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TOWER MULTIPLEXING AND SLOW WEAK MIXING

BY

TERRENCE ADAMS (Laurel, MD)

Abstract. A technique is presented for multiplexing two ergodic measure preserving transformations together to derive a third limiting transformation. This technique is used to settle a question regarding rigidity sequences of weak mixing transformations. Namely, given any rigidity sequence for an ergodic measure preserving transformation, there exists a weak mixing transformation which is rigid along the same sequence. This establishes a wide range of rigidity sequences for weakly mixing dynamical systems.

1. Introduction. Fix a Lebesgue probability space. Endow the set of invertible measure preserving transformations with the weak topology. It is well known that both the properties of weak mixing and rigidity are generic properties in this topological space [15]. This is interesting since the key behaviors of these two properties contrast greatly. Weak mixing occurs when a system equitably spreads mass throughout the probability space for most times. Rigidity occurs when a system evolves to resemble the identity map infinitely often. Since both of these behaviors exist simultaneously in a large class of transformations, it is natural to ask what types of rigidity sequences are realizable by weak mixing transformations. Here we resolve this question by showing that all rigidity sequences are realizable by the class of weak mixing transformations.

THEOREM 1.1. *Given any ergodic measure preserving transformation R on a Lebesgue probability space, and any rigid sequence ρ_n for R, there exists a weak mixing transformation T on a Lebesgue probability space such that T is rigid on ρ_n.*

Prior to proving this main result, we present a new and direct method for combining two invertible ergodic finite measure preserving transformations to obtain a third limiting transformation. The technique iteratively utilizes the Kakutani–Rokhlin lemma ([18], [21]). A measure preserving transformation T on a separable probability space $(X, \mathbb{B}, \mu)$ is ergodic if any invariant measurable set A has measure 0 or 1. In particular, $TA = A$ implies $\mu(A) = 0$ or 1.

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Lemma 1.2 (Kakutani 1943, Rokhlin 1948). *Let $T : X \to X$ be an ergodic measure preserving transformation on a nonatomic probability space $(X, \mathcal{B}, \mu)$, h a positive integer and $\epsilon > 0$. There exists a measurable set $B \subset X$ such that $B, TB, \ldots, T^{h-1}B$ are pairwise disjoint and $\mu(\bigcup_{i=0}^{h-1} T^i(B)) > 1-\epsilon$. The collection $\{B, TB, \ldots, T^{h-1}B\}$ is referred to as a Rokhlin tower of height h for the transformation T.*

Clearly, this lemma demonstrates that any ergodic measure preserving transformation can be approximated arbitrarily well by periodic transformations in an appropriate topology (i.e. uniform topology); see Halmos [16], [15], Rokhlin [22], Katok and Stepin [20]. Much of the early work in this regard focuses on the topological genericity of specific properties of measure perserving transformations. In [20], results are presented on rates of approximation by periodic transformations, and connections with dynamical properties. Recent research of Kalikow demonstrates the utility of developing a general theory of Rokhlin towers [19]. Also, it is clear from the Kakutani–Rokhlin lemma that any ergodic measure preserving tranformation can be approximated arbitrarily well by another ergodic measure preserving transformation from any isomorphism class. This observation is utilized repeatedly in this work.

Two input transformations R and S are multiplexed together to derive an output transformation T with prescribed properties. The multiplexing operation is defined using an infinite chain of measure-theoretic isomorphisms. In the case where R is ergodic and rigid, and S weak mixing, we present a method for unbalanced multiplexing of R and S. Over time, transformations isomorphic to R are used on a higher proportion of the measure space, as the action by S dissipates over time. We refer to this process informally as slow weak mixing.

A measure preserving transformation $T : X \to X$ is *weak mixing* if for all measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^i A \cap B) - \mu(A)\mu(B)| = 0.$$

Clearly, if T is weak mixing, then T is ergodic. Also, T is weak mixing if and only if T has only 1 as an eigenvalue, and all eigenfunctions are constant almost everywhere. An ergodic measure preserving transformation R is rigid on a sequence $\rho_n \to \infty$ if for any measurable set A,

$$\lim_{n\to\infty} \mu(T^{\rho_n} A \triangle A) = 0.$$

The sequence ρ_n is called a *rigidity sequence* for R.

Several forms of rigidity have been studied in both ergodic theory and topological dynamics. In the case of topological dynamics, both rigidity and uniform rigidity are considered. Uniform rigidity was introduced in [13] and given a specific generic characterization. In [17], it is shown that the notion of uniform rigidity is mutually exclusive from measurable weak mixing on a Cantor set. In particular, every finite measure preserving weak mixing transformation has a representation that is not uniformly rigid. Weak mixing and rigidity have been studied for interval exchange transformations: see [6] and [3] for recent results in this regard. Rigid, weak mixing transformations have been studied in the setting of infinite measure preserving transformations, as well as nonsingular transformations. Mildly mixing transformations are finite measure preserving transformations that do not contain a rigid factor. These are the transformations which yield ergodic products with any infinite conservative ergodic transformation [12]. See [1], [2], [4] and the references therein for results related to notions of weak mixing and rigidity for infinite measure preserving or nonsingular transformations. The notion of IP-sequences was introduced by Furstenberg and Weiss in connection with rigid transformations. There has been recent research on IP-rigidity sequences (i.e. IP-sequences which form a rigid sequence) for weak mixing transformations. See [4] and [14] for results on IP-rigidity.

The notion of rigidity was extended to α-rigidity by Friedman [10]. Transformations are constructed which are α-rigid and $(1-\alpha)$-partial mixing for any $0 < \alpha < 1$. See [11] and [8] for further research on α-rigid transformations. Many of these notions have been studied for more general group actions. See [5] for a survey of weak mixing group actions. Since our results depend mainly on the use of Lemma 1.2 which extends to more general groups (i.e. amenable, abelian), there should exist an extension of techniques provided in this work to a wider class of groups. Since some of the principles provided in this work appear new, we focus exclusively on the case of measure preserving $\mathbb{Z}$-actions on $[0,1)$ with Lebesgue measure.

For a recent comprehensive account on rigidity sequences, we recommend recent publications [4] and [7]. Both of these works provide much detail on the current understanding of rigidity for weak mixing transformations.

2. Towerplex constructions.

The main result is established constructively using Lemma 1.2. Given two transformations R and S, we define a third transformation T which is constructed as a blend of R and S, such that T acts more like R, asymptotically. We will define a sequence of positive integers $h_n, n \in \mathbb{N}$, and a sequence of real numbers $\epsilon_n > 0$ such that $\sum_{n=1}^{\infty} 1/h_n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Also, let r_n and s_n for $n \in \mathbb{N}$ be sequences of real numbers satisfying $0 \leq r_n, s_n \leq 1$.

2.1. Initialization.

Suppose R and S are ergodic measure preserving transformations defined on a Lebesgue probability space $(X, \mu, \mathbb{B})$. Partition

X into two equal sets X_1 and Y_1 (i.e. $\mu(X_1) = \mu(Y_1) = 1/2$). Initialize R_1 isomorphic to R and S_1 isomorphic to S to operate on X_1 and Y_1, respectively. Define $T_1(x) = R_1(x)$ for $x \in X_1$ and $T_1(x) = S_1(x)$ for $x \in Y_1$. Produce Rohklin towers of height h_1 with residual less than $\epsilon_1/2$ for each of R_1 and S_1. In particular, let I_1, J_1 be the base of the R_1-tower and S_1-tower such that

$$\mu\Big(\bigcup_{i=0}^{h_1-1} R_1^i I_1\Big) > \frac{1}{2}(1-\epsilon_1) \quad \text{and} \quad \mu\Big(\bigcup_{i=0}^{h_1-1} S_1^i J_1\Big) > \frac{1}{2}(1-\epsilon_1).$$

Let $X_1^* = X_1 \setminus \bigcup_{i=0}^{h_1-1} R_1^i(I_1)$ and $Y_1^* = Y_1 \setminus \bigcup_{i=0}^{h_1-1} S_1^i(J_1)$ be the residuals for the R_1 and S_1 towers, respectively. Choose $I_1' \subset I_1$ and $J_1' \subset J_1$ such that

$$\mu(I_1') = r_1\mu(I_1) \quad \text{and} \quad \mu(J_1') = s_1\mu(J_1).$$

Set

$$X_2 = \Big(X_1 \setminus \Big[\bigcup_{i=0}^{h_1-1} R_1^i(I_1')\Big]\Big) \cup \Big[\bigcup_{i=0}^{h_1-1} S_1^i(J_1')\Big],$$

$$Y_2 = \Big(Y_1 \setminus \Big[\bigcup_{i=0}^{h_1-1} S_1^i(J_1')\Big]\Big) \cup \Big[\bigcup_{i=0}^{h_1-1} R_1^i(I_1')\Big].$$

We will define second stage transformations $R_2 : X_2 \to X_2$ and $S_2 : Y_2 \to Y_2$. First, it may be necessary to add or subtract measure from the residuals so that X_2 is scaled properly to define R_2, and Y_2 is scaled properly to define S_2.

2.2. Tower rescaling. In the case where $\mu(I_1') \neq \mu(J_1')$, we give a procedure for transferring measure between the towers and the residuals. This is done in order to consistently define R_2 and S_2 on the new inflated or deflated towers. Let $a = \mu(\bigcup_{i=0}^{h_1-1} R_1^i I_1)$ and $b = h_1(\mu(J_1') - \mu(I_1'))$. Let c be the scaling factor and d represent the amount of measure transferred between $\bigcup_{i=0}^{h_1-1} S_1^i(J_1')$ and X_1^*. The sign of d indicates the direction of the measure transfer. Thus, $a + b - d = ca$ and $1/2 - a + d = c(1/2 - a)$. The goal is to solve two unknowns d and c in terms of the other values. Hence, $d = (1-2a)b$ and $c = 1 + 2b$.

2.2.1. *R-rescaling.* If $d > 0$, define $I_1^* \subset J_1'$ such that $\mu(I_1^*) = d/h_1$. Let $X_1' = X_1^* \cup \bigcup_{i=0}^{h_1-1} R_1^i(I_1^*)$. If $d = 0$, set $X_1' = X_1^*$. If $d < 0$, transfer measure from X_1^* to the tower. Choose disjoint sets $I_1^*(0), I_1^*(1), \ldots, I_1^*(h_1-1)$ contained in X_1^* such that $\mu(I_1^*(i)) = -d/h_1$. Denote $I_1^* = I_1^*(0)$. Begin by defining a μ-measure preserving map α_1 such that $I_1^*(i+1) = \alpha_1(I_1^*(i))$ for $i = 0, 1, \ldots, h_1 - 2$. In this case, let $X_1' = X_1^* \setminus \bigcup_{i=0}^{h_1-1} I_1^*(i)$.

2.2.2. *S-rescaling.* The direction mass is transferred depends on the sign of b above. If $d > 0$, then $\mu(J_1') > \mu(I_1')$ and mass is transferred from the residual Y_1^* to the S_1-tower. Choose disjoint sets $J_1^*(0), J_1^*(1), \ldots, J_1^*(h_1-1)$

contained in Y_1^* such that $\mu(J_1^*(i)) = d/h_1$. Denote $J_1^* = J_1^*(0)$. Begin by defining a μ-measure preserving map β_1 such that $J_1^*(i+1) = \beta_1(J_1^*(i))$ for $i = 0, 1, \ldots, h_1 - 2$. In this case, let $Y_1' = Y_1^* \setminus \bigcup_{i=0}^{h_1-1} J_1^*(i)$. If $d = 0$, set $Y_1' = Y_1^*$. If $d < 0$, transfer measure from the S_1-tower to the residual Y_1^*. Define $J_1^* \subset J_1 \setminus J_1'$ such that $\mu(J_1^*) = -d/h_1$. Let $Y_1' = Y_1^* \cup \bigcup_{i=0}^{h_1-1} S_1^i(J_1^*)$.

Note that if $d \neq 0$, then both ϵ_1 and $\mu(X_1^*)$ may be chosen small enough (relative to r_1) to ensure the following solutions lead to well-defined sets and mappings. For subsequent stages, assume ϵ_n is chosen small enough to force well-defined rescaling parameters, transfer sets and mappings R_n, S_n.

2.3. Stage 2 construction. We have specified three cases: $d > 0$, $d = 0$ and $d < 0$. The case $d = 0$ can be handled along with the case $d > 0$. This gives two essential cases. Note that the case $d < 0$ is analogous to $d > 0$, with the roles of R_1 and S_1 reversed. However, due to a key distinction in the handling of the R-rescaling and the S-rescaling, it is important to clearly define R_2 and S_2 in both cases.

Case 2.1 ($d \geq 0$). Define $\tau_1 : X_1' \to X_1^*$ as a measure preserving map between the normalized spaces $(X_1', \mathbb{B} \cap X_1', \mu/\mu(X_1'))$ and $(X_1^*, \mathbb{B} \cap X_1^*, \mu/\mu(X_1^*))$. Extend τ_1 to the new tower base,

$$\tau_1 : [I_1 \setminus I_1'] \cup [J_1' \setminus I_1^*] \to I_1,$$

so that τ_1 preserves normalized measure between

$$\frac{\mu}{\mu([I_1 \setminus I_1'] \cup [J_1' \setminus I_1^*])} \quad \text{and} \quad \frac{\mu}{\mu(I_1)}.$$

Define τ_1 on the remainder of the tower consistently by

$$\tau_1(x) = \begin{cases} R_1^i \circ \tau_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1, \\ R_1^i \circ \tau_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1' \setminus I_1^*) \text{ for } 0 \leq i < h_1. \end{cases}$$

Define $R_2 : X_2 \to X_2$ as $R_2 = \tau_1^{-1} \circ R_1 \circ \tau_1$. Note that

$$R_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i(J_1' \setminus I_1^*) \text{ for } 0 \leq i < h_1 - 1, \\ R_1(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1 - 1. \end{cases}$$

Clearly, R_2 is isomorphic to R_1 and R.

Define $\psi_1 : Y_1' \to Y_1^*$ as a measure preserving map between the normalized spaces $(Y_1', \mathbb{B} \cap Y_1', \mu/\mu(Y_1'))$ and $(Y_1^*, \mathbb{B} \cap Y_1^*, \mu/\mu(Y_1^*))$. Extend ψ_1 to the new tower base,

$$\psi_1 : [J_1 \setminus J_1'] \cup J_1^* \cup I_1' \to J_1,$$

so that ψ_1 preserves normalized measure between

$$\frac{\mu}{\mu([J_1 \setminus J_1'] \cup J_1^* \cup I_1')} \quad \text{and} \quad \frac{\mu}{\mu(J_1)}.$$

Define ψ_1 on the remainder of the tower consistently by

$$\psi_1(x) = \begin{cases} S_1^i \circ \psi_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1 \setminus J_1') \text{ for } 0 \le i < h_1, \\ S_1^i \circ \psi_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \le i < h_1, \\ \beta_1^i \circ \psi_1 \circ \beta_1^{-i}(x) & \text{if } x \in J_1^*(i) \text{ for } 0 \le i < h_1. \end{cases}$$

In this case, define $S_2 : Y_2 \to Y_2$ by $S_2 = \psi_1^{-1} \circ S_1 \circ \psi_1$. Note that

$$S_2(x) = \begin{cases} R_1(x) & \text{if } x \in R_1^i I_1' \text{ for } 0 \le i < h_1 - 1, \\ S_1(x) & \text{if } x \in S_1^i(J_1 \setminus J_1') \text{ for } 0 \le i < h_1 - 1, \\ \beta_1(x) & \text{if } x \in J_1^*(i) \text{ for } 0 \le i < h_1 - 1, \\ \psi_1^{-1} \circ S_1 \circ \psi_1(x) & \text{if } x \in Y_1' \cup S_1^{h_1-1}(J_1 \setminus J_1') \cup R_1^{h_1-1} I_1' \cup \beta_1^{h_1-1} J_1^*, \end{cases}$$

and S_2 is isomorphic to S_1 and S.

Case 2.2 ($d < 0$). Define $\tau_1 : X_1' \to X_1^*$ as a measure preserving map between the normalized spaces $(X_1', \mathbb{B} \cap X_1', \mu/\mu(X_1'))$ and $(X_1^*, \mathbb{B} \cap X_1^*, \mu/\mu(X_1^*))$. Extend τ_1 to the new tower base,

$$\tau_1 : [I_1 \setminus I_1'] \cup I_1^* \cup J_1' \to I_1,$$

so that τ_1 preserves normalized measure between

$$\frac{\mu}{\mu([I_1 \setminus I_1'] \cup I_1^* \cup J_1')} \quad \text{and} \quad \frac{\mu}{\mu(I_1)}.$$

Define τ_1 on the remainder of the tower consistently by

$$\tau_1(x) = \begin{cases} R_1^i \circ \tau_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \le i < h_1, \\ R_1^i \circ \tau_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1') \text{ for } 0 \le i < h_1, \\ \alpha_1^i \circ \tau_1 \circ \alpha_1^{-i}(x) & \text{if } x \in I_1^*(i) \text{ for } 0 \le i < h_1. \end{cases}$$

In this case, define $R_2 : X_2 \to X_2$ by

$$R_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i J_1' \text{ for } 0 \le i < h_1 - 1, \\ R_1(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \le i < h_1 - 1, \\ \alpha_1(x) & \text{if } x \in I_1^*(i) \text{ for } 0 \le i < h_1 - 1, \\ \tau_1^{-1} \circ R_1 \circ \tau_1(x) & \text{if } x \in X_1' \cup R_1^{h_1-1}(I_1 \setminus I_1') \cup S_1^{h_1-1} J_1' \cup \alpha_1^{h_1-1} I_1^*. \end{cases}$$

Clearly, R_2 is isomorphic to R_1 and R.

Define $\psi_1 : Y_1' \to Y_1^*$ as a measure preserving map between the normalized spaces $(Y_1', \mathbb{B} \cap Y_1', \mu/\mu(Y_1'))$ and $(Y_1^*, \mathbb{B} \cap Y_1^*, \mu/\mu(Y_1^*))$. Extend ψ_1 to the new tower base,

$$\psi_1 : [J_1 \setminus (J_1' \cup J_1^*)] \cup I_1' \to J_1,$$

so that ψ_1 preserves normalized measure between

$$\frac{\mu}{\mu([J_1 \setminus (J_1' \cup J_1^*)] \cup I_1')} \quad \text{and} \quad \frac{\mu}{\mu(J_1)}.$$

Define ψ_1 on the remainder of the tower consistently by

$$\psi_1(x) = \begin{cases} S_1^i \circ \psi_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1 \setminus [J_1' \cup J_1^*]) \text{ for } 0 \le i < h_1, \\ S_1^i \circ \psi_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \le i < h_1. \end{cases}$$

Define $S_2 : Y_2 \to Y_2$ by $S_2 = \psi_1^{-1} \circ S_1 \circ \psi_1$. Note that

$$S_2(x) = \begin{cases} R_1(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \le i < h_1 - 1, \\ S_1(x) & \text{if } x \in S_1^i(J_1 \setminus [J_1' \cup J_1^*]) \text{ for } 0 \le i < h_1 - 1. \end{cases}$$

The transformation S_2 is isomorphic to S_1 and S.

Define T_2 as

$$T_2(x) = \begin{cases} R_2(x) & \text{if } x \in X_2, \\ S_2(x) & \text{if } x \in Y_2. \end{cases}$$

Clearly, neither T_1 nor T_2 are ergodic. For T_1, X_1 and Y_1 are ergodic components, and X_2, Y_2 are ergodic components for T_2. See the appendix for a pictorial of the multiplexing operation used to produce R_2 and S_2 from R_1, S_1 and the intermediary maps defined in this section.

2.4. General multiplexing operation. For $n \ge 1$, suppose that R_n and S_n have been defined on X_n and Y_n respectively. Construct Rohklin towers of height h_n for each R_n and S_n, and such that I_n is the base of the R_n tower, J_n is the base of the S_n tower, and

$$\mu\Big(\bigcup_{i=0}^{h_n-1} R_n^i I_n\Big) + \mu\Big(\bigcup_{i=0}^{h_n-1} S_n^i J_n\Big) > 1 - \epsilon_n.$$

Let $I_n' \subset I_n$ be such that $\mu(I_n') = r_n \mu(I_n)$. Similarly, suppose $J_n' \subset J_n$ with $\mu(J_n') = s_n \mu(J_n)$.

We define R_{n+1} and S_{n+1} by switching the subcolumns

$$\{I_n', R_n(I_n'), R_n^2(I_n'), \ldots, R_n^{h_n-1}(I_n')\}$$

and

$$\{J_n', S_n(J_n'), S_n^2(J_n'), \ldots, S_n^{h_n-1}(J_n')\}.$$

Let

$$X_{n+1} = \Big[\bigcup_{i=0}^{h_n-1} R_n^i(I_n \setminus I_n')\Big] \cup \Big[\bigcup_{i=0}^{h_n-1} S_n^i J_n'\Big] \cup \Big[X_n \setminus \bigcup_{i=0}^{h_n-1} R_n^i I_n\Big],$$

$$Y_{n+1} = \Big[\bigcup_{i=0}^{h_n-1} S_n^i(J_n \setminus J_n')\Big] \cup \Big[\bigcup_{i=0}^{h_n-1} R_n^i I_n'\Big] \cup \Big[Y_n \setminus \bigcup_{i=0}^{h_n-1} S_n^i J_n\Big].$$

As in the initial case, it may be necessary to transfer measure between each column and its respective residual. We can follow the same algorithm as above, and define maps τ_n, α_n, ψ_n and β_n. Thus, we get the following definitions:

CASE 2.3 ($d \geq 0$).

$$\tau_n(x) = \begin{cases} R_n^i \circ \tau_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n \setminus I_n') \text{ for } 0 \leq i < h_n, \\ R_n^i \circ \tau_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n' \setminus I_1^*) \text{ for } 0 \leq i < h_n, \end{cases}$$

$$R_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i(J_n' \setminus I_n^*) \text{ for } 0 \leq i < h_n - 1, \\ R_n(x) & \text{if } x \in R_n^i(I_n \setminus I_n') \text{ for } 0 \leq i < h_n - 1, \\ \tau_n^{-1} \circ R_n \circ \tau_n(x) & \text{if } x \in X_n' \cup R_n^{h_n-1}(I_n \setminus I_n') \cup S_n^{h_n-1}(J_n' \setminus I_n^*), \end{cases}$$

and $R_{n+1} = \tau_n^{-1} \circ R_n \circ \tau_n$.

$$\psi_n(x) = \begin{cases} S_n^i \circ \psi_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n \setminus J_n') \text{ for } 0 \leq i < h_n, \\ S_n^i \circ \psi_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n') \text{ for } 0 \leq i < h_n, \\ \beta_n^i \circ \psi_n \circ \beta_n^{-i}(x) & \text{if } x \in J_n^*(i) \text{ for } 0 \leq i < h_n, \end{cases}$$

$$S_{n+1}(x) = \begin{cases} R_n(x) & \text{if } x \in R_n^i I_n' \text{ for } 0 \leq i < h_n - 1, \\ S_n(x) & \text{if } x \in S_n^i(J_n \setminus J_n') \text{ for } 0 \leq i < h_n - 1, \\ \beta_n(x) & \text{if } x \in J_n^*(i) \text{ for } 0 \leq i < h_n - 1, \\ \psi_n^{-1} \circ S_n \circ \psi_n(x) & \text{if } x \in Y_n' \cup S_n^{h_n-1}(J_n \setminus J_n') \cup R_n^{h_n-1} I_n' \cup \beta_n^{h_n-1} J_n^*, \end{cases}$$

and $S_{n+1} = \psi_n^{-1} \circ S_n \circ \psi_n$.

CASE 2.4 ($d < 0$).

$$\tau_n(x) = \begin{cases} R_n^i \circ \tau_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n \setminus I_n') \text{ for } 0 \leq i < h_n, \\ R_n^i \circ \tau_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n') \text{ for } 0 \leq i < h_n, \\ \alpha_n^i \circ \tau_n \circ \alpha_n^{-i}(x) & \text{if } x \in I_n^*(i) \text{ for } 0 \leq i < h_n, \end{cases}$$

$$R_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i J_n' \text{ for } 0 \leq i < h_n - 1, \\ R_n(x) & \text{if } x \in R_n^i(I_n \setminus I_n') \text{ for } 0 \leq i < h_n - 1, \\ \alpha_n(x) & \text{if } x \in I_n^*(i) \text{ for } 0 \leq i < h_n - 1, \\ \tau_n^{-1} \circ R_n \circ \tau_n(x) & \text{if } x \in X_n' \cup R_n^{h_n-1}(I_n \setminus I_n') \cup S_n^{h_n-1} J_n' \cup \alpha_n^{h_n-1} I_n^*, \end{cases}$$

and $R_{n+1} = \tau_n^{-1} \circ R_n \circ \tau_n$.

$$\psi_n(x) = \begin{cases} S_n^i \circ \psi_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n \setminus [J_n' \cup J_n^*]) \text{ for } 0 \le i < h_n, \\ S_n^i \circ \psi_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n') \text{ for } 0 \le i < h_n, \end{cases}$$

$$S_{n+1}(x) = \begin{cases} R_n(x) & \text{if } x \in R_n^i(I_n') \text{ for } 0 \le i < h_n - 1, \\ S_n(x) & \text{if } x \in S_n^i(J_n \setminus [J_n' \cup J_n^*]) \text{ for } 0 \le i < h_n - 1, \\ \psi_n^{-1} \circ S_n \circ \psi_n(x) & \text{if } x \in Y_n' \cup S_n^{h_n-1}(J_n \setminus [J_n' \cup J_n^*]) \cup R_n^{h_n-1}(I_n'), \end{cases}$$

and $S_{n+1} = \psi_n^{-1} \circ S_n \circ \psi_n$.

2.5. The limiting transformation. Define the transformation $T_{n+1} : X_{n+1} \cup Y_{n+1} \to X_{n+1} \cup Y_{n+1}$ by

$$T_{n+1}(x) = \begin{cases} R_{n+1}(x) & \text{if } x \in X_{n+1}, \\ S_{n+1}(x) & \text{if } x \in Y_{n+1}. \end{cases}$$

The set where $T_{n+1} \ne T_n$ is determined by the top levels of the Rokhlin towers, the residuals and the transfer sets. Note that the transfer set has measure $|d|$. Since this set is used to adjust the size of the residuals between stages, it can be bounded below by a constant multiple of ϵ_n. Thus, there is a fixed constant κ, independent of n, such that $T_{n+1}(x) = T_n(x)$ except for x in a set of measure less than $\kappa(\epsilon_n + 1/h_n)$. Since $\sum_{n=1}^{\infty}(\epsilon_n + 1/h_n) < \infty$, $T(x) = \lim_{n\to\infty} T_n(x)$ exists almost everywhere, and preserves normalized Lebesgue measure. Without loss of generality, we may assume κ and h_n are chosen such that if

$$E_n = \{x \in X : T_{n+1}(x) \ne T_n(x)\}$$

then $\mu(E_n) < \kappa\epsilon_n$ for $n \in \mathbb{N}$. In the following section, additional structure and conditions are implemented to ensure that T inherits properties from R and S, and is also ergodic.

For the remainder of this paper, assume the parameters are chosen so that

- $\lim_{n\to\infty} r_n = 0$;
- $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} s_n = \infty$;
- $\lim_{n\to\infty} \mu(Y_n) = 0$;
- $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

2.6. Isomorphism chain consistency. In the following sections, rigidity and ergodicity will be established on sets from a refining sequence of partitions. For $n \in \mathbb{N}$, let P_n be a refining sequence of finite partitions which generates the sigma algebra. By refining P_n further if necessary, assume $X_n, Y_n, X_n^*, Y_n^* \in P_n$. Also, assume $R_n^i(I_n'), R_n^i(I_n \setminus I_n'), S_n^i(J_n'), S_n^i(J_n \setminus J_n')$ are elements of P_n for $0 \le i < h_n$. Finally, assume that for $0 \le i < h_n - 1$, if $p \in P_n$ and $p \subset R_n^i(I_n)$ then $R_n(p) \in P_n$. Likewise, assume that for $0 \le i < h_n - 1$, if $p \in P_n$ and $p \subset S^i(J_n)$ then $S_n(p) \in P_n$. Previously, we required that τ_n map certain finite orbits from the R_n and S_n towers to a corresponding orbit in the R_{n+1} tower. In this section, further regularity is imposed on τ_n relative to P_n to ensure dynamical properties of R_n are inherited by R_{n+1}.

Let

$$P'_n = \Big\{p \in P_n : p \subset \bigcup_{i=0}^{h_n-1} R_n^i(I_n \setminus I'_n)\Big\}.$$

For each of the following three cases, we impose the corresponding restriction on τ_n:

- for $d = 0$ and $p \in P'_n$, τ_n is the identity map (i.e. $\tau_n(p) = p$);
- for $d > 0$ and $p \in P'_n$, $\tau_n(p) \subset p$;
- for $d < 0$ and $p \in P'_n$, $p \subset \tau_n(p)$.

This can be accomplished by uniformly distributing the appropriate mass from the sets $R_n^i(I_n^*)$ using τ_n. Note that τ_n either preserves Lebesgue measure in the case $d = 0$, or contracts sets relative to Lebesgue measure in the case $d > 0$, or inflates measure in the case $d < 0$. In all three cases, for $p \in P'_n$,

$$\frac{\mu(p)}{\mu(\tau_n(p))} = \frac{\mu(X_{n+1})}{\mu(X_n)}.$$

It is straightforward to verify that for any set A measurable relative to P'_n,

$$\mu(A \triangle \tau_n A) < \left|\frac{\mu(X_{n+1})}{\mu(X_n)} - 1\right|.$$

The properties of τ_n allow approximation of R_{n+1} by R_n indefinitely over time. This is needed to establish our rigidity sequence for the limiting transformation T. This lemma is not required for establishing ergodicity, but for convenience we will reuse it to prove that our limiting T is ergodic.

Lemma 2.5. *Suppose $\delta > 0$ and $n \in \mathbb{N}$ is chosen such that*

$$\left|\frac{\mu(X_{n+1})}{\mu(X_n)} - 1\right| < \frac{\delta}{7}, \qquad r_n + \epsilon_n + \mu(Y_n) < \frac{\delta}{7}.$$

Then for $A, B \in P_n$ and $i \in \mathbb{N}$, the following hold:

1. $|\mu(R_{n+1}^i A \cap B) - \mu(A)\mu(B)| < |\mu(R_n^i A \cap B) - \mu(A)\mu(B)| + \delta$;
2. $\mu(R_{n+1}^i A \triangle A) < \mu(R_n^i A \triangle A) + \delta$.

Proof. For $A, B \in P_n$, let

$$A' = \bigcup_{p \in P'_n} p \cap A \quad \text{and} \quad B' = \bigcup_{p \in P'_n} p \cap B.$$

Since $\mu(\bigcup_{j=0}^{h_n-1} R_n^j(I'_n)) = h_n \mu(I'_n) < r_n$ and $\mu(X_n^*) < \epsilon_n$, we have $\mu(A \triangle A') < r_n + \epsilon_n < \delta/7$. Likewise, $\mu(B \triangle B') < \delta/7$. Since $|\mu(X_{n+1})/\mu(X_n) - 1|$

$< \delta/7$, we have $\mu(A \triangle \tau_n A) < \delta/7$. By applying the triangle inequality several times, we can get our approximations. Below is a sequence of quantities to chain through such that consecutive values in the chain are less than $\delta/7$ apart:

$$\begin{aligned}\mu(R_{n+1}^i A \cap B) \to \mu(R_{n+1}^i A \cap B') \to \mu(R_{n+1}^i A' \cap B') = \mu(\tau_n^{-1} R_n^i \tau_n A' \cap B') \\ \to \mu(R_n^i \tau_n A' \cap \tau_n B') \to \mu(R_n^i \tau_n A' \cap B') \\ \to \mu(R_n^i A' \cap B') \to \mu(R_n^i A' \cap B) \to \mu(R_n^i A \cap B).\end{aligned}$$

Each arrow in the chain signifies less than $\delta/7$ difference. Hence,

$$|\mu(R_{n+1}^i A \cap B) - \mu(R_n^i A \cap B)| < \delta,$$

which implies

$$|\mu(R_{n+1}^i A \cap B) - \mu(A)\mu(B)| < |\mu(R_n^i A \cap B) - \mu(A)\mu(B)| + \delta.$$

The second part of the lemma can be proven in a similar fashion using the triangle inequality, or chaining through the following six approximations.

$$\begin{aligned}\mu(R_{n+1}^i A \triangle A) \to \mu(R_{n+1}^i A \triangle A') \to \mu(R_{n+1}^i A' \triangle A') = \mu(\tau_n^{-1} R_n^i \tau_n A' \triangle A') \\ \to \mu(R_n^i \tau_n A' \triangle \tau_n A') \to \mu(R_n^i \tau_n A' \triangle A') \\ \to \mu(R_n^i A' \triangle A') \to \mu(R_n^i A' \triangle A) \to \mu(R_n^i A \triangle A).\end{aligned}$$

Since each arrow indicates a difference less than $\delta/7$, it follows that

$$|\mu(R_{n+1}^i A \triangle A) - \mu(R_n^i A \triangle A)| < \delta.$$

This completes the proof of the lemma. ∎

3. Establishing rigidity. Suppose that ρ_n is a rigidity sequence for R. In this section, we define parameters such that T is rigid on ρ_n.

3.1. Waiting for rigidity. Let δ_n be a sequence of positive real numbers such that $\lim_{n\to\infty} \delta_n = 0$. Since $T_n|_{X_n} = R_n$ is rigid, choose a natural number $M_n^1 > \max\{h_{n-1}, M_{n-1}^1\}$ such that for $N \geq M_n^1$, and $A \in P_{n-1} \cap X_n$,

$$\mu(R_n^{\rho_N} A \triangle A) < \delta_n.$$

Choose ϵ_n such that

$$\epsilon_n M_n^1 < \epsilon_{n-1}. \tag{3.1}$$

Also, without loss of generality, assume $h_n > M_n^1$. Below we show that this choice of ϵ_n is sufficient to produce $T(x) = \lim_{n\to\infty} T_n(x)$ rigid on ρ_n. First, we provide a diagram and heuristic description of our method for establishing rigidity on ρ_n.

Fig. 1. Rigidity timeline

3.2. The key idea. To establish rigidity of T, we can focus on the asymptotic rigidity of T on the intervals $(M_n^1, M_{n+1}^1]$. We have chosen M_n^1 sufficiently large such that rigidity "kicks in" for R_n and $\rho_i > M_n^1$. Lemma 2.5 allows us to approximate R_n by R_{n+1} as ρ_i becomes closer to M_{n+1}^1. The fact that we can choose ϵ_{n+1} arbitrarily small compared to $1/M_{n+1}^1$ allows us to carry over the approximation to T. A precise proof is given below.

3.3. Rigidity proof. If $E_{n+1} = \{x \in X : T_{n+2}(x) \neq T_{n+1}(x)\}$ and

$$E_{n+1}^1 = \bigcup_{i=0}^{M_{n+1}^1 - 1} [T_{n+2}^{-i} E_{n+1} \cup T_{n+1}^{-i} E_{n+1}]$$

then $\mu(E_{n+1}^1) < 2M_{n+1}^1 \kappa \epsilon_{n+1}$. For $x \notin E_{n+1}^1$, $T_{n+2}^i(x) = T_{n+1}^i(x)$ for $0 \leq i \leq M_{n+1}^1$. Let $\hat{E}_{n+1}^1 = \bigcup_{k=n+1}^{\infty} E_k^1$. For $x \notin \hat{E}_{n+1}^1$ and $0 \leq i \leq M_{n+1}^1$, $T^i(x) = T_{n+1}^i(x)$. Also, by (3.1),

$$\mu(\hat{E}_{n+1}^1) < \sum_{k=n+1}^{\infty} 2M_k^1 \kappa \epsilon_k < \sum_{k=n+1}^{\infty} 2\kappa \epsilon_{k-1} \to 0$$

as $n \to \infty$.

Proof of rigidity. Let A be a set in P_{n_1} for some n_1, and let $\delta > 0$. Choose $n_2 \geq n_1$ such that for $n \geq n_2$,

- $|\mu(X_{n+1})/\mu(X_n) - 1| < \delta/28$;
- $r_n + \epsilon_n + \mu(Y_n) < \delta/28$;
- $\delta_n < \delta/6$;
- $\sum_{i=n_2}^{\infty} 2\kappa\epsilon_i < \delta/12$.

For $n > n_2$, let $M_n^1 < N \leq M_{n+1}^1$, $A_1 = A \setminus \hat{E}_{n+1}^1$ and $A_2 = A \cap X_n$. Thus,

$$\begin{aligned}\mu(T^{\rho_N} A \triangle A) &\leq \mu(T^{\rho_N} A \triangle T^{\rho_N} A_1) + \mu(T^{\rho_N} A_1 \triangle A) \\ &= \mu(A \triangle A_1) + \mu(R_{n+1}^{\rho_N} A_1 \triangle A) \\ &< \delta/4 + \mu(R_{n+1}^{\rho_N} A_1 \triangle R_{n+1}^{\rho_N} A) + \mu(R_{n+1}^{\rho_N} A \triangle A) \\ &< \delta/2 + \mu(R_{n+1}^{\rho_N} A \triangle A).\end{aligned}$$

By Lemma 2.5,

$$\begin{aligned}\mu(T^{\rho_N} A \triangle A) &< \delta/2 + \mu(R_{n+1}^{\rho_N} A \triangle A) < 3\delta/4 + \mu(R_n^{\rho_N} A \triangle A) \\ &\leq 3\delta/4 + \mu(R_n^{\rho_N} A \triangle R_n^{\rho_N} A_2) + \mu(R_n^{\rho_N} A_2 \triangle A_2) + \mu(A_2 \triangle A) \\ &< 3\delta/4 + 2\mu(Y_n) + \delta_n < \delta.\end{aligned}$$

Therefore, ρ_n is a rigidity sequence for T. ■

4. Ergodicity. A measure preserving transformation T on a Lebesgue space is *ergodic* if any invariant set has measure zero or one. It is well known this is equivalent to the mean and pointwise ergodic theorem. For our purposes, we use the following equivalent condition of ergodicity: for all measurable sets A and B,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mu(T^i A \cap B) = \mu(A)\mu(B).$$

Let P_n, $n \in \mathbb{N}$, be a sequence of finite refining partitions as defined in the previous section. Using approximation, T is ergodic if the previous condition holds for all natural numbers n and sets A and B from P_n.

4.1. Ergodic parameter choice. Let δ_n be a sequence of positive real numbers such that $\lim_{n\to\infty} \delta_n = 0$. Since $T_n|_{X_n} = R_n$ is ergodic, choose a natural number $M_n = M_n^2$ such that for $N \geq M_n$, and sets $A, B \in P_{n-1} \cap X_n$,

$$\left| \frac{1}{N} \sum_{i=0}^{N-1} \frac{\mu(T_n^i A \cap B)}{\mu(X_n)} - \frac{\mu(A)\mu(B)}{\mu(X_n)^2} \right| < \delta_n.$$

Note that

$$\begin{aligned}&\left| \frac{1}{N} \sum_{i=0}^{N-1} \mu(T_n^i A \cap B) - \mu(A)\mu(B) \right| \\ &\qquad = \mu(X_n) \left| \frac{1}{N} \sum_{i=0}^{N-1} \frac{\mu(T_n^i A \cap B)}{\mu(X_n)} - \frac{\mu(A)\mu(B)}{\mu(X_n)} \right| \\ &\qquad \leq \mu(X_n) \left| \frac{1}{N} \sum_{i=0}^{N-1} \frac{\mu(T_n^i A \cap B)}{\mu(X_n)} - \frac{\mu(A)\mu(B)}{\mu(X_n)^2} \right| \\ &\qquad\quad + \left| \frac{\mu(A)\mu(B)}{\mu(X_n)} - \mu(A)\mu(B) \right| \\ &\qquad < \delta_n + \frac{\mu(Y_n)}{\mu(X_n)}\end{aligned}$$

Choose ϵ_n such that

$$\epsilon_n M_n < \epsilon_{n-1}. \tag{4.1}$$

4.2. Approximation. As previously, set $E_{n+1} = \{x \in X : T_{n+2}(x) \neq T_{n+1}(x)\}$. Let

$$E^2_{n+1} = \bigcup_{i=0}^{M_{n+1}-1} [T^{-i}_{n+2} E_{n+1} \cup T^{-i}_{n+1} E_{n+1}].$$

Thus, $\mu(E^2_{n+1}) < 2M_{n+1}\kappa\epsilon_{n+1}$. For $x \notin E^2_{n+1}$, $T^i_{n+2}(x) = T^i_{n+1}(x)$ for $0 \leq i \leq M_{n+1}$. Let $\hat{E}^2_{n+1} = \bigcup_{k=n+1}^{\infty} E^2_k$. For $x \notin \hat{E}^2_{n+1}$ and $0 \leq i \leq M_{n+1}$, $T^i(x) = T^i_{n+1}(x)$. Also, by (4.1),

$$\mu(\hat{E}^2_{n+1}) < \sum_{k=n+1}^{\infty} 2M_k\kappa\epsilon_k < \sum_{k=n+1}^{\infty} 2\kappa\epsilon_{k-1} \to 0$$

as $n \to \infty$.

Proof of ergodicity. Let A and B be sets in P_{n_1} for some n_1, and let $\delta > 0$. Choose $n_2 \geq n_1$ such that for $n \geq n_2$,

- $|\mu(X_{n+1})/\mu(X_n) - 1| < \delta/28$;
- $r_n + \epsilon_n + \mu(Y_n) < \delta/28$;
- $\delta_n + \mu(Y_n)/\mu(X_n) < \delta/4$;
- $\sum_{i=n_2}^{\infty} 2\kappa\epsilon_i < \delta/12$.

For $n > n_2$, let $M_n < N \leq M_{n+1}$, $A_1 = A \setminus \hat{E}^2_{n+1}$ and $B_1 = B \setminus \hat{E}^2_{n+1}$. Then

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(T^iA \cap B) - \mu(A)\mu(B)\right| \\
&\quad\leq \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(T^iA \cap B) - \frac{1}{N}\sum_{i=0}^{N-1} \mu(T^iA_1 \cap B)\right| \\
&\qquad + \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(T^iA_1 \cap B) - \mu(A)\mu(B)\right| \\
&\quad\leq \frac{1}{N}\sum_{i=0}^{N-1} |\mu(T^iA \cap B) - \mu(T^iA_1 \cap B)| \\
&\qquad + \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(R^i_{n+1}A_1 \cap B) - \mu(A)\mu(B)\right| \\
&\quad< \mu(\hat{E}^2_{n+1}) + \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(R^i_{n+1}A_1 \cap B) - \mu(R^i_{n+1}A \cap B)\right| \\
&\qquad + \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(R^i_{n+1}A \cap B) - \mu(A)\mu(B)\right| \\
&\quad< \frac{\delta}{4} + \frac{\delta}{4} + \left|\frac{1}{N}\sum_{i=0}^{N-1} \mu(R^i_{n+1}A \cap B) - \mu(A)\mu(B)\right|.
\end{aligned}$$

Since $A, B \in P_n$, by Lemma 2.5 we have

$$\left|\frac{1}{N}\sum_{i=0}^{N-1}\mu(T^iA\cap B)-\mu(A)\mu(B)\right|$$
$$<\frac{\delta}{2}+\frac{1}{N}\sum_{i=0}^{N-1}|\mu(R_{n+1}^iA\cap B)-\mu(A)\mu(B)|$$
$$<\frac{3\delta}{4}+\frac{1}{N}\sum_{i=0}^{N-1}|\mu(R_n^iA\cap B)-\mu(A)\mu(B)|<\delta.$$

Since δ is chosen arbitrarily, and the above holds for any $n > n_2$ and $M_n < N \le M_{n+1}$, we conclude that T is ergodic. ∎

5. Weak mixing. Since the weak mixing component is dissipative, and the resulting transformation inherits its rigidity properties from R, we do not focus on multiplexing with general weak mixing transformations. Instead, we set S equal to the famous Chacon transformation. It is defined via cutting and stacking, and considered the earliest construction demonstrated to be weak mixing and not mixing. See [9] for a precise definition. For the remainder of this paper, assume both R and S are defined on $([0,1),\mu,\mathbb{B})$ where μ is Lebesgue measure. In this section, we further specify h_n and switching sets $C_n = \bigcup_{i=0}^{h_n-1} R_n^i(I_n')$ for $n \in \mathbb{N}$. As in previous sections, all conditions imposed are easily satisfied by choosing a faster growing sequence of tower heights h_n. No upper bounds are imposed on the growth rate of h_n.

5.1. Switching set definition. For each $k \in \mathbb{N}$ and $n > k$, denote $U_k^n = \bigcup_{j=k}^{n-1} C_j$, $V_k^n = (U_k^n)^c$ and $\dot{V}_k^n = V_k^n \cap X_n$. Since R_n is ergodic on X_n, r_n is fixed, and C_n predominantly represents long orbits of R_n, it follows that h_n may be chosen sufficiently large such that C_n is nearly conditionally independent of $\dot{V}_k^n$ for each $k < n$.

Precisely, define h_n and C_n so that

$$\left|\frac{\mu(C_n\cap\dot{V}_k^n)}{\mu(X_n)}-\frac{\mu(C_n)\mu(\dot{V}_k^n)}{\mu(X_n)^2}\right|\le\frac{1}{2}\mu(C_n)\mu(\dot{V}_k^n). \tag{5.1}$$

LEMMA 5.1. *For each $k \in \mathbb{N}$, $\lim_{n\to\infty}\mu(V_k^n) = 0$.*

Proof. Suppose the claim is not true, and there exists $k_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty}\mu(V_k^n) > 0.$$

Since $\lim_{n\to\infty}\mu(Y_n) = 0$, we can choose $k_1 > k_0$ such that $\mu(Y_j) < \frac{1}{2}\mu(V_{k_1}^{k_1+n})$ for $j \ge k_1$ and $n \in \mathbb{N}$. Thus,

$$\frac{\mu(C_{k_1+1} \cap \dot{V}_{k_1}^{k_1+1})}{\mu(X_{k_1+1})} \geq \frac{\mu(C_{k_1+1})\mu(\dot{V}_{k_1}^{k_1+1})}{\mu(X_{k_1+1})^2} - \frac{1}{2}\mu(C_{k_1+1})\mu(\dot{V}_{k_1}^{k_1+1}),$$

$$\begin{aligned}\mu(C_{k_1+1} \cap \dot{V}_{k_1}^{k_1+1}) &\geq \mu(V_{k_1}^{k_1+1})\frac{\mu(V_{k_1}^{k_1+1} \cap X_{k_1+1})}{\mu(V_{k_1}^{k_1+1})}\mu(C_{k_1+1}) \\ &\quad \times \left[\frac{1}{\mu(X_{k_1+1})} - \frac{\mu(X_{k_1+1})}{2}\right] \\ &> \frac{1}{4}\mu(C_{k_1+1})\mu(V_{k_1}^{k_1+1}).\end{aligned}$$

Hence,

$$\begin{aligned}\mu(V_{k_1}^{k_1+2}) &= \mu(V_{k_1}^{k_1+1}) - \mu(C_{k_1+1} \cap V_{k_1}^{k_1+1}) \\ &< \mu(V_{k_1}^{k_1+1})\left[1 - \frac{1}{4}\mu(C_{k_1+1})\right] \\ &< \left(1 - \frac{1}{4}\mu(C_{k_1})\right)\left(1 - \frac{1}{4}\mu(C_{k_1+1})\right).\end{aligned}$$

Extending this inductively produces

$$\mu(V_{k_1}^{k_1+n}) < \prod_{i=0}^{n-1}\left(1 - \frac{1}{4}\mu(C_{k_1+i})\right).$$

Note that

$$\mu(C_n) = \mu(I'_n)h_n = \frac{\mu(I'_n)}{\mu(I_n)}\mu(I_n)h_n = r_n\mu(X_n).$$

Since $\sum_{n=1}^{\infty} r_n = \infty$ and $\lim_{n\to\infty}\mu(X_n) = 1$, we have $\sum_{n=1}^{\infty}\mu(C_n) = \infty$. This is sufficient to force

$$\lim_{n\to\infty}\prod_{i=0}^{n-1}\left(1 - \frac{1}{4}\mu(C_{k_1+i})\right) = 0,$$

which proves our claim by contradiction. ∎

The previous claim establishes that almost every point falls in infinitely many sets C_n.

PROPERTY 5.2. $\mu(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} C_i) = 1$.

5.2. Multiplexing Chacon's transformation. Chacon's transformation S is typically defined using cutting and stacking [9]. Initialize $I_1^0 = [0, 2/3)$ and $\mathcal{C}_1 = I_1^0$. Cut I_1 into three pieces of equal width, $I_2^0 = [0, 2/9)$, $I_2^1 = [2/9, 4/9), I_2^3 = [4/9, 2/3)$, and add a single spacer $I_2^2 = [2/3, 8/9)$ above interval I_2^1. Stack into a single column $\mathcal{C}_2 = \langle I_2^0, I_2^1, I_2^2, I_2^3\rangle$. Define S as the linear map from I_2^i to I_2^{i+1} for $i = 0, 1, 2$. Let $H_n = (3^n - 1)/2$ be the height of column $\mathcal{C}_n$. Obtain $\mathcal{C}_{n+1}$ by cutting $\mathcal{C}_n$ into three subcolumns of equal width, $\mathcal{C}_n^0, \mathcal{C}_n^1, \mathcal{C}_n^2$, adding one spacer above the second subcolumn and stacking left to right. Again, S maps each level linearly to the level directly above it. Also, notice that the height of $\mathcal{C}_{n+1}$ equals $H_{n+1} = 3H_n + 1 = (3^{n+1} - 1)/2$. The main property we utilize in this work is related to one of its limit joinings.

LEMMA 5.3. *Let S be Chacon's transformation. Given any two measurable sets, A and B,*

$$\lim_{n\to\infty} \mu(S^{H_n} A \cap B) = (\mu(A \cap B) + \mu(S^{-1} A \cap B))/2.$$

Proof. Each column $\mathcal{C}_n$, $n \in \mathbb{N}$, has a single level of spacer above precisely half the mass of the top level of $\mathcal{C}_n$. This includes the spacers added when $\mathcal{C}_n$ is cut into three subcolumns, as well as the infinitely many spacers added when $\mathcal{C}_{n+1}, \mathcal{C}_{n+2}, \ldots$ are cut into three subcolumns and stacked. Thus, S^{H_n} maps half of each level to the same level, and maps the other half to the level directly below itself. This establishes the lemma for sets consisting of a finite union of levels. Since the levels of the columns form a refining sequence of partitions which generate the sigma algebra, the lemma follows by approximation. ∎

5.3. Weak mixing stage. Now we define S_n inductively to ensure the final transformation T is weak mixing. Let S_1 be the Chacon transformation defined on Y_1. Suppose $S_n \simeq S$ has been defined on Y_n. Now we specify the manner in which S_{n+1} should be defined.

5.3.1. *Local approximation of switching sets.* Choose natural number $k_n > n$ such that for each $i = 0, 1, \ldots, h_n - 1$, there exists a finite collection of indices $\hat{K}_n^i$ and dyadic intervals $K_n^i(j)$, $j \in \hat{K}_n^i$, such that $\mu(K_n^i(j)) = 1/2^{k_n}$ and $K_n^i = \bigcup_{j\in\hat{K}_n^i} K_n^i(j)$ satisfies $\mu(R_n^i I_n' \triangle K_n^i) < (\frac{\epsilon_n}{h_n})^2 \mu(I_n')$. Let

$$\hat{G}_n^i = \{j \in \hat{K}_n^i : \mu(R_n^i I_n' \cap K_n^i(j)) > (1 - \epsilon_n/h_n)\mu(K_n^i(j))\}.$$

It is not difficult to show $\mu(\bigcup_{j\in\hat{G}_n^i} K_n^i(j)) > (1 - \epsilon_n/h_n)\mu(I_n')$. Set $G_n^i = \bigcup_{j\in\hat{G}_n^i} K_n^i(j)$. For each $n \in \mathbb{N}$, define

$$D_n = \bigcup_{\ell=0}^{h_n-1} G_n^\ell.$$

Note that

$$\mu(C_n \setminus D_n) < \sum_{\ell=0}^{h_n-1} \frac{\epsilon_n}{h_n} = \epsilon_n.$$

Next, we show almost every point falls in infinitely many D_n.

PROPERTY 5.4. $\mu(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty D_i) = 1$.

Proof. Given $\epsilon > 0$, choose $N = N(\epsilon) \in \mathbb{N}$ such that $\sum_{n=N}^\infty \epsilon_n < \epsilon$. Thus,

$$\mu\Big(\bigcup_{n=N}^{\infty} D_n\Big) \geq \mu(\bigcup_{n=N}^{\infty} C_n) - \sum_{n=N}^{\infty} \mu(C_n \setminus D_n)$$
$$> 1 - \sum_{n=N}^{\infty} \epsilon_n > 1 - \epsilon.$$

Since ϵ is arbitrarily small, we have $\mu(\bigcup_{n=N}^{\infty} D_n) = 1$, and Property 5.4 is established. ■

5.3.2. *Weak mixing component.* The main goal in this work is to demonstrate how properties of a given ergodic transformation can be transferred to produce a tailored weak mixing transformation. Since the weak mixing component will dissipate over time, we do not focus on introducing general properties using S. Instead, we set S to the Chacon transformation inside our towerplex construction. Thus, S_n will be isomorphic to Chacon's transformation. By Lemma 5.3, for each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that for each $i = 0, 1, \ldots, h_n - 1$, $j \in \hat{K}_n^i$ and $A = K_n^i(j)$,

$$\Big|\mu(S_{n+1}^{H_{m_n}} A \cap A) - \frac{1}{2}\mu(A)\Big| < \epsilon_n \mu(A)$$

and

$$\Big|\mu(S_{n+1}^{H_{m_n}} A \cap S^{-1}(A) - \frac{1}{2}\mu(A)\Big| < \epsilon_n \mu(A).$$

Let $w_n = \min\{\mu(K_\ell^i(j)) > 0 : 1 \leq \ell \leq n,\ 0 \leq i \leq h_n - 1,\ j \in \hat{K}_\ell^i\}$. Choose h_{n+1} such that

$$h_{n+1} > \frac{H_{m_n}}{\epsilon_n w_n}. \tag{5.2}$$

6. Slow weak mixing theorem. In this final section, we prove our main result using the towerplex constructions. First, we give explicit parameters r_n and s_n that can be used to generate our rigid weak mixing examples. Let

$$r_n = \frac{\mu(I_n')}{\mu(I_n)} = \frac{1}{2(n+2)} \quad \text{and} \quad s_n = \frac{\mu(J_n')}{\mu(J_n)} = \frac{1}{2}.$$

Thus, the switching sets have measure

$$\mu\Big(\bigcup_{i=0}^{h_n-1} R_n^i(I_n')\Big) = \frac{\mu(X_n) - \mu(X_n^*)}{2(n+2)} \quad \text{and} \quad \mu\Big(\bigcup_{i=0}^{h_n-1} S_n^i(J_n')\Big) = \frac{\mu(Y_n) - \mu(Y_n^*)}{2}$$

for $n \in \mathbb{N}$. This implies

$$\mu(Y_{n+1}) = \frac{1}{2(n+2)}[(n+1)\mu(Y_n) + 1] + \kappa_n \epsilon_n$$

where $|\kappa_n|$ is bounded for all $n \in \mathbb{N}$. If all residuals had zero mass, then $\kappa_n \epsilon_n = 0$ and by induction

$$\mu(X_n) = \frac{n}{n+1} \quad \text{and} \quad \mu(Y_n) = \frac{1}{n+1}.$$

In the case the residuals are not null, the next lemma yields

$$\lim_{n\to\infty} \mu(X_n) = 1, \qquad \lim_{n\to\infty} \mu(Y_n) = 0.$$

The parameters given here are called the *canonical towerplex settings.*

LEMMA 6.1. *If real numbers $\epsilon_n > 0$ are chosen sufficiently small for $n \in \mathbb{N}$, then a canonical towerplex construction, given by $r_n = 1/(2(n+2))$ and $s_n = 1/2$, has the property, for $n \in \mathbb{N}$,*

$$\frac{1}{n+2} < \mu(Y_n) < \frac{1}{n}. \tag{6.1}$$

Proof. The function $f(y) = (1/2(n+2))[(n+1)y + 1]$ has a fixed point at $y = 1/(n+3)$. If $y > 1/(n+3)$, then $f(y) > 1/(n+3)$. Thus, if ϵ_n is sufficiently small, and $\mu(Y_n) > 1/(n+2)$, then $\mu(Y_{n+1}) > 1/(n+3)$. This establishes the first inequality from (6.1).

To prove the second inequality, assume $y = \mu(Y_n) < 1/n$ for fixed $n \in \mathbb{N}$. Thus,

$$\begin{aligned} f(y) &< \frac{1}{2(n+2)}\left[(n+1)\frac{1}{n} + 1\right] = \frac{1}{2(n+2)}\left[2 + \frac{1}{n}\right] \\ &= \frac{1}{n+2} + \frac{1}{2n(n+2)} = \frac{1}{n+1} + \frac{1-n}{2n(n+1)(n+2)} \leq \frac{1}{n+1}. \end{aligned}$$

Therefore, if ϵ_n is sufficiently small, then $\mu(Y_{n+1}) < \frac{1}{n+1}$. ∎

Now, we are ready to prove our main theorem.

THEOREM 6.2. *Given an ergodic measure preserving transformation R on a Lebesgue probability space, and a rigid sequence ρ_n for R, there exists a weak mixing transformation T on a Lebesgue probability space such that T is rigid on ρ_n.*

Proof. Much of the details have been established in the previous sections. In particular, the conditions imposed in each of the sections on ergodicity, rigidity and weak mixing, are consistent. Essentially, $\epsilon_n \to 0$ arbitrarily fast, which is possible since only the extra mass from successive Rokhlin towers is bounded by ϵ_n. Also, each section imposes a lower bound on the growth rate of the tower heights h_n, but no upper bound. Appendix 8 lists conditions that can be used to support the explicit proofs. Below, we need to complete the argument that T is weak mixing.

Suppose $f \neq 0$ is an eigenfunction for T with eigenvalue λ. Since we established that T is ergodic, we may assume $|f|$ is a constant. Without loss of generality, assume $|f| = |\lambda| = 1$. Given $\delta > 0$, there exists a set Λ_δ of positive measure such that $|f(x)-f(y)|<\delta$ for $x,y\in\Lambda_\delta$. Let Λ'_δ be the set of Lebesgue density points of Λ_δ. In particular, if

$$\Lambda'_\delta=\left\{x\in\Lambda_\delta:\lim_{\eta\to 0}\frac{\mu(\Lambda_\delta\cap(x-\eta,x+\eta))}{2\eta}=1\right\},$$

then $\mu(\Lambda'_\delta)=\mu(\Lambda_\delta)>0$. Choose $x\in\Lambda'_\delta\cap\overline{D}$. Choose $\eta'>0$ such that

$$\frac{\mu(\Lambda_\delta\cap(x-\eta,x+\eta))}{2\eta}>1-\delta\quad\text{for } \eta<\eta'.$$

Choose $n\in\mathbb{N}$ such that $1/2^{k_n}<\eta'$, $\sum_{i=n}^\infty\epsilon_i<\delta$ and $x\in D_n$. There exists $i=i(x)$ such that $x\in G_n^i$, and subsequently $j=j(x)$ such that $x\in K_n^i(j)$. Let $\eta_x=\max\{|y-x|:y\in K_n^i(j)\}$. Note

$$\eta_x<\eta'\quad\text{and}\quad\frac{\mu(\Lambda_\delta\cap(x-\eta_x,x+\eta_x))}{2\eta_x}>1-\delta.$$

Thus,

$$\begin{aligned}\mu(\Lambda_\delta\cap K_n^i(j))&>\mu(K_n^i(j))-2\eta_x\delta\mu(K_n^i(j))-2\delta\mu(K_n^i(j))\\&\geq(1-2\delta)\mu(K_n^i(j)).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\mu(S_{n+1}^{H_{m_n}}(\Lambda_\delta\cap K_n^i(j))\cap(\Lambda_\delta\cap K_n^i(j)))-\frac{1}{2}\mu(\Lambda_\delta\cap K_n^i(j))\right|\\&\quad\leq|\mu(S_{n+1}^{H_{m_n}}(\Lambda_\delta\cap K_n^i(j))\cap(\Lambda_\delta\cap K_n^i(j)))-\mu(S_{n+1}^{H_{m_n}}(K_n^i(j))\cap K_n^i(j))|\\&\qquad+\left|\mu(S_{n+1}^{H_{m_n}}(K_n^i(j))\cap(K_n^i(j)))-\frac{1}{2}\mu(K_n^i(j))\right|\\&\qquad+\left|\frac{1}{2}\mu(K_n^i(j))-\frac{1}{2}\mu(\Lambda_\delta\cap K_n^i(j))\right|\\&\quad<4\delta\mu(K_n^i(j))+\epsilon_n\mu(K_n^i(j))+\delta\mu(K_n^i(j))=(5\delta+\epsilon_n)\mu(K_n^i(j)).\end{aligned}$$

We wish to establish that T is weak mixing, and T does not equal S_{n+1} everywhere. In particular, T may differ from S_{n+1} on the top levels of the towers of height $h_{n+1},h_{n+2},\ldots$, on the accompanying residuals, and on the transfer sets. However, we have chosen the growth of the tower heights sufficient to ensure that the set where T and S_{n+1} may differ will be small relative to interval, $K_n^i(j)$. Thus,

$$\mu(\{x\in Y_{n+1}:Tx\neq S_{n+1}x\})<\sum_{i=n+1}^\infty\left[\frac{1}{h_i}+4\epsilon_i\right]<\sum_{i=n}^\infty\left[\frac{5\epsilon_i w_n}{H_{m_i}+1}\right].$$

This implies

$$\begin{aligned}&\mu(\{x\in Y_{n+1}:T^ix\neq S_{n+1}^ix,\ i=1,\ldots,H_{m_n}+1\})\\&\qquad\qquad<w_n(H_{m_n}+1)\sum_{i=n}^\infty\frac{5\epsilon_i}{H_{m_i}+1}<5w_n\sum_{i=n}^\infty\epsilon_i<5\delta w_n.\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\mu(T^{H_{m_n}}(\varLambda_\delta \cap K_n^i(j)) \cap (\varLambda_\delta \cap K_n^i(j))) - \frac{1}{2}\mu(\varLambda_\delta \cap K_n^i(j))\right| \\
&\qquad \le |\mu(T^{H_{m_n}}(\varLambda_\delta \cap K_n^i(j)) \cap (\varLambda_\delta \cap K_n^i(j))) \\
&\qquad\quad - \mu(S_{n+1}^{H_{m_n}}(\varLambda_\delta \cap K_n^i(j)) \cap (\varLambda_\delta \cap K_n^i(j)))| \\
&\qquad\quad + \left|\mu(S_{n+1}^{H_{m_n}}(\varLambda_\delta \cap K_n^i(j)) \cap (\varLambda_\delta \cap K_n^i(j))) - \frac{1}{2}\mu(\varLambda_\delta \cap K_n^i(j))\right| \\
&\qquad < 5\delta w_n + (5\delta + \epsilon_n)\mu(K_n^i(j)) \le (10\delta + \epsilon_n)\mu(K_n^i(j)).
\end{aligned}$$

For δ and ϵ sufficiently small, there exists $x_1 \in \varLambda_\delta \cap K_n^i(j)$ such that $T^{H_{m_n}}x_1 \in \varLambda_\delta \cap K_n^i(j)$, and there exists $x_2 \in \varLambda_\delta \cap K_n^i(j)$ such that $T^{H_{m_n}+1}x_2 \in \varLambda_\delta \cap K_n^i(j)$. Thus,

$$|\lambda^{H_{m_n}} f(x_1) - f(x_1)| = |f(T^{H_{m_n}}x_1) - f(x_1)| < \delta,$$
$$|\lambda^{H_{m_n}+1} f(x_2) - f(x_2)| = |f(T^{H_{m_n}+1}x_2) - f(x_2)| < \delta.$$

Hence,

$$|\lambda^{H_{m_n}} - 1| < \frac{\delta}{|f(x_1)|} = \delta \quad \text{and} \quad |\lambda^{H_{m_n}+1} - 1| < \frac{\delta}{|f(x_2)|} = \delta.$$

Therefore,

$$|\lambda - 1| = |\lambda^{H_{m_n}+1} - \lambda^{H_{m_n}}| \le |\lambda^{H_{m_n}+1} - 1| + |\lambda^{H_{m_n}} - 1| < 2\delta.$$

Since $\delta > 0$ may be chosen arbitrarily small, it follows that $\lambda = 1$. Since it was established that T is ergodic in an earlier section, f must be a constant. Therefore, T is weak mixing. ■

Our theorem establishes the following corollaries which answer questions raised in the ground-breaking works [4] and [7].

Corollary 6.3. *Given any ergodic measure preserving transformation R on a Lebesgue probability space with discrete spectrum, and a rigidity sequence ρ_n for R, there exists a weak mixing transformation T with rigidity sequence ρ_n. In particular, for any $k \in \mathbb{N}$, $k \ge 2$, there exists a weak mixing transformation with k^n, $n \in \mathbb{N}$, as a rigidity sequence.*

The next corollary gives an explicit characterization of "large" rigid sequences for weak mixing transformations. While this corollary appears known in [2], our characterization gives a general concrete method for establishing "large" rigidity sequences of weak mixing transformations. Given a sequence $\mathcal{A}$, define the density function $g_{\mathcal{A}} : \mathbb{N} \to [0,1]$ such that $g_{\mathcal{A}}(k) = |\mathcal{A} \cap \{1,\dots,k\}|/k$.

Corollary 6.4. *Given any function $f : \mathbb{N} \to (0,\infty)$ such that*

$$\lim_{n\to\infty} f(n) = 0,$$

there exists a weak mixing transformation with rigidity sequence $\mathcal{A}$ *such that*

$$\lim_{n\to\infty} \frac{f(n)}{g_{\mathcal{A}}(n)} = 0.$$

Also, there exist weak mixing transformations with rigidity sequences ρ_n *satisfying*

$$\lim_{n\to\infty} \frac{\rho_{n+1}}{\rho_n} = 1.$$

Proof. Let α be an irrational number and R_α the rotation by $2\pi\alpha$ on the unit circle. Given $\epsilon > 0$, define $\mathcal{A}(\epsilon) = \{j \in \mathbb{N} : |\exp(2\pi\alpha j) - 1| < \epsilon\}$, and for $n \in \mathbb{N}$, define $\mathcal{A}(\epsilon, n) = \mathcal{A}(\epsilon) \cap \{1, \dots, n\}$. For $\bar{\epsilon} = \{\epsilon_1 > \epsilon_2 > \cdots > 0\}$, let $\mathcal{A}(\bar{\epsilon}) = \bigcup_{n=1}^{\infty} \mathcal{A}(\epsilon_n, n)$. If $\lim_{n\to\infty} \epsilon_n = 0$ and $\mathcal{A}(\bar{\epsilon})$ is infinite, then $\mathcal{A}(\bar{\epsilon})$ forms a rigidity sequence for R_α. Let $f : \mathbb{N} \to (0, \infty)$ be such that $\lim_{n\to\infty} f(n) = 0$. Since $\mathcal{A}(1/2^i)$ has positive density for $i \in \mathbb{N}$, there exists $j_i \in \mathbb{N}$ such that for all $j \geq j_i$,

$$\frac{|\mathcal{A}(1/2^i, j)|}{j} > 2^i f(j).$$

For $k \in \mathbb{N}$, choose $i = i_k \in \mathbb{N}$ such that $j_i + 1 \leq k \leq j_{i+1}$. Set $\epsilon_k = 1/2^i$ and let $\mathcal{A} = \mathcal{A}(\bar{\epsilon})$. Thus,

$$g_{\mathcal{A}}(k) = \frac{|\mathcal{A} \cap \{1, \dots, k\}|}{k} \geq \frac{|\mathcal{A}(\epsilon_k, k)|}{k} > 2^i f(k).$$

Hence,

$$\frac{f(k)}{g_{\mathcal{A}}(k)} < \frac{1}{2^i} \quad \text{for } i = i_k.$$

This confirms that $\lim_{k\to\infty} f(k)/g_{\mathcal{A}}(k) = 0$. Since $\mathcal{A}$ is a rigidity sequence for R_α, Theorem 6.2 shows that $\mathcal{A}$ is a rigidity sequence for a weak mixing transformation. The second assertion of Corollary 6.4 can be established in a similar manner. Since ergodic rotations on the unit circle have rigid sequences ρ_n such that $\lim_{n\to\infty} \rho_{n+1}/\rho_n = 1$, weak mixing transformations admit such rigid sequences as well. ∎

Previously, it was established that denominators from convergents of continued fractions serve as rigidity sequences for weak mixing transformations. A partial result was provided in [7] for restricted convergents, and then a general result was established in [4]. In this paper, we extend these results by showing that any rigidity sequence for an ergodic rotation on the unit circle is also a rigidity sequence for a weak mixing transformation. This includes sequences q_n formed from the denominators of convergents p_n/q_n of an irrational α.

Corollary 6.5. *Let* $\alpha \in (0, 1)$ *be any irrational number, and let* ρ_n *be a sequence of natural numbers satisfying*

$$\lim_{n\to\infty} |\exp(2\pi i\alpha\rho_n) - 1| = 0.$$

Then there exists a weak mixing transformation T such that ρ_n is a rigidity sequence for T.

Appendix A. Towerplex pictorial. This appendix provides an illustration of towers for R_1, S_1, and the multiplexing operation applied to obtain towers for R_2 and S_2. The picture below represents only the case where $d_R > 0$ and $d_S < 0$. The other cases are handled as described in the section on towerplex constructions. Also, the general case of deriving R_{n+1} and S_{n+1} from R_n and S_n is analogous to the initial multiplexing operation for deriving R_2 and S_2.

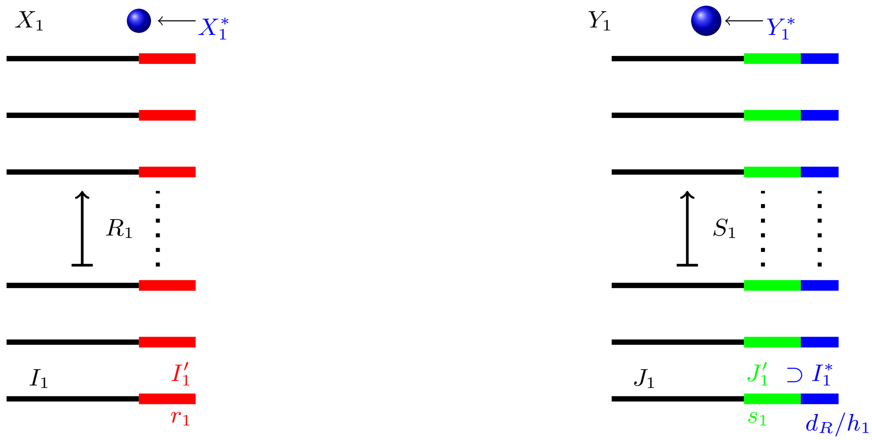

Fig. 2. Towers for T and S prior to subcolumn switching

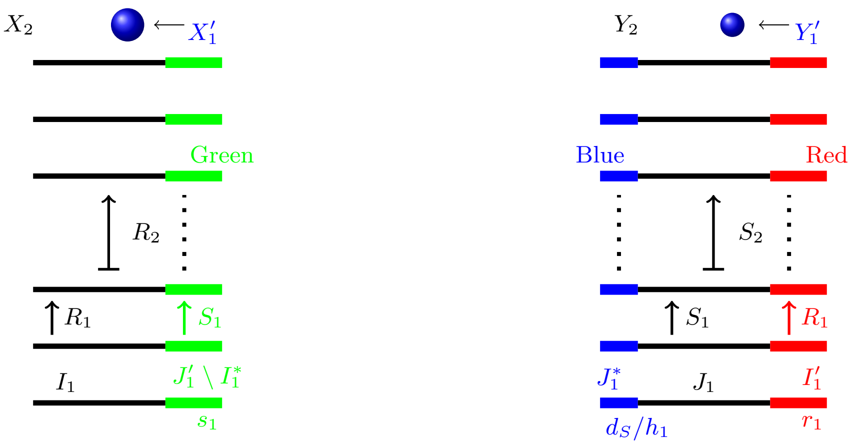

Fig. 3. Towers for T and S after subcolumn switching

The transformations R_2 and S_2 are derived from R_1 and S_1 by switching the red subcolumn with the green subcolumn. The switching of these sets is the main multiplexing operation, and the corresponding subcolumns are called switching sets. In order to preserve maps isomorphic to R and S, and avoid redefining R_1 or S_1 on most of the probability space, it may be necessary to transfer measure between the towers and residuals. This is a rescaling operation, and these sets are referred to as transfer sets. In the case where $d_R > 0$, the blue colored subcolumn I_1^* from $J_1' \subset Y_1$ is absorbed into X_1'. For $d_S < 0$, mass is removed from Y_1^* and added as a blue subcolumn to define S_2.

Appendix B. Towerplex conditions. Below is a list of explicit conditions that can be used to prove Theorem 6.2.

- $\lim_{n\to\infty} r_n = 0$;
- $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} s_n = \infty$;
- $\lim_{n\to\infty} \mu(Y_n) = 0$;
- $\epsilon_n \max\{M_n^1, M_n^2\} < \epsilon_{n-1}$;
- $h_{n-1} < M_n^1, M_n^2 < h_n$;
- h_n sufficiently large such that equation (5.1) holds;
- $h_{n+1}\epsilon_n w_n > H_{m_n} + 1$;
- $\epsilon_{n+1}(H_{m_n} + 1) < \epsilon_n w_n$;
- $H_{m_{n+1}} \geq H_{m_n}$.

If $r_n = 1/2(n+2)$ and $s_n = 1/2$, and ϵ_n is sufficiently small such that Lemma 6.1 holds, then we have a canonical towerplex construction.

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Terrence Adams
U.S. Government
9161 Sterling Drive
Laurel, MD 20723, U.S.A.
E-mail: terry@ganita.org

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