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α-STABLE RANDOM WALK HAS MASSIVE THORNS

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Abstract. We introduce and study a class of random walks defined on the integer lattice $\mathbb{Z}^d$—a discrete space and time counterpart of the symmetric α-stable process in $\mathbb{R}^d$. When $0 < \alpha < 2$ any coordinate axis in $\mathbb{Z}^d$, $d \geq 3$, is a non-massive set whereas any cone is massive. We provide a necessary and sufficient condition for a thorn to be a massive set.

1. Introduction

Motivating questions. This paper is motivated by the following two closely related questions:

1. Assuming that the probability ϕ on the group $\mathbb{Z}^d$ is symmetric and its support generates the whole $\mathbb{Z}^d$, what is the possible decay of the Green function
$$G(x) = \sum_{n\geq 0} \phi^{(n)}(x)$$
as x tends to infinity?
2. If ϕ is as above, which sets are massive (recurrent) with respect to the random walk driven by ϕ?

Recall that the answer to the first question is known when ϕ is symmetric, has finite second moment and $d \geq 3$. Indeed, it is proved in Spitzer [19] (see also Saloff-Coste and Hebisch [11] for the case of general finitely generated groups) that $G(x) \sim c(\phi)\|x\|^{2-d}$ at infinity. If the second moment of ϕ is infinite but ϕ belongs to the domain of attraction of the α-stable law with $d/2 < \alpha < \min\{d, 2\}$, then $G(x) \sim c(\phi)\|x\|^{\alpha-d} l(\|x\|)$ at infinity, where l is an appropriately chosen slowly varying function (see Williamson [20]). However, there are many symmetric probabilities ϕ for which the behaviour of the Green function G at infinity is not known.

In the present paper we use discrete subordination, a natural technique developed in Bendikov and Saloff-Coste [3] that produces interesting examples of probabilities ϕ for which one can estimate the behaviour of the Green

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function G at infinity. This in turn allows us to describe massiveness of some interesting classes of infinite sets. For instance, we give necessary and sufficient conditions for a thorn to be a massive set (see Section 4). Massiveness of thorns for a simple random walk in $\mathbb{Z}^d$, $d \geq 4$, was studied in the celebrated paper of Itô and McKean [12].

The main idea behind this technique is the well-known idea of subordination in the context of continuous time Markov semigroups, but the applications we have in mind require some adjustments and variations. The results we obtain shed some light on the questions formulated above. The present paper is concerned with examples where ϕ has neither finite support nor finite second moment.

Subordinated random walks. In the case of continuous time Markov processes, subordination is a well-known and useful procedure of obtaining a new process from an original process. The new process may differ very much from the original one, but its properties can be understood in terms of the original process. The best known application of this concept is to obtain a symmetric stable process from a Brownian motion (see e.g. Bendikov [1]).

From the probabilistic point of view, a new process $(Y_t)_{t>0}$ is obtained from the original process $(X_t)_{t>0}$ by setting $Y_t = X_{\varsigma_t}$, where the "subordinator" $(\varsigma_s)_{s>0}$ is a non-decreasing Lévy process taking values in $(0, \infty)$ and independent of $(X_t)_{t>0}$ (see e.g. Feller [8, Section X.7]).

From the analytical point of view, the transition function $h_\varsigma(t, x, B)$ of the new process is obtained as a time average of the transition function $h(t, x, B)$ of the original process, that is,

$$h_\varsigma(t, x, B) = \int_0^\infty h(s, x, B)\, d\mu_t(s).$$

In this formula $\mu_t(s)$ is the distribution of the random variable ς_t. Subordination was first introduced by Bochner in the context of semigroup theory (see [8, footnote, p. 347]).

Ignoring technical details, the minus infinitesimal generator $\mathcal{B}$ of the process $(Y_t)_{t>0}$ is a function of the minus infinitesimal generator $\mathcal{A}$ of the process $(X_t)_{t>0}$, that is, $\mathcal{B} = \psi(\mathcal{A})$ (see Jacob [13, Chapters 3 & 4] for a detailed discussion).

A discrete time version of subordination in which the functional calculus equation $\mathcal{B} = \psi(\mathcal{A})$ serves as the defining starting point has been considered by Bendikov and Saloff-Coste [3]. Given a probability ϕ on $\mathbb{Z}^d$ consider the random walk $X = \{X(n)\}_{n\geq 0}$ driven by ϕ. In its simplest form, discrete subordination is the consideration of a probability $\varPhi$ defined as a convex linear combination of the convolution powers $\phi^{(n)}$. That is,

$$\Phi = \sum_{n\geq 1} c_n\, \phi^{(n)},$$

where $c_n \geq 0$ and $\sum_{n\geq 1} c_n = 1$. We easily find that

$$\Phi^{(n)} = \sum_{k\geq n} \Big(\sum_{k_1+\cdots+k_n=k} \prod_{i=1}^{n} c_{k_i} \Big) \phi^{(k)}.$$

The probabilistic interpretation is as follows: Let (R_i) be a sequence of i.i.d. integer valued random variables which are independent of X and such that $\mathbb{P}(R_i = k) = c_k$. Set $\tau_n = R_1 + \cdots + R_n$. Then

$$\mathbb{P}(\tau_n = k) = \sum_{k_1+\cdots+k_n=k} \prod_{i=1}^{n} c_{k_i}$$

and $\Phi^{(n)}$ is the law of $Y(n) = X(\tau_n)$.

Another way to introduce the notion of discrete subordination is to use Markov generators. Let P be the operator of convolution by ϕ. The operator $L = I - P$ may be considered as minus the Markov generator of the associated random walk. For a proper function ψ we want to define a "subordinated" random walk with Markov generator $-\psi(L)$. The appropriate class of functions is the class of Bernstein functions (see the book of Schilling, Song and Vondraček [18]).

Recall that $\psi \in C^\infty(\mathbb{R}^+)$ is called a *Bernstein function* if it is nonnegative and $(-1)^{n-1}\psi^{(n)}(x) \geq 0$ for all $x > 0$ and all $n \in \mathbb{N}$. The set of all Bernstein functions is denoted by $\mathcal{BF}$. Each $\psi \in \mathcal{BF}$ has a representation

$$\psi(\theta) = a + b\theta + \int_{(0,\infty)} (1 - e^{-\theta s})\, d\nu(s) \tag{1.1}$$

for some constants $a, b \geq 0$ and some measure ν (the *Lévy measure*) such that

$$\int_{(0,\infty)} \min\{1, s\}\, d\nu(s) < \infty.$$

Proposition 1.1 ([3, Proposition 2.3]). *Assume that ψ is a Bernstein function with representation* (1.1) *such that $\psi(0) = 0$, $\psi(1) = 1$ and set*

$$\begin{aligned} c(\psi, 1) &= b + \int_{(0,\infty)} te^{-t}\, d\nu(t), \\ c(\psi, n) &= \frac{1}{n!} \int_{(0,\infty)} t^n e^{-t}\, d\nu(t), \qquad n > 1. \end{aligned} \tag{1.2}$$

Let ϕ be a probability on $\mathbb{Z}^d$. Let P be the operator of convolution with ϕ and set

$$P_\psi = I - \psi(I - P). \tag{1.3}$$

Then P_ψ is the convolution with a probability Φ defined as

$$\Phi = \sum_{n\geq 1} c(\psi, n)\phi^{(n)}. \tag{1.4}$$

EXAMPLE 1.2. The power function $\psi_\alpha(s) = s^{\alpha/2}$, $\alpha \in (0, 2)$, belongs to the class $\mathcal{BF}$. Its Lévy density $\nu_\alpha(t)$ is given by

$$\nu_\alpha(t) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} t^{-1-\alpha/2}.$$

The probabilities $c(\psi_\alpha, n)$ are given by

$$c(\psi_\alpha, n) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \frac{\Gamma(n-\alpha/2)}{\Gamma(n+1)} \sim \frac{\alpha/2}{\Gamma(1-\alpha/2)} n^{-1-\alpha/2}.$$

Choosing $\psi = \psi_\alpha$ in Proposition 1.1, we see that the Markov generators of the initial and new random walks are related by the equation

$$I - P_{\psi_\alpha} = (I-P)^{\alpha/2}.$$

DEFINITION 1.3. Let $X = \{X(n)\}_{n\geq 0}$ be the random walk driven by ϕ. The random walk with the transition operator P_ψ defined at (1.3) will be called the *ψ-subordinated random walk* and denoted by $X_\psi = \{X_\psi(n)\}_{n\geq 0}$. When $\psi = \psi_\alpha$ and $X = S$ is the simple random walk, we call X_ψ the *α-stable random walk* and denote it by S_α.

It is straightforward to show that the increments of S_α belong to the domain of attraction of the α-stable law. This justifies the name "α-stable random walk".

Notation. For any two non-negative functions f and g, $f(r) \sim g(r)$ at a means that $\lim_{r\to a} f(r)/g(r) = 1$, $f(x) = O(g(x))$ if $f(x) \leq Cg(x)$ for some constant $C > 0$, and $f(x) \asymp g(x)$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$.

2. Green function's asymptotics.

Let S be the simple random walk and $\psi \in \mathcal{BF}$. Assuming that the subordinated random walk S_ψ is transient we study the asymptotic behaviour of its Green function G_ψ.

We will use the following technical assumption: $\psi(0) = 0$, $\psi(1) = 1$ and

$$\psi(\lambda) = \lambda^{\alpha/2}/l(1/\lambda), \tag{2.1}$$

where $0 < \alpha < 2$ and $l(\lambda)$ varies slowly at infinity.

Recall that a function f defined in a neighbourhood of 0 is said to *vary regularly of index β at* 0 if for all $\lambda > 1$,

$$\lim_{x\to 0} \frac{f(\lambda x)}{f(x)} = \lambda^\beta.$$

When $\beta = 0$, one says that f *varies slowly* at 0. Any regularly varying function of index β is of the form $f(x) = x^\beta l(x)$, where l is slowly varying.

For example, each of the following functions varies regularly at 0 of index β: $x^\beta(\log 1/x)^\delta$, $x^\beta \exp\{(\log 1/x)^\delta\}$, $0 < \delta < 1$, etc.

A function F defined in a neighbourhood of ∞ is said to *vary regularly of index β at ∞* if $f(x) = F(1/x)$ varies regularly of index $-\beta$ at 0.

Let $c(\psi, k)$, $k \in \mathbb{N}$, be the probabilities defined at (1.2). For $k \leq 0$ we set $c(\psi, k) = 0$ and consider a random walk $\tau = (\tau_n)_{n\geq 0}$ on $\mathbb{Z}$ whose increments $\tau_{n+1} - \tau_n$ have distribution $c = \{c(\psi, k)\}_{k\in\mathbb{Z}}$. The random walk τ has non-negative increments, in particular it is transient. Let

$$C(B) = \sum_{k\geq 0} c^{(k)}(B), \qquad B \subset \mathbb{Z},$$

be its potential measure; here $c^{(k)}$ is the Dirac measure concentrated at 0 when $k = 0$ and the k-fold convolution of the probability c when $k \geq 1$. Setting $C(n) = C(\{n\})$ we obtain $C(n) = 0$ for $n < 0$, $C(0) = 1$ and

$$C(n) = \sum_{k=1}^{n} c(\psi, k) C(n - k), \quad n \geq 1.$$

Recall that $\psi \in \mathcal{BF}$ is called a *special Bernstein function*, written $\psi \in \mathcal{SBF}$, if $\lambda/\psi(\lambda)$ is also a Bernstein function. Evidently $\psi_\alpha \in \mathcal{SBF}$ whereas $\psi(\lambda) = 1 - e^{-\lambda}$ does not belong to $\mathcal{SBF}$. In particular, the inclusion $\mathcal{SBF} \subset \mathcal{BF}$ is proper.

LEMMA 2.1. *Let $\psi \in \mathcal{BF}$ satisfy* (2.1). *The strong renewal property*

$$C(n) \sim \frac{1}{\Gamma(\alpha/2)} n^{\alpha/2-1} l(n) \qquad \text{as } n \to \infty \tag{2.2}$$

holds in the following two cases:

(i) $\psi \in \mathcal{BF}$ *and* $1 < \alpha < 2$,
(ii) $\psi \in \mathcal{SBF}$ *and* $0 < \alpha < 2$.

Proof. Define an auxiliary function $M(x)$, $x \in \mathbb{R}$, as

$$M(x) = \sum_{k\leq x} C(k).$$

Observe that M is a right-continuous step function having jumps at integers. More precisely $M(x) = 0$ for $x < 0$, $M(x) = C(0)$ for $0 \leq x < 1$, $M(x) = C(0) + C(1)$ for $1 \leq x < 2$ etc. We will compute its Laplace–Stieltjes transform

$$\mathcal{L}(M)(\lambda) = \int_{\mathbb{R}} e^{-\lambda x}\, dM(x) = \sum_{k=0}^{\infty} e^{-\lambda k} C(k).$$

We have

$$\mathcal{L}(M)(\lambda) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} e^{-\lambda k} c^{(n)}(k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} e^{-\lambda k}\, \mathbb{P}(\tau_n = k)$$
$$= \sum_{n=0}^{\infty} (\mathbb{E}(e^{-\lambda\tau_1}))^n = \frac{1}{1-\mathbb{E}(e^{-\lambda\tau_1})}. \tag{2.3}$$

We claim that

$$\mathbb{E}(e^{-\lambda\tau_1}) = 1 - \psi(1-e^{-\lambda}). \tag{2.4}$$

Indeed, by Proposition 1.2,

$$\mathbb{E}(e^{-\lambda\tau_1}) = \sum_{k=1}^{\infty} e^{-\lambda k}\, c(\psi, k).$$

Using (1.1) and the fact that $\psi(1) = 1$ we obtain

$$1-\psi(1-e^{-\lambda}) = 1 - b(1-e^{-\lambda}) - \int_{(0,\infty)} (1-e^{-t(1-e^{-\lambda})})\, d\nu(t)$$
$$= 1 - \Big(b + \int_{(0,\infty)} (1-e^{-t})\, d\nu(t)\Big) + be^{-\lambda}$$
$$+ \int_{(0,\infty)} e^{-t} \sum_{n=1}^{\infty} \frac{t^n e^{-n\lambda}}{n!}\, d\nu(t)$$
$$= be^{-\lambda} + \sum_{n\ge1} \frac{1}{n!}\Big(\int_{(0,\infty)} e^{-t} t^n\, d\nu(t)\Big) e^{-\lambda n} = \sum_{n\ge1} c(\psi, n) e^{-\lambda n},$$

as desired.

It follows that

$$\mathcal{L}M(\lambda) = \frac{1}{\psi(1-e^{-\lambda})}.$$

Hence, by (2.1) we obtain

$$\mathcal{L}M(\lambda) \sim \lambda^{-\alpha/2} l(1/\lambda) \quad \text{as } \lambda \to 0^+.$$

By Karamata's Tauberian Theorem [4, Theorem 1.7.1],

$$M(x) \sim \frac{1}{\Gamma(1+\alpha/2)} x^{\alpha/2}\, l(x) \quad \text{as } x \to \infty. \tag{2.5}$$

By [4, Theorem 8.7.3], condition (2.5) is equivalent to

$$\sum_{k=n}^{\infty} c(\psi, k) \sim \frac{n^{-\alpha/2}}{l(n)\Gamma(1-\alpha/2)} \quad \text{as } n \to \infty. \tag{2.6}$$

Moreover, recall that

$$C(0) = 1, \qquad C(n) = \sum_{k=1}^{n} c(\psi, k) C(n-k), \qquad n > 1. \tag{2.7}$$

The celebrated Garsia–Lamperti theorem [9, Theorem 1.1] says that (2.7) and (2.6) imply that, when $1 < \alpha < 2$,

$$C(n) \sim \Gamma\left(1 - \frac{\alpha}{2}\right) \frac{\sin(\pi\alpha/2)}{\pi} n^{\alpha/2-1} l(n) \qquad \text{as } n \to \infty.$$

Using Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

we obtain (2.2).

Let us turn to the proof of (ii). Since $\psi \in \mathcal{SBF}$, we have

$$\frac{1}{\psi(\lambda)} = b + \int_0^\infty e^{-\lambda t} u(t)\, dt \tag{2.8}$$

for some $b \geq 0$ and some non-increasing function $u\colon (0,\infty) \to (0,\infty)$ satisfying $\int_0^1 u(t)\, dt < \infty$ (see [18, Theorem 11.3]). Set $\Phi(\lambda) = 1/\psi(\lambda)$ and observe that by (2.1),

$$\Phi(\lambda) \sim \lambda^{-\alpha/2} l(1/\lambda) \qquad \text{as } \lambda \to 0.$$

Applying both the Karamata Tauberian Theorem [4, Theorem 1.7.1] and the Monotone Density Theorem we obtain

$$u(t) \sim \frac{1}{\Gamma(\alpha/2)} t^{\alpha/2-1} l(t) \qquad \text{as } t \to \infty. \tag{2.9}$$

On the other hand,

$$\mathcal{L}(M)(\lambda) = \Phi(1 - e^{-\lambda}) = \sum_{k \geq 0} \frac{(-1)^k \Phi^{(k)}(1)}{k!} e^{-\lambda k},$$

whence by the uniqueness of the Laplace transform we obtain

$$C(k) = \frac{1}{k!} \int_0^\infty t^k e^{-t} u(t)\, dt, \qquad k \in \mathbb{N}.$$

We claim that

$$C(k) = \frac{1}{k!} \int_{k/2}^{2k} t^k e^{-t} u(t)\, dt + O((2/e)^k).$$

To see this, observe that the function $t \mapsto t^k e^{-t}$ is unimodal with maximum at $t = k$. Hence for a, b and k large enough we have

$$\int_1^a t^k e^{-t} u(t)\,dt \le a^k e^{-a} \int_1^a u(t)\,dt, \qquad a < k,$$

$$\int_b^\infty t^k e^{-t} u(t)\,dt \le b^{k+1} e^{-b} \int_b^\infty \frac{u(t)}{t}\,dt, \quad b > k.$$

In particular, choosing $a = k/2$, $b = 2k$ and applying (2.9) we obtain

$$\frac{1}{k!}\Big(\int_0^{k/2} t^k e^{-t} u(t)\,dt + \int_{2k}^\infty t^k e^{-t} u(t)\,dt\Big) = O((2/e)^k),$$

which evidently proves the claim.

Once again applying (2.9) we get

$$\frac{1}{k!}\int_{k/2}^{2k} t^k e^{-t} u(t)\,dt \sim \frac{l(k)}{k!\Gamma(\alpha/2)} \int_{k/2}^{2k} t^{k+\alpha/2-1} e^{-t}\,dt.$$

It is straightforward to show that

$$\frac{1}{k!}\int_{k/2}^{2k} t^{k+\alpha/2-1} e^{-t}\,dt = \frac{1}{k!}\int_0^\infty t^{k+\alpha/2-1} e^{-t}\,dt + O((2/e)^k).$$

Altogether, the above shows that

$$C(k) \sim \frac{l(k)\Gamma(k+\alpha/2)}{\Gamma(\alpha/2)\Gamma(k+1)} \sim \frac{1}{\Gamma(\alpha/2)} k^{\alpha/2-1} l(k). \ \blacksquare$$

REMARK 2.2. Recall that in the continuous time setting to each function $\psi \in \mathcal{BF}$ is associated a unique convolution semigroup $(\eta_t)_{t>0}$ of measures supported on $[0,\infty)$ such that

$$\mathcal{L}\eta_t(\lambda) = e^{-t\psi(\lambda)}.$$

A function $\psi \in \mathcal{SBF}$ is characterized by the fact that the potential measure $U = \int_0^\infty \eta_t\,dt$ restricted to $(0,\infty)$ is absolutely continuous with respect to the Lebesgue measure and its density $u(t)$ is a decreasing function. *Whether this is true in the discrete time setting, i.e. whether the sequence $C(k)$ is decreasing, is an open question at the time of writing.*

We present here some partial answer to this question. Recall that a function $\psi \in \mathcal{BF}$ is called a *complete Bernstein function*, written $\psi \in \mathcal{CBF}$, if its Lévy measure ν is absolutely continuous with respect to the Lebesgue measure and its density $\nu(s)$ is completely monotone, i.e.

$$\nu(s) = \int_{[0,\infty)} e^{-st}\,\mu(dt),$$

Observe that in fact μ is supported on $(0,\infty)$ and satisfies

$$\int_{(0,\infty)} \min(t^{-1}, t^{-2})\, \mu(dt) < \infty.$$

The inclusion $\mathcal{CBF} \subset \mathcal{SBF}$ is proper. For all this we refer to [18].

THEOREM 2.3. *For* $\psi \in \mathcal{CBF}$ *the renewal sequence* $\{C(k)\}_{k\in\mathbb{N}}$ *defined as*

$$C(0) = 1 \quad \text{and} \quad C(k) = \sum_{n=0}^{k} c(\psi, n) C(k-n), \quad k \geq 1,$$

is decreasing.

Proof. The proof is in four steps.

CLAIM 1. *There exists a measure* m *on* $(0,\infty)$ *such that*

$$c(\psi,1) = b + \int_{(0,\infty)} e^{-2r}\, m(dr),$$

$$c(\psi,n) = \int_{(0,\infty)} e^{-(n+1)r}\, m(dr), \qquad n > 1.$$

We consider the case $n > 1$. Since $\psi \in \mathcal{CBF}$,

$$\begin{aligned} c(\psi,n) &= \frac{1}{n!} \int_{(0,\infty)} t^n e^{-t} \nu(t)\, dt \\ &= \frac{1}{n!} \int_{(0,\infty)} dt\, t^n e^{-t} \int_{(0,\infty)} e^{-st}\, \mu(ds) \\ &= \int_{(0,\infty)} \mu(ds)\, \frac{1}{n!} \int_{(0,\infty)} t^n e^{-t(1+s)}\, dt = \int_{(0,\infty)} \frac{\mu(\,ds)}{(1+s)^{n+1}}. \end{aligned}$$

Substitution $\log(1+s) = r$ gives

$$c(\psi,n) = \int_{(0,\infty)} e^{-(n+1)r}\, m(dr),$$

as desired.

CLAIM 2. $\{c(\psi,n)\}_{n\in\mathbb{N}}$ *satisfies*

$$c(\psi, n-1) c(\psi, n+1) > c(\psi,n)^2, \qquad n > 1.$$

It is enough to consider the case $n > 2$. We apply Claim 1:

$$\begin{aligned} c(\psi,n-1)c(\psi,n+1) &= \int_{(0,\infty)} m(\,ds) \int_{(0,\infty)} m(dt)\, e^{-\{ns+(n+2)t\}} \\ &= \int_{(0,\infty)} m(\,ds) \int_{(0,\infty)} m(dt)\, e^{-(n+1)(s+t)} e^{s-t} \end{aligned}$$

$$= \int_{(0,\infty)} m(ds) \int_{(0,\infty)} m(dt)\, e^{-(n+1)(s+t)} \cosh(s-t)$$
$$> \int_{(0,\infty)} m(ds) \int_{(0,\infty)} m(dt)\, e^{-(n+1)(s+t)} = c(\psi, n)^2.$$

The strict inequality follows from the fact that, by (2.1), m is not a Dirac measure.

CLAIM 3. $\{C(n)\}_{n\in\mathbb{N}}$ *satisfies*

$$C(n-1)C(n+1) > C(n)^2, \qquad n > 1.$$

Indeed, we have

$$C(n) = \sum_{k=1}^{n} c(\psi, k)C(n-k), \qquad n \geq 1,$$

and

$$c(\psi, n-1)c(\psi, n+1) > c(\psi, n)^2, \qquad n > 1.$$

The remarkable de Bruijn–Erdős theorem [7, Theorem 1] yields the desired result.

Finally, we prove that $\{C(k)\}_{k\in\mathbb{N}}$ is a decreasing sequence. By Claim 3, the sequence $C(k+1)/C(k)$ increases. Assume that $C(k_0+1)/C(k_0) \geq 1$ for some $k_0 \in \mathbb{N}$. Then there are some $a > 1$ and $N \in \mathbb{N}$ such that $C(k+1)/C(k) > a$ for all $k \geq N$. It follows that $C(n) \geq a^{n-N}C(N)$ for all $n \geq N$. This is a contradiction, because for all $n \geq 0$,

$$C(n) = \mathbb{P}(\exists k : \tau_k = n) \leq 1,$$

so $C(k)$ decreases. ∎

Let $p(n, x)$ be a transition function of the simple random walk S. By $p_\psi(n, x)$ we denote a transition function of the subordinated random walk S_ψ and by G_ψ its Green function,

$$p_\psi(n, x) = \sum_{k=1}^{\infty} p(k, x)\mathbb{P}(\tau_n = k),$$
$$G_\psi(x) = \sum_{n=1}^{\infty} p_\psi(n, x) = \sum_{k=1}^{\infty} p(k, x)C(k).$$

THEOREM 2.4. *Assume that* $\psi \in \mathcal{BF}$ *satisfies* (2.1) *with* $0 < \alpha < d$ *and that* (2.2) *holds. Then*

$$G_\psi(x) \sim \frac{C_{d,\alpha}}{\|x\|^d \psi(1/\|x\|^2)} \qquad \text{as } x \to \infty,$$

where

$$C_{d,\alpha} = \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma(\alpha/2)} \Gamma\left(\frac{d-\alpha}{2}\right).$$

Proof. Recall that $p(k,x)$ is the k-step transition probability of the simple random walk started at 0. Since $p(k,x)=0$ for $k<\|x\|/\sqrt{d}$, we have

$$\begin{aligned} G_\psi(x) &= \sum_{k\ge \|x\|/\sqrt{d}} C(k)p(k,x) \\ &= \underbrace{\sum_{k>\|x\|^2/A} C(k)p(k,x)}_{I_1} + \underbrace{\sum_{\|x\|/\sqrt{d}\le k\le \|x\|^2/A} C(k)p(k,x)}_{I_2}, \end{aligned}$$

where $A>1$ is a constant which will be specified later.

Our further analysis is based on the results of G. F. Lawler [14, Section 1.2]. We write $n \leftrightarrow x$ when $n+x_1+\cdots+x_d$ is *even*. Set

$$q(n,x) = 2\left(\frac{d}{2\pi n}\right)^{d/2} e^{-d\|x\|^2/(2n)}$$

and define the error function

$$E(n,x) = \begin{cases} p(n,x)-q(n,x) & \text{if } n\leftrightarrow x, \\ 0 & \text{if } n \not\leftrightarrow x. \end{cases}$$

By [14, Theorem 1.2.1],

$$|E(k,x)| \le c_1\|x\|^{-2}k^{-d/2} \tag{2.10}$$

for some $c_1>0$ and all $k\ge 1$.

To study I_1 we may assume that $x\leftrightarrow 0$; then $p(2k+1,x)=0$ for all $k\ge 1$. Writing I_1 in the form

$$I_1 = \underbrace{\sum_{2k>\|x\|^2/A} C(2k)q(2k,x)}_{I_{11}} + \underbrace{\sum_{2k>\|x\|^2/A} C(2k)E(2k,x)}_{I_{12}}$$

and using (2.2) and (2.10) we obtain

$$\begin{aligned} I_{12} &\le c_2 \sum_{k>\|x\|^2/A} k^{\alpha/2-1} l(k) \frac{k^{-d/2}}{\|x\|^2} \\ &\sim c_2 \int_{\|x\|^2/A}^{\infty} t^{\alpha/2-d/2-1} l(t)\,dt \quad \text{as } x\to\infty \end{aligned} \tag{2.11}$$

for some constant $c_2 > 0$. By [4, Proposition 1.5.10],

$$\int_{\|x\|^2/A}^{\infty} t^{\alpha/2-d/2-1} l(t)\, dt \sim \frac{2}{d-\alpha} A^{(d-\alpha)/2} \|x\|^{\alpha-d} l(\|x\|^2) \qquad \text{as } x \to \infty.$$

It follows that

$$\lim_{\|x\|\to\infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_{12} = 0.$$

Similarly, when $\|x\| \to \infty$,

$$\begin{aligned} I_{11} &\sim \frac{2\left(\frac{d}{2\pi}\right)^{d/2}}{\Gamma(\alpha/2)} \sum_{2k>\|x\|^2/A} (2k)^{\alpha/2-1} l(2k)(2k)^{-d/2} \exp\left\{\frac{-d\|x\|^2}{4k}\right\} \\ &\sim \frac{\left(\frac{d}{2\pi}\right)^{d/2}}{\Gamma(\alpha/2)} \int_{\|x\|^2/A}^{\infty} t^{\alpha/2-d/2-1} \exp\left\{\frac{-d\|x\|^2}{2t}\right\} l(t)\, dt. \end{aligned}$$

Applying [4, Proposition 4.1.2] we obtain

$$I_{11} \sim \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma(\alpha/2)} \|x\|^{\alpha-d}\, l(\|x\|^2) \int_0^{Ad/2} s^{d/2-\alpha/2-1} e^{-s}\, ds.$$

It follows that

$$\lim_{\|x\|\to\infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_{11} = \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma(\alpha/2)} \int_0^{dA/2} s^{d/2-\alpha/2-1} e^{-s}\, ds =: C_1(A).$$

To estimate I_2 we use the Gaussian upper bound from [11, Theorem 2.1]:

$$\begin{aligned} I_2 &\le c_3 \sum_{\|x\|/\sqrt{d} \le k \le \|x\|^2/A} k^{\alpha/2-1} l(k) k^{-d/2} \exp\left\{\frac{-\|x\|^2}{c_4 k}\right\} \\ &\sim c_3 \int_{\|x\|/\sqrt{d}}^{\|x\|^2/A} t^{\alpha/2-d/2-1} \exp\left\{\frac{-\|x\|^2}{c_4 t}\right\} l(t)\, dt \\ &= \frac{c_3}{c_4^{\alpha/2-d/2}} \|x\|^{\alpha-d} \int_{A/c_4}^{\sqrt{d}\|x\|/c_4} s^{d/2-\alpha/2-1} e^{-s} l\left(\frac{\|x\|^2}{c_4 s}\right) ds \\ &\le \frac{c_3}{c_4^{\alpha/2-d/2}} \|x\|^{\alpha-d} \int_{A/c_4}^{\infty} s^{d/2-\alpha/2-1} e^{-s} l\left(\frac{\|x\|^2}{c_4 s}\right) ds \end{aligned}$$

for some constants $c_3, c_4 > 0$. Next we apply [4, Theorem 1.5.6] and the

Dominated Convergence Theorem:

$$\limsup_{\|x\|\to\infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} I_2 \le \frac{c_3}{c_4^{\alpha/2-d/2}\Gamma(\alpha/2)} \int_{A/c_2}^{\infty} s^{d/2-\alpha/2-1} e^{-s}\, ds =: C_2(A).$$

Altogether, the above shows that, for any fixed $A > 1$,

$$\limsup_{\|x\|\to\infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} G_\psi(x) \le C_1(A) + C_2(A),$$

$$\liminf_{\|x\|\to\infty} \frac{\|x\|^{d-\alpha}}{l(\|x\|^2)} G_\psi(x) \ge C_1(A).$$

Finally, we have

$$\lim_{A\to\infty} C_2(A) = 0,$$

$$\lim_{A\to\infty} C_1(A) = \left(\frac{d}{2}\right)^{\alpha/2} \frac{\pi^{-d/2}}{\Gamma(\alpha/2)} \Gamma\left(\frac{d-\alpha}{2}\right). \blacksquare$$

REMARK 2.5. One useful observation is that if we assume that the function ψ satisfies

$$\psi(\theta) \asymp \theta^{\alpha/2}/l(1/\theta) \quad \text{at } 0$$

and belongs to the class $\mathcal{SBF}$, then following the line of reasoning of Lemma 2.1 and Theorem 2.4, we obtain

$$C(k) \asymp k^{\alpha/2-1} l(k) \quad \text{at } \infty$$

and

$$G_\psi(x) \asymp \|x\|^{\alpha-d} l(\|x\|^2) \quad \text{at } \infty.$$

Whether this is true when $\psi \in \mathcal{BF}\setminus\mathcal{SBF}$ is an open question at the time of writing. In the closely related paper [2] some partial results in this direction are obtained.

3. Massive sets

Basic definitions. Let $X = \{X(n)\}_{n\ge 0}$ be a transient random walk on $\mathbb{Z}^d$. Let B be a proper subset of $\mathbb{Z}^d$ and p_B the hitting probability of B. The set B is called *massive* or *recurrent* if $p_B(x) = 1$ for all $x \in \mathbb{Z}^d$, and *non-massive* otherwise.

Let $\pi_B(x)$ be the probability that the random walk X starting from x visits the set B infinitely many times. The set B is massive if and only if $\pi_B \equiv 1$; for non-massive B, π_B is identically 0.

Let $G(x,y)$ be the Green function of X. In general, the function p_B is excessive, so it can be written in the form

$$p_B = G\varrho_B + \pi_B.$$

When B is a non-massive set, i.e. $\pi_B \equiv 0$, p_B is a potential. It is called the *equilibrium potential* of B, and ϱ_B is the *equilibrium distribution*. When B is non-massive, the *capacity* of B is defined as

$$\mathrm{Cap}(B) = \sum_{y \in B} \varrho_B(y).$$

This quantity can also be computed as

$$\mathrm{Cap}(B) = \sup\Big\{\sum_{y \in B} \varrho(y) : \varrho \in \Xi_B\Big\},$$

where

$$\Xi_B = \{\varrho \geq 0 : \operatorname{supp} \varrho \subset B \text{ and } G\varrho \leq 1\}.$$

For all this we refer to Spitzer [19, Chapter VI].

Test of massiveness. Assume that the Green function $G(x)$ is of the form

$$G(x) = \frac{a(x)}{\chi(\|x\|)}, \qquad x \neq 0, \tag{3.1}$$

where χ is a non-decreasing function satisfying the doubling condition

$$\chi(2\theta) \leq C\chi(\theta) \quad \text{ for all } \theta > 0 \text{ and some } C > 1, \tag{3.2}$$

and $c_1 \leq a(x) \leq c_2$ for some $c_1, c_2 > 0$ uniformly in x.

For a set B define the following sequence of sets:

$$B_k = \{x \in B : 2^k \leq \|x\| < 2^{k+1}\}, \quad k = 0, 1, \dots.$$

Theorem 3.1. *A set B is non-massive if and only if*

$$\sum_{k=0}^{\infty} \frac{\mathrm{Cap}(B_k)}{\chi(2^k)} < \infty.$$

To prove this statement, crucial in fact in our study, we use assumptions (3.1) and (3.2) and follow step by step the classical proof by Spitzer [19, Section 26, T1].

Example 3.2. Let S be the simple random walk in $\mathbb{Z}^3$. The set $B = \mathbb{Z}_+ \times \{0\} \times \{0\}$ is S-massive. Moreover, its proper subset $\mathcal{P} \times \{0\} \times \{0\}$, where $\mathcal{P}$ is the set of primes, is massive (see [12], [15]).

Let $0 < \alpha < 2$ and S_α be the α-stable random walk in $\mathbb{Z}^3$. We claim that the set $B = \mathbb{Z}_+ \times \{0\} \times \{0\}$ is not massive. To prove this we apply Theorem 3.1 with $\chi(\theta) = \theta^{3-\alpha}$. Let $|B_k|$ be the cardinality of B_k. Since

$\mathrm{Cap}(B_k) \le |B_k|$, we have

$$\sum_{k=0}^{\infty} \frac{\mathrm{Cap}(B_k)}{\chi(2^{k+1})} \le \sum_{k=0}^{\infty} \frac{|B_k|}{2^{(k+1)(3-\alpha)}} \le \sum_{k=0}^{\infty} \sum_{n:\,(n,0,0)\in B_k} \frac{1}{n^{3-\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^{3-\alpha}} < \infty.$$

EXAMPLE 3.3. Let B be the hyperplane $\{x \in \mathbb{Z}^d : x_1 = 0\}$, $d \ge 3$. We claim that:

(i) If $0 < \alpha < 1$, then B is non-massive with respect to S_α.
(ii) If $1 \le \alpha \le 2$, then B is massive with respect to S_α.

Let $s_\alpha(n)$ be the projection of $S_\alpha(n)$ on the x_1-axis. Evidently B is S_α-massive if and only if the random walk $\{s_\alpha(n)\}$ is recurrent. The characteristic function of the random variable $S_\alpha(1)$ is

$$H_\alpha(\theta) = 1 - \Bigl(1 - \frac{1}{d}\sum_{j=1}^{d} \cos\theta_j\Bigr)^{\alpha/2}, \qquad \theta \in \mathbb{R}^d.$$

It follows that the characteristic function $h_\alpha(\xi)$ of $s_\alpha(1)$ is

$$h_\alpha(\xi) = 1 - d^{-\alpha/2}(1-\cos\xi)^{\alpha/2}, \qquad \xi \in \mathbb{R}.$$

Let $p(n)$ be the probability of return to 0 in n steps defined by the random walk $\{s_\alpha(n)\}$. Then taking the inverse Fourier transform we obtain

$$p(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (h_\alpha(\xi))^n \, d\xi.$$

It follows that

$$\sum_{n\ge 0} p(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\xi}{1-h_\alpha(\xi)} \asymp \int_0^1 \frac{d\xi}{\xi^\alpha} < \infty$$

if and only if $0 < \alpha < 1$. By the well known criterion of transience, $s_\alpha(n)$ is transient.

4. Thorns. In this section we assume that the dimension d of the lattice $\mathbb{Z}^d$ satisfies $d \ge 3$. For $x = (x_1, \ldots, x_{d-1}, x_d)$ we set $x' = (x_1, \ldots, x_{d-1})$ and write $x = (x', x_d)$. A *thorn* $\mathcal{T}$ is defined as

$$\mathcal{T} = \{(x', x_d) \in \mathbb{Z}^d : \|x'\| \le t(x_d),\ x_d \ge 1\},$$

where $t(n)$ is a non-decreasing sequence of positive numbers. We study S_α-massiveness of $\mathcal{T}$.

The problem of massiveness of thorns with respect to the simple random walk was studied in Itô and McKean [12]. When $d = 3$ a thorn $\mathcal{T}$ is S-massive, because the straight line is S-massive. Hence for the simple random walk one assumes that $d \geq 4$.

We denote by $\mathrm{Cap}_\alpha(B)$ the S_α-capacity of the set $B \subset \mathbb{Z}^d$, whereas $\widetilde{\mathrm{Cap}}_\alpha(A)$ stands for the capacity of the set $A \subset \mathbb{R}^d$, associated with the rotationally invariant α-stable process.

PROPOSITION 4.1. *Assume that* $\limsup_{n\to\infty} t(n)/n = \delta > 0$. *Then the thorn* $\mathcal{T}$ *is* S_α*-massive for any* $0 < \alpha < 2$ *and* $d \geq 3$.

Proof. The sequence $t(n)$ is non-decreasing, so by the assumption, $\limsup_{n\to\infty} t(2^n)/2^n \geq \delta/2$. Hence $t(2^n)/2^n > \delta/3$ for infinitely many n. For such n consider the sets

$$\mathcal{T}_n = \mathcal{T} \cap \{x \in \mathbb{Z}^d : 2^n \leq \|x\| < 2^{n+1}\}. \tag{4.1}$$

Let B_n be the ball of radius $\delta 2^{n-2}$ centred at $(0, \ldots, 0, 3\cdot 2^{n-1})$ (see Figure 1).

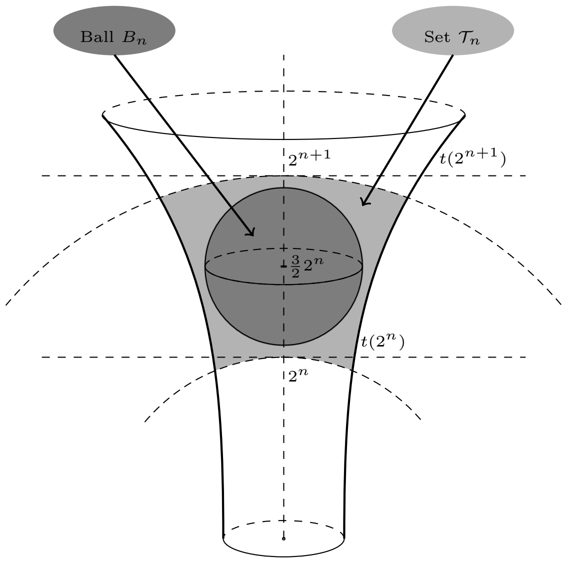

Fig. 1. Ball inscribed in the thorn

Since $t(2^n) > \delta/3 \cdot 2^n > \delta 2^{n-2}$, we have $B_n \subset \mathcal{T}_n$, whence

$$\mathrm{Cap}_\alpha(\mathcal{T}_n) \geq \mathrm{Cap}_\alpha(B_n).$$

By the inequality (5.7) of Section 5, for some $c > 0$,

$$\mathrm{Cap}_\alpha(B_n) \geq c2^{(n-2)(d-\alpha)}.$$

It follows that

$$\sum_{n\geq 0} \frac{\mathrm{Cap}_\alpha(\mathcal{T}_n)}{2^{n(d-\alpha)}} = \infty.$$

By Theorem 3.1, the thorn $\mathcal{T}$ is massive. ∎

REMARK 4.2. Using the capacity bounds given in Section 5 and following the same line of reasoning as in the proof of Proposition 4.1, we show that the thorn $\mathcal{T}$ satisfying $\limsup t(n)/n > 0$ is S_ψ-massive for any special Bernstein function ψ which satisfies the assumptions in Theorem 2.4. When $\lim t(n)/n = 0$, S_ψ-massiveness of $\mathcal{T}$ is a delicate question. In this generality this question is opened at present.

Next we study the case

$$\lim_{n\to\infty} \frac{t(n)}{n} = 0. \tag{4.2}$$

Our reasoning is based on the criterion of massiveness given in Theorem 3.1 but requires more advanced tools than those in the proof of Proposition 4.1. More precisely, we need upper and lower bounds of the α-capacity of non-spherically symmetric sets, long cylinders for instance.

Let $\mathcal{F}_L$ be a cylinder of height L with the unit disc as base,

$$\mathcal{F}_L = \{(x', x_d) \in \mathbb{R}^d : \|x'\| \le 1,\ 0 < x_d \le L\}.$$

PROPOSITION 4.3. *There exist constants $c_0, c_1 > 0$ which depend only on d and α such that*

$$c_0 L \le \widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_L) \le c_1 L, \quad L \ge 1.$$

Proof. For the upper bound we write $L = k + m$, where $k = [L]$ and $m = L - [L]$. Then, for some $c_1 > 0$,

$$\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_L) \le k\,\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_1) + \widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_m) \le c_1 L.$$

To obtain the lower bound we define

$$D_i = \{(x', x_d) \in \mathbb{R}^d : \|x'\| \le 1,\ i-1 \le x_d \le i\}, \qquad i \ge 1.$$

Let μ_i be the equilibrium measure of D_i, i.e. $\mu_i(D_i) = \widetilde{\mathrm{Cap}}_\alpha(D_i)$. We have

$$\widetilde{G_\alpha}\mu_{i+1}(x) = \widetilde{G_\alpha}\mu_1(x - ie_d),$$

where $\widetilde{G_\alpha}$ is the Green function associated with the symmetric α-stable process in $\mathbb{R}^d$ and $e_d = (0, 0, \dots, 1)$. Without loss of generality we can assume that L is an integer. Define the following measure:

$$\sigma = \mu_1 + \cdots + \mu_L.$$

Clearly $\sigma(\mathbb{R}^d) = L\,\widetilde{\mathrm{Cap}}_\alpha(D_1)$.

We claim that

$$\widetilde{G_\alpha}\sigma \le K < \infty. \tag{4.3}$$

Indeed, we have $G_\alpha\mu_1(x) \le 1$ for all x, and

$$\lim_{\|x\|\to\infty} \|x\|^{d-\alpha}\widetilde{G_\alpha}\mu_1(x) < C$$

for some constant $C > 0$. It follows that

$$\sum_{i>0} \widetilde{G_\alpha}\mu_1(x - ie_d) \le C \sum_{i>0} \|x - ie_d\|^{\alpha-d} \wedge 1.$$

Observe that the series above converges uniformly in x, which proves the claim.

The inequality (4.3) in turn implies the lower bound

$$\widetilde{\mathrm{Cap}_\alpha}(F_L) \ge \sigma(F_L)/K = \frac{L}{K}\,\widetilde{\mathrm{Cap}_\alpha}(D_1). \ \blacksquare$$

Define

$$\begin{aligned}
\mathcal{F}_n^- &= \left\{(x', x_d) \in \mathbb{R}^d : \|x'\| \le t(2^n),\ \frac{4}{3}\cdot 2^n \le x_d < \frac{3}{4}\cdot 2^{n+1}\right\},\\
\mathcal{F}_n^+ &= \left\{(x', x_d) \in \mathbb{R}^d : \|x'\| \le t(2^{n+1}),\ \frac{3}{4}\cdot 2^n \le x_d < \frac{4}{3}\cdot 2^{n+1}\right\},\\
F_n^- &= \mathcal{F}_n^- \cap \mathbb{Z}^d \quad \text{and} \quad F_n^+ = \mathcal{F}_n^+ \cap \mathbb{Z}^d.
\end{aligned}$$

Let $Q(b)$ be the cube $[0,1]^d$ centred at b. For any set $B \subset \mathbb{Z}^d$, we denote by $\widetilde{B}$ the subset of $\mathbb{R}^d$ defined as

$$\widetilde{B} = \bigcup_{b\in B} Q(b). \tag{4.4}$$

THEOREM 4.4. *Under the assumption* (4.2), *the thorn* $\mathcal{T}$ *is* S_α*-massive if and only if the series*

$$\sum_{n>0} \left(\frac{t(2^n)}{2^n}\right)^{d-\alpha-1} \tag{4.5}$$

diverges.

Before embarking on the proof of Theorem 4.4 we illustrate the statement by the following example. Consider the thorn $\mathcal{T}$ with $t(n) = n/(\log(1+n))^\beta$, $\beta > 0$. Then $\mathcal{T}$ is S_α-massive if and only if $\beta \le 1/(d-\alpha-1)$.

Proof. Assume that the series (4.5) is convergent. We will show that the set $\mathcal{T}$ is non-massive. For any compact set $A \subset \mathbb{R}^d$ and for any $s > 0$ the following scaling property holds:

$$\widetilde{\mathrm{Cap}_\alpha}(sA) = s^{d-\alpha}\,\widetilde{\mathrm{Cap}_\alpha}(A), \tag{4.6}$$

(see e.g. Sato [17, Example 42.17]). Using Proposition 4.3, the assumption (4.2) and the equation (4.6), for enough large n we have

$$\begin{aligned}\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^+) &= \widetilde{\mathrm{Cap}}_\alpha(t(2^{n+1})\cdot \mathcal{F}_n^+/t(2^{n+1}))\\ &= t(2^{n+1})^{d-\alpha}\cdot \widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^+/t(2^{n+1}))\\ &\le c_1 t(2^{n+1})^{d-\alpha}\cdot t(2^{n+1})^{-1}\cdot\left(\frac{4}{3}\cdot 2^{n+1}-\frac{3}{4}\cdot 2^n\right)\\ &\le c_2 t(2^{n+1})^{d-\alpha-1}\cdot 2^{n+1}\end{aligned}$$

for some $c_1, c_2 > 0$. Let $\mathcal{T}_n$ be as in (4.1). Since $\mathcal{T}_n \subset F_n^+$ (see Figure 2),

$$\mathrm{Cap}_\alpha(\mathcal{T}_n) \le \mathrm{Cap}_\alpha(F_n^+).$$

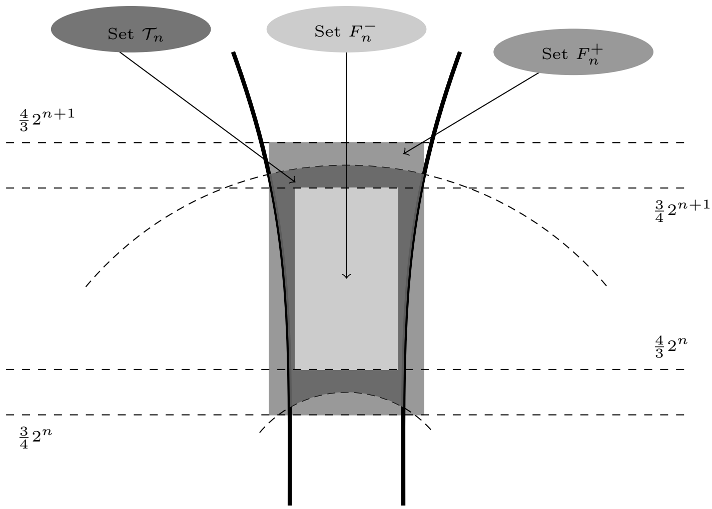

Fig. 2. Two cylinders inscribed in and circumscribed around the thorn

By Theorem 5.2 of Section 5,

$$c_3\,\widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^+) \le \mathrm{Cap}_\alpha(F_n^+) \le c_4\,\widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^+), \tag{4.7}$$

for some $c_3, c_4 > 0$. Using again Proposition 4.3 we obtain

$$c_5\,\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^+) \le \widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^+) \le c_6\,\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^+) \tag{4.8}$$

for some $c_5, c_6 > 0$. Altogether, the above shows that

$$\sum_{n>0}\frac{\mathrm{Cap}_\alpha(\mathcal{T}_n)}{2^{n(d-\alpha)}} \le c_7\sum_{n>0}\left(\frac{t(2^{n+1})}{2^{n+1}}\right)^{d-\alpha-1} < \infty,$$

as desired.

Conversely, assume that the series (4.5) is divergent. We will show that the set $\mathcal{T}$ is massive. Applying Proposition 4.3, the assumption (4.2) and the equation (4.6) we have

$$\begin{aligned}\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^-) &= \widetilde{\mathrm{Cap}}_\alpha(t(2^n)\cdot \mathcal{F}_n^-/t(2^n))\\ &= t(2^n)^{d-\alpha}\cdot \widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^-/t(2^n))\\ &\geq c_1' t(2^n)^{d-\alpha}\cdot t(2^n)^{-1}\cdot \left(\frac{3}{4}\cdot 2^{n+1} - \frac{4}{3}\cdot 2^n\right)\\ &\geq c_2' t(2^n)^{d-\alpha-1}\cdot 2^n\end{aligned}$$

for some $c_1', c_2' > 0$. Since $F_n^- \subset \mathcal{T}_n$ (see Figure 2),

$$\mathrm{Cap}_\alpha(\mathcal{T}_n) \geq \mathrm{Cap}_\alpha(F_n^-).$$

Similarly to (4.7) and (4.8) we get

$$c_3' \,\widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^-) \leq \mathrm{Cap}_\alpha(F_n^-) \leq c_4' \,\widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^-)$$

and

$$c_5' \,\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^-) \leq \widetilde{\mathrm{Cap}}_\alpha(\widetilde{F}_n^-) \leq c_6' \,\widetilde{\mathrm{Cap}}_\alpha(\mathcal{F}_n^-)$$

for some constants $c_3', c_4', c_5', c_6' > 0$. Thus, at last,

$$\sum_{n>0} \frac{\mathrm{Cap}_\alpha(\mathcal{T}_n)}{2^{n(d-\alpha)}} \geq c_7' \sum_{n>0} \left(\frac{t(2^n)}{2^n}\right)^{d-\alpha-1} = \infty,$$

as desired. ∎

5. Two comparisons. Let ψ be a special Bernstein function (see Remark 4.2). Let B_ψ be a Lévy process in $\mathbb{R}^d$ obtained by subordination of the Brownian motion B. Let S_ψ be the random walk obtained by subordination of the simple random walk S. Let $\widetilde{G_\psi}$ (resp. G_ψ) be the Green function of B_ψ (resp. S_ψ).

In what follows we assume that $\psi \in \mathcal{BF}$ satisfies the conditions of Theorem 2.4.

PROPOSITION 5.1. *The function $\widetilde{G_\psi}$ has the following asymptotics:*

$$\widetilde{G_\psi}(x) \sim \frac{A_{d,\alpha}}{\|x\|^d \psi(1/\|x\|^2)} \qquad \textit{as } x \to \infty,$$

where

$$A_{d,\alpha} = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)}.$$

In particular,

$$\widetilde{G_\psi}(x) \sim (2/d)^{\alpha/2} G_\psi(x) \qquad \textit{as } x \to \infty.$$

Proof. As ψ is a special Bernstein function, the potential measure U associated with the corresponding (continuous time) subordinator has a monotone density $u(t)$ (see e.g. [6, Chapter V, Theorem 5.1]). Since $\mathcal{L}(U)(\lambda) =$

$1/\psi(\lambda)$, the Karamata theorem implies that the density function $u(t)$ satisfies

$$u(t) \sim \frac{1}{\Gamma(\alpha/2)} t^{\alpha/2-1} l(t) \quad \text{at } \infty.$$

Recall that, by definition,

$$\widetilde{G_\psi}(x) = \int_0^\infty (4\pi t)^{-d/2} \exp\left\{-\frac{\|x\|^2}{4t}\right\} u(t)\,dt,$$

whence, as $\|x\| \to \infty$,

$$\begin{aligned}
\widetilde{G_\psi}(x) &= 4^{-1}\pi^{-d/2}\|x\|^{2-d} \int_0^\infty s^{d/2-2} e^{-s} u\left(\frac{\|x\|^2}{4s}\right) ds \\
&\sim 4^{-1}\pi^{-d/2}\|x\|^{2-d} \int_0^\infty s^{d/2-2} e^{-s} u(\|x\|^2) \left(\frac{1}{4s}\right)^{\alpha/2-1} ds \\
&= 2^{-\alpha}\pi^{-d/2}\|x\|^{2-d} u(\|x\|^2) \int_0^\infty s^{d/2-\alpha/2-1} e^{-s}\,ds \\
&\sim \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} \|x\|^{\alpha-d} l(\|x\|^2).
\end{aligned}$$

Combining this result with that of Theorem 2.4 we obtain the claimed comparison of the Green functions $\widetilde{G_\psi}$ and G_ψ. ■

Let $\widetilde{\mathrm{Cap}_\psi}(A)$ be the capacity of a set $A \subset \mathbb{R}^d$ associated with the process B_ψ. Recall that by definition (see e.g. [5])

$$\widetilde{\mathrm{Cap}_\psi}(A) = \sup\{\mu(A) : \mu \in \mathcal{K}_A\},$$

where $\mathcal{K}_A$ is the class of measures supported by A and such that

$$\widetilde{G_\psi}\mu(\xi) = \int_A \widetilde{G_\psi}(\xi - \eta)\,\mu(d\eta) \le 1 \quad \text{for all } \xi \in \mathbb{R}^d.$$

Let $\mathrm{Cap}_\psi(B)$ be the capacity of a set $B \subset \mathbb{Z}^d$ associated with the process S_ψ. Similarly

$$\mathrm{Cap}_\psi(B) = \sup\Big\{\sum_{y \in B} \phi(y) : \phi \in \Xi_B\Big\},$$

where

$$\Xi_B = \{\phi \ge 0 : \operatorname{supp}\phi \subset B \text{ and } G_\psi\phi \le 1\}.$$

Theorem 5.2. *Let B be a bounded subset of $\mathbb{Z}^d$. Let $\widetilde{B}$ be defined at* (4.4). *There exist constants $c_1, c_2 > 0$, which depend only on d and ψ, and such* *that*

$$c_1\,\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}) \le \mathrm{Cap}_\psi(B) \le c_2\,\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}).$$

Proof. Take $a, b \in B$. Let $Q(a)$ be the cube $[0,1]^d$ centred at $a \in B$. Let $d\eta$ be the Lebesgue measure in $\mathbb{R}^d$. By Proposition 5.1 and radial monotonicity of $\widetilde{G_\psi}$, we can find a constant $c_2 > 0$ which does not depend on a and b, and such that for $\xi \in Q(a)$ and $\eta \in Q(b)$,

$$\int_{Q(b)} \widetilde{G_\psi}(\xi - \eta)\, d\eta \le c_2 G_\psi(a-b). \tag{5.1}$$

Let E be the equilibrium distribution of B associated with the random walk S_ψ. We define a new measure by setting

$$d\nu(\eta) = \sum_{b\in B} E(b)\mathbf{1}_{Q(b)}(\eta)\, d\eta.$$

Using (5.1) we compute the potential $\widetilde{G_\psi}\nu$:

$$\begin{aligned}\int_{\widetilde{B}} \widetilde{G_\psi}(\xi-\eta)\, d\nu(\eta) &= \sum_{b\in B}\int_{Q(b)} \widetilde{G_\psi}(\xi-\eta)E(b)\, d\eta \\ &\le c_2 \sum_{b\in B} G_\psi(a-b)E(b) = c_2 G_\psi E(a) \le c_2.\end{aligned}$$

Thus, the measure $c_2^{-1}\nu$ belongs to the class $\mathcal{K}_{\widetilde{B}}$, therefore

$$\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}) \ge \frac{1}{c_2}\nu(\widetilde{B}). \tag{5.2}$$

On the other hand,

$$\nu(\widetilde{B}) = \int_{\widetilde{B}} d\nu(\eta) = \sum_{b\in B}\int_{Q(b)} E(b)\, d\eta = \sum_{b\in B} E(b) = \mathrm{Cap}_\psi(B). \tag{5.3}$$

Combining (5.2) and (5.3) we obtain

$$\mathrm{Cap}_\psi(B) = \nu(\widetilde{B}) \le c_2\,\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}).$$

For the converse we use again Proposition 5.1 and radial monotonicity of $\widetilde{G_\psi}$. Let $a, b \in B$. Choose $c_1 > 0$, which does not depend on a and b, such that for $\xi \in Q(a)$ and $\eta \in Q(b)$,

$$c_1 G_\psi(a-b) \le \widetilde{G_\psi}(\xi-\eta). \tag{5.4}$$

Let $\widetilde{E}$ be the equilibrium measure of $\widetilde{B}$, i.e. $\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}) = \widetilde{E}(\widetilde{B})$. Define a distribution ϱ supported by B as

$$\varrho(b) = \widetilde{E}(Q(b)), \qquad b \in B.$$

Let

$$p = c_1 G_\psi \varrho.$$

Using (5.4) we get

$$p(a) \le \sum_{b\in B} \widetilde{G_\psi}(\xi-\eta)\varrho(b) \le \int_{\widetilde{B}} \widetilde{G_\psi}(\xi-\eta)\,d\widetilde{E}(\eta) \le 1.$$

It follows that $c_1\varrho \in \Xi_B$, whence

$$\mathrm{Cap}_\psi(B) \ge c_1\varrho(B). \tag{5.5}$$

Computing $\varrho(B)$ we obtain

$$\varrho(B) = \sum_{b\in B}\varrho(b) = \sum_{b\in B}\int_{Q(b)} d\widetilde{E}(\eta) = \int_{\widetilde{B}} d\widetilde{E}(\eta) = \widetilde{E}(\widetilde{B}). \tag{5.6}$$

From (5.5) and (5.6) we deduce that

$$\mathrm{Cap}_\psi(B) \ge c_1\varrho(B) = c_1\widetilde{E}(\widetilde{B}) = c_1\,\widetilde{\mathrm{Cap}_\psi}(\widetilde{B}). \ \blacksquare$$

COROLLARY 5.3. *Let $B(0,r)\subset\mathbb{Z}^d$ be a ball of radius $r>0$ centred at* 0. *Then*

$$cr^d\psi(1/r^2) \le \mathrm{Cap}_\psi(B(0,r)) \le Cr^d\psi(1/r^2)$$

for some constants $c, C > 0$ and all $r > 0$. In particular,

$$cr^{d-\alpha} \le \mathrm{Cap}_\alpha(B(0,r)) \le Cr^{d-\alpha}. \tag{5.7}$$

Proof. Assume $d\ge 3$. Let ϕ be the Lévy exponent of B_ψ, that is,

$$\mathbb{E}e^{i\xi B_\psi(t)} = e^{-t\phi(\xi)}, \quad \xi\in\mathbb{R}^d.$$

Since B_ψ is a subordinated Brownian motion, we have

$$\phi(\xi) = \psi(\|\xi\|^2), \quad \xi\in\mathbb{R}^d.$$

The function $\phi(s)$ is increasing, whence [10, Proposition 3] applies in the form

$$\widetilde{\mathrm{Cap}_\psi}(B(0,r)) \asymp \psi(r^{-2})r^d.$$

Finally, Theorem 5.2 yields the desired result.

When $d\le 2$ we proceed as follows. We use [6, Proposition 5.55],

$$\widetilde{\mathrm{Cap}_\psi}(B(0,r)) \asymp \frac{r^d}{\int_{B(0,r)}\widetilde{G_\psi}(x)\,dx},$$

and [6, Proposition 5.56],

$$\int_{B(0,r)} \widetilde{G_\psi}(x)\,dx \asymp \mathbb{E}^0\tau_{B(0,r)},$$

where $\tau_{B(0,r)}$ is the B_ψ-first exit time from the ball $B(0,r)$. We use [16, Theorem 1 and p. 954],

$$\mathbb{E}^0\tau_{B(0,r)} \asymp \frac{1}{h(r)},$$

where

$$h(r) = \int_{\mathbb{R}^d} \left(\frac{\|x\|^2}{r^2} \wedge 1 \right) d\nu(x)$$

and ν is the Lévy measure associated with the Lévy exponent ϕ, see [16, Section 3]. By [10, Corollary 1],

$$h(r) \asymp \psi(r^{-2}). \blacksquare$$

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