+}e$ of A into itself, with the obvious notation. Then $[\theta] \in \mathrm{Aut}\, A$ (with $[\theta]^{-1} = [-\theta]$) and $C_A([\theta]) \ge T \oplus \ker\theta$. If also $\phi \in H$, then

$$(t + e)[\theta][\phi] = (t + e\theta + e\phi) + e = (t + e)[\theta + \phi].$$

Set $\Delta = \{[\theta] : \theta \in H\}$. Then $\Delta \le \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, A$.

Suppose in addition that T has infinite exponent and $E = \langle e \rangle \oplus E_1$ for some $e \ne 0$ and some $E_1 \le E$. Suppose $t \in T$ and $m > 0$ with $mt \ne 0$. There exists θ in H with $e\theta = t$ and $E_1\theta = \{0\}$ and then $(me)[\theta] = mt + me \ne me$. Since T has infinite exponent we see that $C_{\langle e \rangle}(\Delta) = \{0\}$. In particular $A/C_A(\Delta)$ is not periodic.

Let $G = \langle x \rangle (A_1 \times A_2)$, where $a \mapsto a_i$ is an isomorphism of A onto the multiplicative copy A_i of A for $i = 1, 2$, where $|x| = 2$ and where $a_1^x = a_2$ and $a_2^x = a_1$ for all a in A. For $a \in A$ let σ_a denote the inner isomorphism of G given by conjugation by a_1 (that is, $g \mapsto (a^{-1})_1 g (a_1)$). Set $\Sigma = \{\sigma_a : a \in A\} \le \mathrm{Aut}\, G$. Clearly $A_1A_2 \le C_G(\sigma_a)$, so in fact $\Sigma \le \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$. Any γ in $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, A$ acts on G via $x\gamma = x$ and $(a_i)\gamma = (a\gamma)_i$ for all $a \in A$ and $i = 1, 2$. Clearly $C_G(\gamma) \ge C_A(\gamma)_1 \times C_A(\gamma)_2$, so $\gamma \in \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$. In particular, in this way, we may regard $\varDelta$ as a subgroup of $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$. Set $\varGamma = \langle \varDelta, \varSigma\rangle \leq \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$.

Let $a, b \in A$, $\delta \in \varDelta$ and $g \in G$. Then

$$x\sigma_a = a_1^{-1}xa_1 = xa_1a_2^{-1} \neq x \quad \text{unless } a = 0,$$
$$g(\sigma_a)^\delta = (a_1^{-1}(g\delta^{-1})a_1)\delta = (a\delta)_1^{-1}g(a\delta)_1 = g\sigma_{a\delta},$$
$$g\sigma_a\sigma_b = b_1^{-1}a_1^{-1}ga_1b_1 = g\sigma_{a+b},$$
$$x\delta = x, \quad \text{so} \quad \delta \neq \sigma_a \quad \text{for all } a \neq 0.$$

Thus $\varSigma$ is $\varDelta$-isomorphic to A via $a \mapsto \sigma_a$ and $\varGamma$ is the split extension of $\varSigma$ by $\varDelta$, and as such is isomorphic to the split extension of A by $\varDelta$. Finally, $A/C_A(\varDelta)$ is not periodic, so $\varSigma/C_\varSigma(\varDelta)$ is not periodic and consequently $\varGamma$ is not centre-by-periodic. We have now proved the following.

Lemma 3.1. *Let G be the wreath product of an abelian group $A = T \times \langle e\rangle \times E_1$, where T is periodic of infinite exponent, e has infinite order and E_1 is torsion-free, by a cyclic group of order 2. Then $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ is not centre-by-periodic.*

Remark 3.2. The arguments above show that $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, A$ can be regarded as a subgroup of $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ and then we have

$$\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G \geq (\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, A)\varSigma \geq (\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, T)\varDelta(\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, E)\varSigma,$$

so the examples above can have very large finite-finitary automorphism groups.

We return now to our additive group $A = T \oplus E = T \oplus \langle e\rangle \oplus E_1$. Set $S = \{t \in T : 2t = 0\}$. Clearly T/S has infinite exponent, assuming we keep our assumption that T does. Let $t \in T$ and $m > 0$ with $mt \notin S$. Define $\theta \in H$ by $e\theta = t$ and $E_1\theta = \{0\}$. Then

$$(me)[\theta] = m(e\theta) + me = mt + me \notin S + me.$$

Thus $[me, [\theta]] \notin S$. Consequently, $(A/S)/C_{A/S}(\varDelta)$ is not periodic.

From now on write A multiplicatively and let $G = \langle x\rangle A$, where x has order 2 and inverts A. Now inversion is central in $\mathrm{Aut}\, A$. Thus the action of $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, A$ on A extends to G by centralizing x and clearly this action on G is finite-finitary. Hence we may assume that $\varDelta \leq \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ with $x \in C_G(\varDelta)$. For $a \in A$ let σ_a denote the inner automorphism $g \mapsto a^{-1}ga$ of G. Clearly $A \leq C_G(\sigma_a)$. Set $\varSigma = \{\sigma_a : a \in A\}$. Then $\varSigma \leq \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ and hence $\varGamma = \langle \varDelta, \varSigma\rangle \leq \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$.

If $a \in A$, then $x\sigma_a = xa^2$. Then the map $a \mapsto \sigma_a$ determines an isomorphism of A/S onto $\varSigma$. Clearly $(\sigma_a)^\delta = \sigma_{a\delta}$ for all $\delta \in \varDelta$. Thus A/S and $\varSigma$ are $\varDelta$-isomorphic and hence $\varSigma/C_\varSigma(\varDelta)$ is not periodic. Therefore $\varGamma$ is not centre-by-periodic. (Also $\varDelta$ centralizes x and $\varSigma$ centralizes A, so

$\Delta \cap \Sigma = \langle 1 \rangle$ and again Γ is the split extension of Σ by Δ.) We have now proved the following.

LEMMA 3.3. *Let $G = \langle x \rangle A$ be the split extension of A by $\langle x \rangle$, where $A = T \times \langle e \rangle \times E_1$ is as in Lemma* 3.1 *and x is the inversion automorphism of A. Then $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ is not centre-by-periodic.*

The minimal case of both Lemmas 3.1 and 3.3 is when A is the direct product of a Prüfer p^∞-group for some prime p and an infinite cyclic group. Thus we have now confirmed statement 1.3 and the claims in the paragraph immediately following it. In both these cases, G is a soluble group of finite abelian total rank and Hirsch numbers 2 and 1 respectively. (A soluble group G has *finite abelian total rank* if G has finite Hirsch number and $\tau(G)$ satisfies the minimal condition on subgroups.) Suppose G is a finite extension of a soluble group of finite abelian total rank. If G has no Prüfer subnormal subgroups, then G is (torsion-free)-by-finite and $\Gamma = \mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ is centre-by-finite by Proposition 1.2. If G has no infinite cyclic subnormal subgroups, then $\mathrm{Fitt}\, G$ is periodic and Γ is locally finite, also by Proposition 1.2. Lemma 3.3 shows that if G contains at least one Prüfer subgroup and at least one infinite cyclic subgroup, then Γ need not be centre-by-periodic, which tidies things up nicely.

4. Examples: centre-by-(locally finite) groups

Proof of Statement 1.1. Let Γ be any FC-group. We claim that $\Gamma \in \mathrm{F}_{\mathbf{F}}$. Let $G = \langle x \rangle(\Gamma_1 \times \Gamma_2)$ be the wreath product of Γ and a cyclic group of order 2, where $|x| = 2$, Γ_1 and Γ_2 are copies of Γ, and x interchanges Γ_1 and Γ_2. It is easy to check that $C_G(\gamma)$ has finite index in G for every $\gamma \in \Gamma_1$ and that $\Gamma_1 \cap \zeta_1(G) = \langle 1 \rangle$. Thus Γ embeds into $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ via any isomorphism of Γ to Γ_1 followed by conjugation on G.

Proof of Proposition 1.5. Suppose Γ is an infinite simple subgroup of $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ for some group G. Note first that Γ is locally finite (since Γ' always is by 2.5). If Γ does not act faithfully on $[G, \Gamma, \Gamma^G]$, it centralizes it. But then Γ stabilizes the series $G \geq [G, \Gamma] \geq [G, \Gamma, \Gamma^G] \geq \langle 1 \rangle$ and consequently is nilpotent. Thus Γ acts faithfully on $[G, \Gamma, \Gamma^G]$ and we may henceforth assume that G is locally finite-normal by Lemma 2.3.

Suppose G is locally nilpotent. Since G is also locally finite-normal, it is hypercentral. If Γ centralizes every upper central factor of G, a simple induction on the central height of G (and the simplicity of Γ) shows the Γ centralizes G, which it does not. Thus we may assume that G is abelian. Suppose Γ centralizes every elementary abelian Γ-invariant section of G. Then since Γ is perfect simple, Γ centralizes G/G^n for every positive integer n. But then Γ centralizes $G/\bigcap_n G^n$ and $\bigcap_n G^n$. This implies that Γ

is abelian, which it is not. Thus we may assume that G is an elementary abelian p-group for some prime p and then $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}\, G$ is the finitary linear group $\mathrm{FAut}_{\mathrm{GF}(p)}(G)$. This settles this case.

Now consider the case where G is not locally nilpotent. Choose $\Delta \leq \Gamma$ finite but not nilpotent. There exists N normal of finite index in G that is centralized by Δ and there exists $H \geq N$ normal in G with $G/H \cong S$ (finite) simple. Set $K = \bigcap(X \triangleleft G : G/X \cong S)$. Clearly K is normal in ΓG. We claim that Γ acts faithfully on some section of G that is residually a specific finite simple group. If not, $[G, \Gamma] \leq K$ and Γ acts faithfully on K. Clearly $(K : K \cap N) < (G : N)$. If $[G, \Gamma]$ is not contained in N we repeat the above step, replacing G and N by K and $K \cap N$ and, if possible, keep going. After a finite number, r say, of steps we arrive at $[G, {}_r\Gamma] \leq N$. But then Δ is nilpotent, which we have assumed otherwise. Thus Γ acts faithfully on some section of G that is residually a specific finite simple group. Consequently, we may assume that G is residually a specific finite simple group S. If S is cyclic, then G is abelian and we are back in the previous case. Thus assume S is a perfect finite simple group.

Let $\Omega = \{S_i : i \in I\}$ be the set of all normal subgroups of G isomorphic to S. We claim that $G = \langle S_i : i \in I\rangle$, from which it follows that G is the direct product of the S_i. Now G is locally finite-normal. Let $X < Y$ be non-trivial finite normal subgroups of G. Then there exist distinct normal subgroups $H_1, \ldots, H_r$ of G with each G/H_j isomorphic to S such that with $H = \bigcap_j H_j$ we have $X \cap H = \langle 1\rangle$. Then $G/H \cong S^{(r)}$ and $XH/H \cong X$ is a normal subgroup of G/H. Hence $X = X_1 \times \cdots \times X_m$ for some $X_j \cong S$. In the same way we have $Y = Y_1 \times \cdots \times Y_n$ with each Y_j isomorphic to S. But X is normal in Y, so each X_j is a Y_k for some $k = k(j)$. Consequently, each X_j is normal in Y, for all such Y. Therefore each X_j is an S_i and therefore $G = \langle S_i : i \in I\rangle$.

The normal subgroups of G are the $\langle S_i : i \in J\rangle$ as J ranges over all possible subsets of I. Clearly Γ embeds into $\mathrm{Sym}(\Omega)$ via its action on G. If N is a normal subgroup of G of finite index, then $N = \langle S_i : i \in J\rangle$ with J cofinite in I, the latter since $G/N \cong \langle S_i : i \in I \setminus J\rangle$. Thus $\Gamma \leq \mathrm{FSym}(\Omega)$. Therefore Γ is a finitary alternating group, not necessarily over Ω itself; this follows from a theorem of Wielandt [16, Satz 9.4]. The finitary alternating groups are finitary linear, in fact over any field, via (infinite) permutation matrices. The proof is complete.

Proof of Statement 1.4. We prove that Hall's countable universal locally finite group is not an $\mathrm{F}_{\mathbf{F}}$-group. If it were it would be isomorphic to some finitary linear group by Proposition 1.5, and hence (see [5, 6.1]) so would every countable locally finite group. However, there are very many such groups that are not isomorphic to finitary linear groups. For example, finitary linear Baer groups are Fitting groups (see the theorem of [12]), and there are countable locally finite Baer p-groups, p a prime, that are not Fitting groups (see [9, Vol. 2, p. 4]).

5. The polycyclic cases

LEMMA 5.1. *Let K and Z be normal subgroups of a group G such that $Z \leq K \cap \zeta_1(G)$ and $K/Z \in \mathbf{PF}$. Suppose $G/C_G(K)$ is finitely generated. Then $K = TZ$, where T is a $\mathbf{PF}$-subgroup normal in G.*

Proof. There are finite sets X and Y with $K = \langle X \rangle Z$, $G = \langle Y \rangle C_G(K)$ and $Y = Y^{-1}$. For each $x \in X$ and $y \in Y$ there exists $z(x, y) \in Z$ with $x^y \in \langle X \rangle z(x, y)$. Set $S = \langle z(x, y) : x \in X \text{ and } y \in Y \rangle$ and $T = \langle X \rangle S$. Then T is finitely generated, $T/(T \cap Z) \in \mathbf{PF}$ and $T \cap Z \leq \zeta_1(T)$. Therefore $T \in \mathbf{PF}$, e.g. by [15, 3.9]. Also G centralizes S, and Y normalizes T. Therefore G normalizes T.

REMARK. In Lemma 5.1 we can weaken the hypothesis that $G/C_G(K)$ is finitely generated to $G/C_G(K/K')$ being finitely generated, which makes 5.1 much more similar to its **Ch** analogue, Lemma 7.3 below. (In Section 5 we do not need the weaker hypothesis, but in Section 7 the stronger hypothesis is definitely insufficient.) To see that this strengthening of 5.1 holds, note that $K/\zeta_1(K) \in \mathbf{PF}$, so $K' \in \mathbf{PF}$ (see [9, Vol. 1, p. 115]). Thus one may pass to G/K' and then 5.1 in its present form applies.

LEMMA 5.2. *For some group G, let Γ be a finitely generated subgroup of $\mathrm{F}_{\mathbf{PF}}\mathrm{Aut}\, G$. The following hold:*

(a) *If G is centre-by-$\mathbf{PF}$, then $G' \in \mathbf{PF}$ and $\Gamma \in \mathbf{L}_{\mathbb{Z}}$.*
(b) *$[G, \Gamma]$ is centre-by-$\mathbf{PF}$.*
(c) *$[G, \Gamma, \Gamma^G] \in \mathbf{PF}$.*
(d) *$\Gamma \in (\mathbf{G} \cap \mathbf{A})\mathbf{L}_{\mathbb{Z}}$.*
(e) *If Γ is soluble-by-finite, then $\Gamma \in \mathbf{PF}$.*
(f) *Γ is residually finite.*

Proof. There exists a normal subgroup N of G with $[N, \Gamma] = \langle 1 \rangle$ and $G/N \in \mathbf{PF}$.

(a) Here $G/\zeta_1(G) \in \mathbf{PF}$. Apply Lemma 5.1 with K, Z and G replaced by G, $N \cap \zeta_1(G)$ and ΓG respectively. Thus $G = TZ$ for some $\mathbf{PF}$-subgroup T normal in ΓG and Z central in ΓG. Then Γ embeds into $\mathrm{Aut}\, T$ and so lies in $\mathbf{L}_{\mathbb{Z}}$ by a theorem of Merzlyakov ([6] or [11, 1.4]). Finally $G' \in \mathbf{PF}$ (see [9, Vol. 1, p. 115]).

(b) Set $C = C_G(N)$ and note that $[G, \Gamma] \leq C$ by Lemma 2.1. Then $C \cap N \leq \zeta_1(C)$ and $C/(C \cap N) \cong CN/N \leq G/N$, so C is centre-by-$\mathbf{PF}$. Part (b) follows.

(c) Applying Lemma 5.1 to $Z = C \cap N \leq C \leq \Gamma C$ we have $C = TZ$ for some **PF**-subgroup T normal in ΓC, and Z central in ΓC. Then $[C, \Gamma]$ equals $[T, \Gamma] \leq T$ and so is a **PF**-group. Also so is C' by [9, Vol. 1, p. 115] again. Consequently, since $[G, \Gamma] \leq C$,

$$[C, \Gamma^G] = [C, [G, \Gamma]\Gamma] \leq [C, \Gamma]C'$$

is also a **PF**-subgroup. Part (c) follows.

(d) Set $\Sigma = C_\Gamma(G/[C, \Gamma^G]) \cap C_\Gamma([C, \Gamma^G])$. Now $G = \langle X \rangle N$ for some finite subset X. If $\sigma \in \Sigma$ and $x \in X$, then $[Nx, \sigma] = \{[x, \sigma]\}$ and $\sigma \mapsto ([x, \sigma])_{x \in X}$ is an embedding of Σ into $A = (\zeta_1([C, \Gamma^G]))^X$ (cf. [5, 1.C.3(a)]). But $[C, \Gamma^G] \in \mathbf{PF}$ and X is finite; therefore A and so Σ are finitely generated abelian groups. Now $\Gamma/C_\Gamma([C, \Gamma^G]) \in \mathbf{L}_\mathbb{Z}$ by Merzlyakov's theorem again, since $[C, \Gamma^G] \in \mathbf{PF}$. Also Γ stabilizes the series $G \geq C \geq [C, \Gamma^G]$, so $\Gamma' \leq C_\Gamma(G/[C, \Gamma^G])$ and $\Gamma/C_\Gamma(G/[C, \Gamma^G])$ is abelian and finitely generated. Trivially such a group lies in $\mathbf{L}_\mathbb{Z}$. Therefore $\Gamma/\Sigma \in \mathbf{L}_\mathbb{Z}$ and part (d) follows.

(e) Soluble $\mathbf{L}_\mathbb{Z}$-groups are always polycyclic (a theorem of Mal'tsev, e.g. see [15, 4.4]) and clearly $(\mathbf{G} \cap \mathbf{A})\mathbf{P} = \mathbf{P}$. Therefore part (d) implies (e).

(f) Now $[C, \Gamma^G] \in \mathbf{PF}$, so if n is a positive integer, then $[C, \Gamma^G]/[C, \Gamma^G]^n$ is finite and $\bigcap_n [C, \Gamma^G]^n = \langle 1 \rangle$. Hence $\bigcap_n C_\Gamma(G/[C, \Gamma^G]^n) = \langle 1 \rangle$ and $C_\Gamma([C, \Gamma^G]/[C, \Gamma^G]^n)$ stabilizes $G \geq C \geq [C, \Gamma^G] \geq [C, \Gamma^G]^n$. Thus $\Gamma/C_\Gamma(G/[C, \Gamma^G]^n)$ is nilpotent-by-finite, is by hypothesis finitely generated, and therefore is residually finite. Consequently, Γ is residually finite.

Clearly essentially the same proof yields the following (although (a) follows from 5.2(a), (b) from 2.1 and then (c)–(f) are immediate from (b) and 5.2(c)–(f).

Lemma 5.3. *For some group G, let Γ be a finitely generated subgroup of* $\mathrm{F}_\mathbf{P} \operatorname{Aut} G$. *The following hold:*

(a) *If G is centre-by-$\mathbf{P}$, then $G' \in \mathbf{P}$ and $\Gamma \in \mathbf{L}_\mathbb{Z}$.*
(b) *$[G, \Gamma]$ is centre-by-$\mathbf{P}$.*
(c) *$[G, \Gamma, \Gamma^G] \in \mathbf{P}$.*
(d) *$\Gamma \in (\mathbf{G} \cap \mathbf{A})\mathbf{L}_\mathbb{Z}$.*
(e) *If Γ is soluble, then $\Gamma \in \mathbf{P}$.*
(f) *Γ is residually finite.*

Note that Theorem 1.6 is an immediate consequence of 5.2(e), and Theorem 1.7 is an immediate consequence of 5.2(d) and 5.2(f).

6. Examples: polycyclic groups. We have already pointed out that $\mathrm{F}_\mathbf{PF} \supseteq \mathrm{F}_\mathbf{F} \cup \mathrm{F}_\mathbf{P}$. If R is a ring with its additive group finitely generated (e.g. $R = \mathbb{Z}$) and if M is any R-module, then the finitary (module) automorphism group $\mathrm{FAut}_R M$ lies in $\mathrm{F}_\mathbf{P}$. In particular $\mathrm{FAut}_\mathbb{Z} A$ lies in $\mathrm{F}_\mathbf{P}$ for every abelian group A. A group G is $\mathbf{X}$-finitary if $G/C_G(g^G) \in \mathbf{X}$ for every $g \in G$; that is, G is $\mathbf{X}$-finitary if and only if $\mathrm{Inn}\, G \le \mathrm{F}_{\mathbf{X}}\mathrm{Aut}\, G$ (assuming $\mathbf{X}$ is quotient-closed). An $\mathbf{F}$-finitary group is just an FC-group and $\mathbf{PF}$-finitary groups in [2] are called *PC-groups*. Note that to say G is $\mathbf{X}$-finitary is not the same as saying $G \in \mathrm{F}_{\mathbf{X}}$, even for $\mathbf{X}$ equal to $\mathbf{F}$, $\mathbf{P}$ or $\mathbf{PF}$: if G is an infinite simple finitary linear group over a finite field, then G lies in $\mathrm{F}_{\mathbf{F}} \cap \mathrm{F}_{\mathbf{P}}$ but clearly cannot be $\mathbf{PF}$-finitary. Of course if G is $\mathbf{X}$-finitary, then $\mathrm{Inn}\, G \in \mathrm{F}_{\mathbf{X}}$. Any $\mathbf{F}$-finitary group Γ lies in $\mathrm{F}_{\mathbf{F}}$ by 1.1. The trick used in the proof of 1.1, namely letting G be the wreath product of Γ with a cyclic group of order 2 and allowing Γ to act on G via the inner automorphisms of G induced by one of the two direct factors of the base group isomorphic to Γ, proves the following.

STATEMENT 6.1. *Every* $\mathbf{PF}$*-finitary* (*resp.* $\mathbf{P}$*-finitary*) *group lies in* $\mathrm{F}_{\mathbf{PF}}$ (*resp.* $\mathrm{F}_{\mathbf{P}}$).

Thus $\mathbf{PF}$-finitary $\mathrm{L}(\mathbf{SF})$-groups are locally $\mathbf{PF}$-groups by 1.6 and 6.1. In fact much more is already known to be true. Write $G \in \mathrm{L}(\triangleleft\, \mathbf{PF})$ if G has a local system of normal $\mathbf{PF}$-subgroups of G, and similarly with $\mathbf{P}$ in place of $\mathbf{PF}$.

THEOREM 6.2 (Franciosi, de Giovanni and Tomkinson [2]). *A group G is* $\mathbf{PF}$*-finitary if and only if* $G \in \mathrm{L}(\triangleleft\, \mathbf{PF})$; *also G is* $\mathbf{P}$*-finitary if and only if* $G \in \mathrm{L}(\triangleleft\, \mathbf{P})$.

Note that the corresponding statement for the class $\mathbf{F}$ is false; that is, being an FC-group is not the same as being locally (finite normal). It is easy to derive a proof of 6.2 from what we have done above.

Proof. Let G be $\mathbf{PF}$-finitary and pick $g \in G$. Set $N = C_G(g^G)$ and $K = C_G(N)$. Now $G/N \in \mathbf{PF}$, so $G = \langle Y \rangle N$ for some finite subset Y of G. Then $G/C_G(Y^G)$ is also in $\mathbf{PF}$. Set $Z = C_K(Y^G)$. Then $K/Z \in \mathbf{PF}$ and $Z \le \zeta_1(G)$, so $K = TZ$ by Lemma 5.1, where T is in $\mathbf{PF}$ and normal in G. But $g \in K$, so $\langle g \rangle T$ is in $\mathbf{PF}$ and is normal in G. Consequently, $\langle g^G \rangle \le \langle g \rangle T$ is also in $\mathbf{PF}$ and therefore $G \in L(\triangleleft\, \mathbf{PF})$.

Conversely, suppose $G \in \mathrm{L}(\triangleleft\, \mathbf{PF})$ and choose $g \in G$. Then $\langle g^G \rangle \in \mathbf{PF}$, so $G/C_G(g^G) \in \mathbf{L}_{\mathbb{Z}}$ by Merzlyakov's theorem. Thus G is $\mathbf{PF}$-finitary by the following lemma.

LEMMA 6.3. *If* $G \le \mathrm{GL}(n, \mathbb{Z})$ *is locally* (*soluble-by-finite*), *then* $G \in \mathbf{PF}$.

Proof. If $X \le Y$ are finitely generated subgroups of G, then $X^0 \le Y^0$ and are soluble, so $M = \bigcup_X X^0$ is a locally soluble, normal subgroup of G with G/M locally finite (see [10, Chapter 5]). Then M is soluble [10, 3.8], so its (Zariski) closure N in G is a soluble normal subgroup of G [10, 5.9 and 5.11] and G/N is isomorphic to a (locally finite) subgroup of $\mathrm{GL}(r, \mathbb{Q})$

for some integer $r \geq 1$ [10, 6.4]. Then G/N is finite by [10, 9.33] and N is polycyclic by Mal'tsev's theorem [15, 4.4].

Returning to the proof of 6.2, if $G \in L(\triangleleft\, \mathbf{P})$, then clearly every $\mathbf{PF}$-image of G is locally soluble and hence polycyclic. Thus G is $\mathbf{P}$-finitary by the $\mathbf{PF}$ case above. If G is $\mathbf{P}$-finitary, then every section of G is $\mathbf{P}$-finitary and also $G \in \mathrm{L}(\triangleleft\, \mathbf{PF})$ by the $\mathbf{PF}$ case. But finite $\mathbf{P}$-finitary groups clearly are soluble, so $G \in \mathrm{L}(\triangleleft\, \mathbf{P})$. All parts of 6.2 are now proved.

7. The Chernikov case. The basic properties in this case were discovered by Yu. D. Polovitskiĭ. In [7] (see also [9, 4.27]), he proved that if G is a group with $G/\zeta_1(G) \in \mathbf{Ch}$, then $G' \in \mathbf{Ch}$, and in [8] he proved that a group G is $\mathbf{Ch}$-finitary if and only if for all $x \in G$ we have $\langle x^G \rangle \in \mathbf{Ch}.\mathbf{G}_1$ and $G/C_G(x^G)$ periodic. ($\mathbf{G}_1$ denotes the class of all cyclic groups.) The latter is not quite in the form using the local operator that we have used in the $\mathbf{F}$, $\mathbf{P}$ and $\mathbf{PF}$ cases and Polovitskiĭ's Second Theorem can be reworked as follows (but see the end of this section for proofs).

Statement 7.1. *A group G is $\mathbf{Ch}$-finitary if and only if G is a union of normal subgroups L satisfying $[L, G] \in \mathbf{Ch}$ with $G/C_G(L)$ periodic.*

Thus if G is a $\mathbf{Ch}$-finitary group, then $G' \leq \tau(G)$, every periodic normal subgroup of G is locally (a normal $\mathbf{Ch}$-subgroup of G) and $\tau(G)$ is the union of all the normal $\mathbf{Ch}$-subgroups of G.

With $\mathrm{F}_{\mathbf{Ch}}$ defined analogously to $\mathrm{F}_{\mathbf{F}}$ and $\mathrm{F}_{\mathbf{P}}$ etc., clearly $\mathrm{F}_{\mathbf{Ch}} \supseteq \mathrm{F}_{\mathbf{F}}$. If Γ is $\mathbf{Ch}$-finitary then $\Gamma \in \mathrm{F}_{\mathbf{Ch}}$ via the usual trick of letting G be the wreath product of Γ by a cyclic group of order 2. As with the $\mathbf{PF}$ case, $\mathrm{F}_{\mathbf{Ch}}$-groups need not by $\mathbf{Ch}$-finitary and for much the same reason. If $G \in \mathbf{Ch}$, then $\mathrm{F}_{\mathbf{Ch}}\mathrm{Aut}\, G = \mathrm{Aut}\, G \in \mathbf{Ch}.\mathbf{L}_{\mathbb{C}}$. Polovitskiĭ's Second Theorem suggests that perhaps we should be working with $\mathbf{Ch}.\mathbf{G}_1$ rather that $\mathbf{Ch}$. However, if $G \in \mathbf{Ch}.\mathbf{G}_1$, then $\mathrm{Aut}\, G$ lies in $\mathbf{Ch}.\mathbf{L}_{\mathbb{C}}$ but need not be $\mathbf{Ch}$-finitary; just consider the extension G of a Prüfer p^∞-group C by its automorphism x given by $a^x = a^{1+p}$ for all $a \in C$, when G' is in $\mathbf{Ch}$ but neither G nor $\mathrm{Inn}\, G$ is $\mathbf{Ch}$-finitary. However $\mathbf{Ch}.\mathbf{G}_1$-groups always lie in $\mathrm{F}_{\mathbf{Ch}}$ (see Lemma 8.3 below).

Suppose G is abelian. Then $\mathrm{F}_{\mathbf{Ch}}\mathrm{Aut}\, G$ is studied in [14], being denoted there by $\mathrm{F}_1\, \mathrm{Aut}_{\mathbb{Z}}\, G$. For example, it is shown in [14] that if G is abelian, then $\mathrm{F}_{\mathbf{Ch}}\mathrm{Aut}\, G$ is locally residually finite [14, 10_1] and is an extension of a locally residually nilpotent group by an $\mathrm{F}_{\mathbf{P}}$-group [14, 2_1]. Further $\mathrm{F}_{\mathbf{P}}\mathrm{Aut}\, G$ is an extension of a Fitting group by a group that is not necessarily in $\mathrm{F}_{\mathbf{Ch}}$, but is quite close to it (see [14, 2^1]). Thus there is some sort of vague relationship between the classes $\mathrm{F}_{\mathbf{P}}$ and $\mathrm{F}_{\mathbf{Ch}}$. Taking G to be a divisible abelian p-group of rank n shows that $\mathrm{GL}(n, \mathbb{Z}_p)$ lies in $\mathrm{F}_{\mathbf{Ch}}$ (where $\mathbb{Z}_p$ denotes the ring of p-adic integers) and in particular that $\mathrm{F}_{\mathbf{Ch}} \cap \mathbf{L}_{\mathbb{C}} \supseteq \mathbf{L}_{\mathbb{Z}}$. Then [10, 2.11] shows that any free metabelian group of finite rank (indeed of rank at most the cardinality of $\mathbb{C}$) embeds into $\mathrm{GL}(2, \mathbb{Z}_p)$ and so lies in $\mathrm{F}_{\mathbf{Ch}}$; presumably this also holds for such groups of any rank.

LEMMA 7.2. *Let G be a $\mathbf{Ch}$-group and Γ a subgroup of* $\mathrm{Aut}\, G$. *If*

(a) *G is abelian or*
(b) *Γ is finitely generated,*

then $\Gamma \in \mathbf{L}_{\mathbb{C}}$.

Proof. (a) Note first that finite extensions of $\mathbf{L}_{\mathbb{C}}$-groups are in $\mathbf{L}_{\mathbb{C}}$ (e.g. [10, 2.3]). Here $G = F \oplus D$, where F is finite and D is divisible. Then $\mathrm{Hom}(D, F) = \{0\}$, $\mathrm{Hom}(F, D)$ is finite and

$$\mathrm{Aut}\, G = ((\mathrm{Aut}\, F) \times (\mathrm{Aut}\, D))(1 + \mathrm{Hom}(F, D)).$$

Also $(\mathrm{Aut}\, F) \times (\mathrm{Aut}\, D)$ lies in $\mathbf{L}_{\mathbb{C}}$ and has finite index in $\mathrm{Aut}\, G$. Part (a) follows.

(b) Let A denote the finite residual of G, so A is divisible abelian and G/A is finite. There is a finite subgroup K of G such that $KA = G$ and such that K and $K\phi$ are conjugate in G for every automorphism ϕ of G (see [5, 3.9]). Let g^* be the inner automorphism of G determined by the element $g \in G$. Then $\mathrm{Aut}\, G = N_{\mathrm{Aut}\, G}(K).G^*$, so if $C = C_{\mathrm{Aut}\, G}(K)$ then CA^* has finite index in $\mathrm{Aut}\, G$.

Now Γ is finitely generated, so $\Gamma \cap CA^*$ is too and $\Gamma \cap CA^* \leq \langle X, Y\rangle$ for some finite subsets X of C and Y of A^*. But A^* is a divisible abelian normal $\mathbf{Ch}$-subgroup of $\mathrm{Aut}\, G$, so $\langle Y^X\rangle$ is finite. Hence $\langle X\rangle$ has finite index in $\langle X, Y\rangle$. Now $\langle X\rangle \leq C$ and $C \cap C_{\mathrm{Aut}\, G}(A) = \langle 1\rangle$ (since $KA = G$), so C embeds into $\mathrm{Aut}\, G/C_{\mathrm{Aut}\, G}(A)$ and hence into $\mathrm{Aut}\, A$, which is an $\mathbf{L}_{\mathbb{C}}$-group, for example by part (a). Therefore C, $\langle X\rangle$, $\langle X, Y\rangle$, $\Gamma \cap CA^*$ and Γ are all in $\mathbf{L}_{\mathbb{C}}$.

We now produce analogues of 5.1 and 5.2.

LEMMA 7.3. *Let K and Z be normal subgroups of a group G such that $Z \leq K \cap \zeta_1(G)$ and $K/Z \in \mathbf{Ch}$. Suppose $G/C_G(K/K')$ is finitely generated. Then $[K, G] \in \mathbf{Ch}$.*

Proof. By Polovitskiĭ's First Theorem $K' \in \mathbf{Ch}$ and K' is clearly normal in G and contained in $[K, G]$. Thus we may pass to G/K' and assume that K is abelian. Then $G = \langle X\rangle C_G(K)$ for some finite set X. If $x \in X$, then $kZ \mapsto [k, x]$ is a homomorphism of K/Z into K and hence $[K, x] \in \mathbf{Ch}$. Let $T = \langle [K, x] : x \in X\rangle \leq [K, G]$. Then $T \in \mathbf{Ch}$. Also $x \in X$ centralizes $K/[K, x]$, so $\langle X\rangle$ centralizes K/T, as trivially so does $C_G(K)$. Therefore $[K, G] = T \in \mathbf{Ch}$.

LEMMA 7.4. *Let H be a $\mathbf{Ch}$-group and A an H-module whose additive group is a $\mathbf{Ch}$-group. Then* $\mathrm{Der}(H, A) = S \times T$, *where S is $\mathbb{Z}$-torsion-free and $T \in \mathbf{Ch}$. If H is divisible, then $T = \langle 0\rangle$.*

Proof. Let K denote the finite residual of H. Now $\mathrm{Aut}_{\mathbb{Z}} A$ is residually finite. Hence K acts trivially on A and $\mathrm{Der}(K, A) = \mathrm{Hom}(K, A)$, which is torsion-free (e.g. [3, p. 182]). If $H = K$, then $\mathrm{Der}(H, A)$ is torsion-free.

Now H/K is finite; let $\mathbf{X}$ be a (finite) transversal of H to K. If $x \in X$, $k \in K$ and $\delta \in \mathrm{Der}(H, A)$, then

$$(xk)\delta = (x\delta)k + k\delta = x\delta + k\delta.$$

Hence $\delta \mapsto \{(x\delta)_{x\in X}, \delta|_K\}$ is an additive embedding of $\mathrm{Der}(H, A)$ into $A^{(X)} \oplus \mathrm{Hom}(K, A)$. Thus the torsion subgroup T of $\mathrm{Der}(H, A)$ embeds into $A^{(X)}$ and as such lies in $\mathbf{Ch}$. Consequently, $\mathrm{Der}(H, A) = S \oplus T$ for some torsion-free S by [3, 21.2 and 27.5].

Lemma 7.5. *For some group G, let Γ be a finitely generated subgroup of $\mathrm{F}_{\mathbf{Ch}}\mathrm{Aut}\, G$. The following hold:*

(a) *If G is centre-by-$\mathbf{Ch}$, then $[G, \Gamma]G' = [G, \Gamma G] \in \mathbf{Ch}$.*
(b) *$[G, \Gamma]$ is centre-by-$\mathbf{Ch}$.*
(c) *$[G, \Gamma, \Gamma^G] \in \mathbf{Ch}$.*
(d) *Γ is residually finite.*
(e) *$\Gamma \in (\mathbf{F}^{-\mathrm{S}} \cap \mathbf{A})\mathbf{L}_{\mathbb{C}}$.*
(f) *Γ is (torsion-free)-by-finite.*
(g) *If $\tau(G) = \langle 1\rangle$, more generally if G has no non-trivial normal $\mathbf{Ch}$-subgroups, then Γ is finitely generated abelian.*

By 6.2 a finitely generated group is $\mathbf{PF}$-finitary if and only if it is $\mathbf{PF}$. Since $\mathbf{G} \cap \mathbf{Ch} = \mathbf{F}$, the $\mathbf{Ch}$ analogue of this, namely that a finitely generated group is $\mathbf{Ch}$-finitary if and only if it is $\mathbf{F}$-finitary (and hence centre-by-finite), is immediate.

Proof. There is a normal subgroup N of G centralized by Γ with $G/N \in \mathbf{Ch}$.

(a) Here also $G/\zeta_1(G) \in \mathbf{Ch}$ and therefore so does G/Z for $Z = N \cap \zeta_1(G)$. Clearly $Z \le \zeta_1(\Gamma G)$. Thus $[G, \Gamma G] \in \mathbf{Ch}$ by 7.3, where we have taken G for K and ΓG for G.

(b) Set $C = C_G(N)$ and note that $[G, \Gamma] \le C$ by 2.1. Now set $Z = C \cap N$. Then $Z = \zeta_1(N) \le \zeta_1(C)$ and $C/Z \cong CN/N \le G/N \in \mathbf{Ch}$. Thus C is centre-by-$\mathbf{Ch}$ and part (b) follows.

(c) Since C is centre-by-$\mathbf{Ch}$, part (a) implies that $[C, \Gamma C] \in \mathbf{Ch}$. But $[G, \Gamma] \le C$, so ΓC is normal in ΓG. Hence $[C, \Gamma^G] \le [C, \Gamma C]$ and part (c) follows.

(d) Let B/N denote the finite residual of G/N. Then G/B and $\Gamma/C_\Gamma(G/B)$ are finite, so $C_\Gamma(G/B)$ is also finitely generated. Thus we may assume that $[G, \Gamma] \le B$.

Suppose $\Gamma = \langle \gamma_1, \dots, \gamma_m\rangle$. Now B/N is a divisible abelian $\mathbf{Ch}$-group. For $r \ge 1$ set $B_r = \{b \in B : b^r \in N\}$. Then each B_r/N is finite. By [5, 3.9] there is a finite subgroup K/N of G/N with $KB = G$ such that $K\gamma$ is conjugate to K, in particular by an element of B, for every γ in Γ. Hence for each i there exists $b_i \in B$ with $K\gamma_i = b_i^{-1}Kb_i$. There exists $s \geq 1$ with each b_i in B_s. Suppose s divides r. Then B_r/N is finite and Γ normalizes KB_r. Hence $\Gamma/C_\Gamma(KB_r)$ embeds into $\mathrm{F}_{\mathbf{F}}\mathrm{Aut}(KB_r)$. As such it is finitely generated and, by 2.3, centre-by-finite. Thus $\Gamma/C_G(KB_r)$ is residually finite. Clearly $\bigcap_{s|r} C_\Gamma(KB_r) = C_\Gamma(G) = \langle 1\rangle$. Therefore Γ is residually finite.

(e) Set $\Sigma = C_\Gamma(G/[C,\Gamma^G]) \cap C_\Gamma([C,\Gamma^G])$. Now Γ stabilizes the series $G \geq C \geq [C,\Gamma^G]$, so $\Gamma/C_\Gamma(G/[C,\Gamma^G])$ is abelian, finitely generated and hence an $\mathbf{L}_{\mathbb{C}}$-group. Also $\Gamma/C_\Gamma([C,\Gamma^G]) \in \mathbf{L}_{\mathbb{C}}$ by 7.2. Consequently, $\Gamma/\Sigma \in \mathbf{L}_{\mathbb{C}}$.

Now N centralizes ΓC, so the map $\sigma \mapsto (gN \mapsto [g,\sigma])$ is a well-defined embedding of Σ into $\mathrm{Der}(G/N, \zeta_1([C,\Gamma^G]))$. Thus the torsion subgroup T of Σ lies in $\mathbf{Ch}$ by 7.4. But Σ is residually finite by part (d), so T is finite. Again by (d) there is a normal subgroup Δ of Γ of finite index with $\Delta \cap T = \langle 1\rangle$. Then $\Delta \cap \Sigma$ is torsion-free abelian. Also $\Delta/(\Delta\cap\Sigma)$ embeds into Γ/Σ and hence is in $\mathbf{L}_{\mathbb{C}}$. Further Γ/Δ is finite. Therefore $\Gamma/(\Delta\cap\Sigma) \in \mathbf{L}_{\mathbb{C}}$ by [10, 2.3].

(f) Finitely generated $\mathbf{L}_{\mathbb{C}}$-groups are (torsion-free)-by-finite [10, 4.8]. Thus part (f) is an easy consequence of part (e).

(g) Here $[G,\Gamma,\Gamma] = \langle 1\rangle$ by part (c), so stability theory shows that Γ is abelian and, by hypothesis, finitely generated.

Note that Theorem 1.8 follows from parts (d) and (e) of Lemma 7.5. Our final three lemmas of Section 7 prove both 7.1 and Polovitskiĭ's Second Theorem.

Lemma 7.6. *Let G be a group. Then $\langle x^G\rangle \in \mathbf{Ch}.\mathbf{G}_1$ for all $x \in G$ if and only if $[\langle x^G\rangle, G] \in \mathbf{Ch}$ for all $x \in G$.*

Proof. Set $H = \langle x^G\rangle$. Clearly $H = \langle x\rangle[H,G]$, so one way round is trivial. Suppose $H \in \mathbf{Ch}.\mathbf{G}_1$ for all $x \in G$. Then $H \cap \tau(G) = \tau(H) \in \mathbf{Ch}$, $H = \langle x\rangle\tau(H)$ and $H/\tau(H)$ is trivial or infinite cyclic. Then $G/\tau(G)$ is torsion-free with each of its cyclic subgroups normal. Therefore $G' \leq \tau(G)$, so $[H,G] \leq H \cap \tau(G) \in \mathbf{Ch}$.

Lemma 7.7. *Let x be an element of a group G and set $H = \langle x^G\rangle$ and $T = \tau(H)$. If $H \in \mathbf{Ch}.\mathbf{G}_1$ and if $G/C_G(T)$ is periodic, then $G/C_G(H) \in \mathbf{Ch}$.*

Proof. Clearly H/T is cyclic, $T \in \mathbf{Ch}$ is normal in G, and $H = \langle x\rangle T$. Also $G/C_G(T) \in \mathbf{Ch}$ by [5, 1.F.3], and $G/C_G(H/T)$ is finite (in fact of order at most 2). Further $(C_G(T) \cap C_G(H/T))/C_G(H)$ embeds into the centre of T by [5, 1.C.3], so $G/C_G(H) \in \mathbf{Ch}$ as claimed.

Lemma 7.8. *Let G be a $\mathbf{Ch}$-finitary group and L a finitely G-generated subgroup of G. Then $G/C_G(L)$ is periodic and $[L,G] \in \mathbf{Ch}$.*

Proof. Set $C = C_G(L)$ and $L = \langle X^G\rangle$, where X is some finite subset. Clearly $C = \bigcap_{x\in X} C_G(x^G)$, so $G/C \in \mathbf{Ch}$. In particular G/C is periodic.

Also $L/(C \cap L) \in \mathbf{Ch}$ and $C \cap L = \zeta_1(L)$, so by Polovitskiĭ's First Theorem $L' \in \mathbf{Ch}$. By passing to G/L' we may assume that $L \leq C$ is abelian.

If $g \in G$, then $L/C_L(g^G) \in \mathbf{Ch}$ and G/C acts on it via conjugation on L. Then G/C acts as a finite group on $L/C_L(g^G)$ by [5, 1.F.3] and hence if K/C is the finite residual of G/C, then G/K is finite, K/C is a divisible abelian $\mathbf{Ch}$-group and

$$[L, K] \leq \bigcap_{g \in G} C_L(g^G) = L \cap \zeta_1(G) = Z \quad \text{say.}$$

If $x \in X$ (recall $L = \langle X^G \rangle$), then the map $kC \mapsto [k, x]$ is a homomorphism of K/C into Z, so $[K, x] \in \mathbf{Ch}$ and is central in G. Set $D = \langle [K, x] : x \in X \rangle \leq Z$. Then D is a central (even divisible) $\mathbf{Ch}$-subgroup of G. By passing to G/D we may assume that K centralizes X. But $C_L(K)$ is normal in G, so $[K, L] = \langle 1 \rangle$ and $K = C$.

There exists a finite set Y with $G = KY$. Then

$$G/C_G(Y^G) = G/\bigcap_{y \in Y} C_G(y^G) \in \mathbf{Ch}$$

and clearly $C_L(Y^G) \leq C_L(KY) = Z$ so $L/Z \in \mathbf{Ch}$. Let W denote the split extension of L by G/K. Then $Z \leq \zeta_1(W)$ and $W/Z \in \mathbf{Ch}.\mathbf{F} = \mathbf{Ch}$. Hence Polovitskiĭ's First Theorem implies that $[L, G] \leq W'$ lies in $\mathbf{Ch}$. The proof is complete.

8. Examples of $\mathrm{F}_{\mathbf{Ch}}$ groups

LEMMA 8.1. *Let R be a finitely generated integral domain of characteristic zero and let n be any positive integer. Then there exists a positive integer m such that for almost all primes p the group* $\mathrm{GL}(n, R)$ *embeds into the automorphism group of the divisible abelian p-group of rank mn.*

Proof. Let F be the quotient field of R. By Noether's Normalization Lemma (e.g. [17, p. 200, Theorem 25]), there exist integers $h > 0$ and $k \geq 0$ and algebraically independent elements $x_1, \ldots, x_k$ (over $\mathbb{Q}$) of $R[h^{-1}] \leq F$ such that R is integral over

$$S = \mathbb{Z}[h^{-1}, x_1, \ldots, x_k] \leq F.$$

But R is a finitely generated ring. Therefore $RS = R[h^{-1}, x_1, \ldots, x_k] \leq F$ is a finitely generated, torsion-free S-module.

Let $K \leq F$ denote the quotient field of S. Then $\dim_K RK = m < \infty$ and there exist elements $y_1, \ldots, y_m$ of RK and some $\alpha \in S \setminus \{0\}$ such that $RK = \bigoplus_i Ky_i$ and $\bigoplus_i \alpha S y_i \leq RS \leq \bigoplus_i S y_i$. Then

$$\bigoplus_i \alpha S[\alpha^{-1}] y_i \leq RS[\alpha^{-1}] \leq \bigoplus_i S[\alpha^{-1}] y_i,$$

so $RS[\alpha^{-1}] = \bigoplus_i S[\alpha^{-1}] y_i$.

Since $\alpha \in S$ there exists $s > 0$ such that $\alpha h^s \in \mathbb{Z}[x_1, \dots, x_k]$, which is a polynomial ring over the x_i's. Hence there exist integers $r_1, \dots, r_k$ with $\beta = \alpha h^s(r_1, \dots r_k) \neq 0$. Let p be any prime exceeding the integer $|\beta|h$. Then in $\mathbb{Z}_{(p)} = \{u/v : u, v \in \mathbb{Z}\ \&\ v \notin \mathbb{Z}p\}$ we see that h and $\beta h^{-s} = \alpha(r_1, \dots, r_k)$ are not congruent to 0 modulo p. In the p-adic integers $\mathbb{Z}_p$ there exist algebraically independent (over the rationals) elements $z_1, \dots, z_k$ in $p\mathbb{Z}_p$. Also

$$\alpha(r_1 + z_1, \dots, r_k + z_k) \equiv_p \alpha(r_1, \dots, r_k) = h^{-s}\beta,$$

which is not congruent to 0 modulo p in $\mathbb{Z}_{(p)}$ and hence not congruent to 0 modulo p in $\mathbb{Z}_p$ either. Thus

$$\alpha(r_1 + z_1, \dots, r_k + z_k)$$

is a unit of $\mathbb{Z}_p$ and hence $S[\alpha^{-1}]$ is isomorphic to the subring

$$\mathbb{Z}[h^{-1}, r_1 + z_1, \dots, r_k + z_k, \alpha(r_1 + z_1, \dots, r_k + z_k)^{-1}]$$

of $\mathbb{Z}_p$.

Let $e_1, \dots, e_n$ be the standard basis of the row vector space $F^{(n)}$. Then

$$\bigoplus_j RS[\alpha^{-1}]e_j = \bigoplus_{i,j} S[\alpha^{-1}]y_i e_j$$

and so $\mathrm{GL}(n, R) \leq \mathrm{GL}(n, RS[\alpha^{-1}])$ embeds into $\mathrm{GL}(mn, S[\alpha^{-1}])$ and hence into $\mathrm{GL}(mn, \mathbb{Z}_p)$. The latter is isomorphic to $\mathrm{Aut}\, D$ for D the divisible abelian p-group of rank mn. The lemma follows.

Corollary 8.2. *If Γ is a finitely generated linear group of characteristic* 0, *then $\Gamma \in \mathrm{F}_{\mathbf{Ch}}$.*

For if $\Gamma \leq \mathrm{GL}(n, F)$, where F is a field of characteristic 0, let R denote the subring of F generated by the entries of the elements of Γ. Then R is finitely generated and Lemma 8.1 applies. Thus we have now proved that $\mathrm{F}_{\mathbf{Ch}} \supseteq \mathbf{G} \cap \mathbf{L}_{\mathbb{C}}$.

Note that 8.2 does not extend to any positive characteristic: if p is a prime let Γ be the wreath product of a cyclic group of order p by an infinite cyclic group. Then Γ is metabelian, 2-generator and isomorphic to a linear group of characteristic p and degree 2 (see [9, 2.16]). However if $\Gamma \in \mathrm{F}_{\mathbf{Ch}}$, then $\Gamma \in (\mathbf{F}^{-\mathrm{S}} \cap \mathbf{A})\mathbf{L}_{\mathbb{C}}$ by Lemma 7.5 and hence $\Gamma \in \mathbf{L}_{\mathbb{C}}$. The latter is false (see [9, 10.21]). Further note that Theorem 1.9 follows from Lemmas 7.2 and 8.1. Finally, we prove the following.

Lemma 8.3. *If Γ is a $\mathbf{Ch}.\mathbf{G}_1$-group, then Γ embeds into* $\mathrm{Aut}\, G$ *for some $\mathbf{Ch}$-group G and in particular lies in $\mathrm{F}_{\mathbf{Ch}}$.*

Proof. Now $\Gamma = A\langle y \rangle$ for some normal $\mathbf{Ch}$-subgroup A of Γ and some $y \in \Gamma$ (we are not assuming that A is abelian). If y has finite order, then $\Gamma \in \mathbf{Ch}$ and hence Γ embeds into the automorphism group of the wreath product $\Gamma \wr \langle x\rangle$, where $|x| = 2$. Clearly here $\Gamma \wr \langle x\rangle \in \mathbf{Ch}$. Thus assume that y has infinite order, so now Γ is the split extension of A by $\langle y\rangle$.

Set $Z = C_{\langle y\rangle}(A)$. If $Z \neq \langle 1\rangle$, then Γ/Z is $\mathbf{Ch}$ and so Γ/Z embeds into $\operatorname{Aut} G_0$ for $G_0 = (\Gamma/Z) \wr \langle x\rangle$. If G_1 is any Prüfer group (for any prime), then $\Gamma/A \cong \langle y\rangle$ is embeddable in $\operatorname{Aut} G_1$. Thus Γ embeds into $\operatorname{Aut}(G_0 \times G_1)$ and clearly $G_0 \times G_1$ is a $\mathbf{Ch}$-group.

Now assume that $Z = \langle 1\rangle$. Set $W = \Gamma \wr \langle x\rangle = \langle x\rangle(\Gamma_1 \times \Gamma_2)$, where $\gamma \mapsto \gamma_i$ is an isomorphism of Γ onto Γ_i and $(\gamma_i)^x = \gamma_{3-i}$ for $i = 1, 2$. Then $\Delta = \langle y_1y_2\rangle A_1 \leq W$ is isomorphic to Γ. Set $G = \langle x\rangle A_1A_2 \leq W$. Clearly $G \in \mathbf{Ch}$. Also A_1A_2 is normal in W and x and y_1y_2 commute, so Δ normalizes G and we have a map of Δ into $\operatorname{Aut} G$ with kernel $C_\Delta(G)$. Now y acts faithfully on A, so y_1y_2 acts faithfully on A_2, and A_1 centralizes A_2. Hence $C_\Delta(A_2) = A_1$. Also if $a \in A \setminus \langle 1\rangle$, then $a_1^{-1}xa_1 = xa_1a_2^{-1} \neq x$. Hence $C_\Delta(x) = \langle y_1y_2\rangle$. Therefore $C_\Delta(G) = \langle 1\rangle$ and consequently $\Gamma \cong \Delta$ embeds into $\operatorname{Aut} G$. The proof is complete.

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