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$(1,4)$-*GROUPS WITH HOMOCYCLIC REGULATOR QUOTIENT OF EXPONENT* p^3

BY

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Abstract. The class of almost completely decomposable groups with a critical typeset of type $(1,4)$ and a homocyclic regulator quotient of exponent p^3 is shown to be of bounded representation type. There are precisely four near-isomorphism classes of indecomposables, all of rank 6.

1. Introduction. Kaplansky once observed that, in essence, anything can happen in torsion-free abelian groups even if the groups have finite rank. Thus to obtain results one has to consider subclasses and, in addition, a weakening of the isomorphism concept proved to be essential. A suitable, nontrivial, yet amenable class is the class of almost completely decomposable groups first introduced by Lady [14]. Every torsion-free abelian group of finite rank is the direct sum of indecomposable groups, but even in the case of almost completely decomposable groups such decompositions are notoriously "pathological". This problem is avoided by restricting the "regulator index" to be a power of a single prime and by employing a modest weakening of isomorphism, also due to Lady, called "near-isomorphism" [15]. In this way one obtains a Remak–Krull–Schmidt category and achieves a classification up to near-isomorphism as soon as the indecomposable groups in the class are found. As was shown in [6] most of these classes contain indecomposable groups of arbitrarily large rank, in which case it is hopeless to try to describe all near-isomorphism classes of indecomposable groups. This leaves some special subclasses that may have a finite number of near-isomorphism classes of indecomposable groups. The class considered in this paper is shown to be such a class and the indecomposables are determined. Earlier work on this topic was done in [2]–[4].

Any torsion-free abelian group G is an additive subgroup of a $\mathbb{Q}$-vector space V. The $\mathbb{Q}$-subspace of V generated by G is denoted $\mathbb{Q}G$, and $\dim(\mathbb{Q}G)$

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is the *rank* of G. A torsion-free abelian group R of finite rank is *completely decomposable* if R is the direct sum of rank-1 groups. Given a completely decomposable group R, we get a decomposition $R = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ where R_ρ is obtained by combining the isomorphic rank-1 summands of R into a summand R_ρ $(\neq 0)$ where ρ is the isomorphism class or *type* of the rank-1 summands of R_ρ. The set $\mathrm{T}_{\mathrm{cr}}(R)$ is the *critical typeset* of R.

A group G is *almost completely decomposable* if it contains a completely decomposable subgroup of finite index. An almost completely decomposable group G contains a well-understood fully invariant completely decomposable subgroup of finite index, the *regulator* $\mathrm{R}(G)$ [12]. The critical typeset of G is the critical typeset of R, $\mathrm{T}_{\mathrm{cr}}(G) = \mathrm{T}_{\mathrm{cr}}(R)$.

Given a finite poset S of p-locally free types, an almost completely decomposable group G is an *(S, p^k)-group* if $S = \mathrm{T}_{\mathrm{cr}}(G)$ and the exponent of the *regulator quotient* $G/\mathrm{R}(G)$ is p^k, i.e., $\exp(G/\mathrm{R}(G)) = p^k$. Two (S, p^k)-groups G and H are *nearly isomorphic* if there is an integer n relatively prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n$ and $gf = n$. A group G is indecomposable if and only if it is nearly isomorphic to an indecomposable group [1]. Moreover, an almost completely decomposable G with $G/\mathrm{R}(G)$ p-primary is, up to near-isomorphism, uniquely a direct sum of indecomposable groups [13], [16, Corollary 10.4.6]. Consequently, a classification of all indecomposable (S, p^k)-groups up to near isomorphism results in a classification of all (S, p^k)-groups up to near isomorphism. Hence, for almost completely decomposable groups G with $G/\mathrm{R}(G)$ p-primary, the main question is to determine the near-isomorphism classes of indecomposable (S, p^k)-groups.

There is an interesting connection between almost completely decomposable groups and representations of finite partially ordered sets [10], [11], [19].

Let G be an almost completely decomposable group with regulator R, critical typeset S, and regulator quotient G/R that is a finite abelian group of exponent p^k. By choosing a basis of G/R and expressing its elements in terms of a basis of R (suitable, called a *p-basis*) one encodes the group in an integral matrix, a "coordinate matrix". In this matrix the entries are determined only modulo p^k, and therefore the matrix may be considered to have coefficients in $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$. Suppose that the coordinate matrix M has n columns and let $S^* = S \cup \{*\}$ where $*$ is incomparable with any element of S. Then we get a representation U_G of S^* in the $\mathbb{Z}_{p^k}$-module $U_0 = (\mathbb{Z}_{p^k})^n$ where each $s \in S$ is assigned a certain summand of U_0 and $*$ is assigned the row space of M. Two representations U_G and U_H are isomorphic (as representations) if and only if G and H are nearly isomorphic, and U_G is indecomposable if and only if G is indecomposable. This is where the terms "bounded representation type" and “unbounded representation type” originate. Details are in [8] and [9]. Moreover, [9] also contains a complete survey of the known and open problems in the subject. The present paper settles one of these open problems.

We denote by $(1,n)$ the poset $\{\tau_0, \tau_1 < \cdots < \tau_n\}$ where τ_0 is incomparable to any one of the other elements. In this paper we study homocyclic $((1,4),p^3)$-groups, where $G \in ((1,4),p^3)$ is *homocyclic* if $G/\mathrm{R}(G)$ is a direct sum of cyclic groups all of the same order, $p^k = \exp(G/\mathrm{R}(G))$. We present a complete collection of near-isomorphism types of indecomposable homocyclic groups in $((1,4),p^3)$. There are precisely four near-isomorphism classes, and all have rank 6. The proof includes finding a normal form for coordinate matrices of $((1,4),p^3)$-groups—see Section 3.

Still unresolved are the homocyclic cases $((1,5),p^3)$ and $((1,3),p^5)$, and the nonhomocyclic cases $((1,2),p^5)$ and $((1,4),p^3)$ [9], [17], [18], [20].

2. Matrices. We deal with integer matrices. A *line* of a matrix is a row or a column. *Transformations* of matrices are successive applications of elementary transformations. Matrices are simplified by making entries equal to 0. While annihilating an entry, other entries that were originally zero may become nonzero; such entries are called *fill-ins* and must be removed, i.e., the original 0 must be restored. There is a fixed exponent p^k and entries may be changed modulo p^k, in particular $p^h = 0$ if $h \geq k$. A *unit* in our context is an integer that is relatively prime to p. An integer matrix $A = [a_{i,j}]$ is called *p-reduced* (modulo p^k) if

(1) there is at most one 1 in a line and all other entries are in $p\,\mathbb{Z}$,
(2) if an entry 1 of A is at the position (i_s, j_s), then $a_{i_s,j} = 0$ for all $j > j_s$ and $a_{i,j_s} = 0$ for all $i < i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\,\mathbb{Z}$ for all $j < j_s$ and all $i > i_s$.

Thus in a p-reduced matrix, the entries left of and below an entry 1 are in $p\,\mathbb{Z}$.

Lemma 2.1 (cf. [7, Lemma 1]). *Let A be an integer matrix.*

(i) *The matrix A can be transformed into a p-reduced matrix by elementary row transformations upward and elementary column transformations to the right and multiplications of lines by units.*
(ii) *If in addition row transformations down are allowed, then A can be transformed into a p-reduced matrix where all entries are* 0 *below an entry* 1.

3. (S,p^k)**-groups and coordinate matrices.** The following terminology is used in this paper. Details, equivalent formulations, and confirmation of assertions can be found in [1] or [16].

Let G be an almost completely decomposable group. The isomorphism types of the regulator $\mathrm{R}(G)$ and the regulator quotient $G/\mathrm{R}(G)$ are near-isomorphism invariants of G. In particular, the rank r of the regulator quotient is an invariant of G. Given a prime p, G is *p-reduced* if the localization $G_{(p)}$ of G at p is a free $\mathbb{Z}_{(p)}$-module, or equivalently, if each type $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$ is *p-locally free*, i.e., $pX \neq X$ for any rank-1 subgroup X of G with $[X] = \tau$, where $[X]$ denotes the isomorphism class of X.

Given a finite poset S of p-locally free types, an almost completely decomposable group G is an *(S,p^k)-group* if $S = \mathrm{T}_{\mathrm{cr}}(G)$ and $\exp(G/\mathrm{R}(G)) = p^k$. An (S,p^k)-group is *homocyclic* if $G/\mathrm{R}(G)$ is a free $\mathbb{Z}_{p^k}$-module.

Let G be an (S,p^k)-group of rank m with regulator $R = \mathrm{R}(G)$. The group G is *clipped* if it has no rank-1 summands. A coordinate matrix of G is obtained by means of bases of R and G/R. Write $R = S_1x_1 \oplus \cdots \oplus S_mx_m$ with $x_i \in R$, $S_i = \{s \in \mathbb{Q} : sx_i \in R\}$, and $p^{-1} \notin S_i$. In this case, $\{x_1, \ldots, x_m\}$ is called a *p-basis* of R.

A matrix $\alpha = [\alpha_{i,j}]$ is a *coordinate matrix* of G modulo R if α is integral, there is a basis $(\gamma_1, \ldots, \gamma_r)$ of G/R, there are representatives $g_i \in G$ of γ_i, and there is a p-basis $\{x_1, \ldots, x_m\}$ of R such that

$$g_i = p^{-k_i}\Big(\sum_{j=1}^{m} \alpha_{i,j}x_j\Big) \quad \text{where} \quad \langle\gamma_i\rangle \cong \mathbb{Z}_{p^{k_i}},\ 1 \le k_i \le k.$$

A coordinate matrix M of G is of size $r \times m$ and coordinate matrices that are congruent modulo p^k describe equal groups.

Since $(\gamma_1, \ldots, \gamma_r)$ is a basis of G/R, a coordinate matrix of size $r \times m$ has (p-)rank r.

Henceforth, let G be a homocyclic $((1,n),p^k)$-group with regulator R and critical typeset $\mathrm{T}_{\mathrm{cr}}(G) = \{\tau_0, \tau_1 < \cdots < \tau_n\}$, a poset of p-locally free types, and $r = \operatorname{rank} G/R$. Each column of a coordinate matrix corresponds to a type. So we speak of a *τ-column* of α. The number $r_\tau(G)$ of τ-columns of α is called the *τ-homogeneous rank* of G. This is a near-isomorphism invariant of G.

We call transformations of rows and of columns of a coordinate matrix of G *allowed* if the transformed coordinate matrix is the coordinate matrix of a near-isomorphic group. Since the regulator quotient is homocyclic, arbitrary row transformations are allowed, and if the columns are ordered as their types, all column transformations to the right are allowed. Hence coordinate matrices can always be transformed into p-reduced form by Lemma 2.1(i).

If the coordinate matrix M is formed with respect to the regulator R, then the submatrices of M formed by all τ_0-columns and the rest matrix both have rank equal to the rank r of the regulator quotient. Conversely, if the coordinate matrix M is formed with respect to a completely decomposable subgroup R of finite index and M satisfies the stated rank conditions, then R is the regulator. These rank conditions are called the *Regulator Criterion.*

4. Standard coordinate matrices. We establish a standard form for coordinate matrices of homocyclic $((1,n),p^k)$-groups. If $A = [A_{i,j}]$ is a block matrix, then we denote by $A_{*,j}$ and by $A_{i,*}$ the jth block column and the ith block row of A, respectively.

One of our main techniques is forming the (*partial*) *Smith Normal Form* as follows. Let $p^h X$ be an integer matrix. If X has entries that are units and if for $p^h X$ arbitrary row and column transformations are allowed, then $p^h X$ can be transformed to $\left[\begin{smallmatrix} p^h I & 0 \\ 0 & p^{h+1}X' \end{smallmatrix}\right]$. In our case $p^3 = 0$, and the possible partial Smith Normal Forms are

$$\begin{bmatrix} I & 0 \\ 0 & pX \end{bmatrix}, \quad \begin{bmatrix} pI & 0 \\ 0 & p^2X \end{bmatrix}, \quad \begin{bmatrix} p^2I & 0 \\ 0 & 0 \end{bmatrix}.$$

A matrix $A = [A_{i,j}]$ with blocks $A_{i,j}$ is said to be *completely reduced* if $A_{i,j} \in \{0, I, pI, p^2I, \ldots\}$ for all i, j.

Another technique is to form iterated Smith Normal Forms as follows. Let $[p^hA_1 \mid p^hA_2 \mid p^hA_3 \mid \ldots]$ be a block matrix. We form the Smith Normal Form $\left[\begin{smallmatrix} p^h I & 0 \\ 0 & p^{h+1}A_1' \end{smallmatrix}\right]$ of p^hA_1, annihilate with p^hI in $[p^hA_2 \mid p^hA_3 \mid \ldots]$ and form the Smith Normal Form $\left[\begin{smallmatrix} p^h I & 0 \\ 0 & p^{h+1}A_2' \end{smallmatrix}\right]$ of the rest of p^hA_2 and annihilate in the rest of the block matrix, and so on. We call this *forming the iterated Smith Normal Form starting with* p^hA_1. Instead of having blocks in a row there may be blocks in a column.

Often we assume ahead of forming Smith Normal Forms that certain lines cannot be 0. This specializes the resulting Smith Normal Form; for instance if p^2X has no 0-row (and we assume $p^3 = 0$), then we get $[p^2I \mid 0]$ instead of $\left[\begin{smallmatrix} p^2 I & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

We improve the notation of [9, Proposition 2] and, for the convenience of the reader, we give an adapted proof.

Proposition 4.1. *Let k and n be natural numbers. A homocyclic $((1,n),p^k)$-group without summands of rank ≤ 3 has a coordinate matrix of the form*

$$(4.1)\qquad [I\|pA\|I]$$

$$= \begin{array}{c}
\left[\begin{array}{c||ccccc||ccccc}
 & pA_{2,1} & 0 & 0 & \cdots & 0 & I(\tau_2) & 0 & \cdots & \cdots & 0 \\
 & pA_{3,1} & pA_{3,2} & 0 & \cdots & 0 & 0 & I(\tau_3) & 0 & \cdots & 0 \\
I & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots \\
 & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots \\
 & pA_{n,1} & pA_{n,2} & \cdots & \cdots & pA_{n,n-1} & 0 & 0 & \cdots & \cdots & I(\tau_n)
\end{array}\right] \\
\begin{array}{ccccccccccc}
\tau_0 & \tau_1 & \tau_2 & & & \tau_{n-1} & \tau_2 & \tau_3 & & & \tau_n
\end{array}
\end{array}.$$

The lower triangular block matrix $[A_{i,j}]$ (here $[pA_{i,j}]$ is stripped to $[A_{i,j}]$) is p-reduced. For the block $pA_{n,1}$ there is a matrix C with $pA_{n,1} = p^2C$. The blocks $pA_{i,j}$ are of the form

$$pA_{i,j} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & pI & 0 \\ 0 & 0 & p^2A'_{i,j} \end{array}\right].$$

Note that block lines may be absent.

In particular, if G is a homocyclic $((1,4),p^3)$-group, then the coordinate matrix has the form

$$(4.2)\qquad \begin{array}{c}
\left[\begin{array}{ccc|ccc|cc||ccc}
0 & pI & 0 & & 0 & & & 0 & I(\tau_2) & & \\
0 & 0 & p^2A & & & & & & & & \\
\hline
pI & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & p^2B_1 & p^2B_2 & 0 & pI & 0 & & 0 & & I(\tau_3) & \\
0 & p^2B_3 & p^2B_4 & 0 & 0 & p^2D & & & & & \\
\hline
p^2C_1 & p^2C_2 & p^2C_3 & pI & 0 & 0 & 0 & 0 & & & \\
p^2C_4 & p^2C_5 & p^2C_6 & 0 & p^2E_1 & p^2E_2 & pI & 0 & & & I(\tau_4) \\
p^2C_7 & p^2C_8 & p^2C_9 & 0 & p^2E_3 & p^2E_4 & 0 & p^2F & & &
\end{array}\right] \\
\begin{array}{cccccc}
\underbrace{\qquad\qquad\qquad}_{\tau_1} & \underbrace{\qquad\qquad\qquad}_{\tau_2} & \underbrace{\qquad\qquad}_{\tau_3} & \tau_2 & \tau_3 & \tau_4
\end{array}
\end{array}.$$

Proof. Let G be given by a coordinate matrix M where the columns are ordered as their types. Let S be the submatrix formed by the τ_0-columns. Since the hypothesis “homocyclic” allows arbitrary row transformations, there is a Smith Normal Form for S, and because G is clipped and by the Regulator Criterion this Smith Normal Form is the identity matrix. Moreover, this identity matrix can be restored after row transformations by column transformations alone, i.e., without changing the rest. Hence $M = [I \mid M']$ and we disregard the leading identity matrix and call M' the coordinate matrix.

The regulator quotient is homocyclic. This allows arbitrary row transformations and Lemma 2.1(ii) applies. So this coordinate matrix M' can be transformed to a p-reduced matrix, and by the Regulator Criterion M' contains columns forming a permutation matrix of size r, where r is the number of rows of M'. We move those columns of the included permutation matrix to the right and rearrange this matrix by row permutations to I. As the coordinate matrix M' is p-reduced we obtain the complete coordinate matrix in the form $[I \mid M'] = [I \mid pA \mid I]$. The identity matrix to the right contains all the remaining units in M'.

We order the columns of pA and of I as their types. So by Lemma 2.1(i) the part A (here pA is stripped to A) can be transformed to a p-reduced matrix. The identity matrix I (to the right) has block structure due to the types. This and the ordering of the columns of pA define a block structure of pA.

Since a τ_n-column in pA is 0, as A is p-reduced, there cannot be a τ_n-column in pA if G is clipped. A τ_1-column in I (to the right) displays a summand of rank 2, hence there is no such column in I. As A is p-reduced, $A_{i,j} = 0$ if $j \geq i$. Thus we get the claimed block matrix for $[I \mid pA \mid I]$.

A $p \in pA_{n,1}$ allows to annihilate in its whole row and in its whole column displaying a summand of rank 3. So there is a matrix C such that $pA_{n,1} = p^2C$.

Since arbitrary column transformations are allowed in the first block column $A_{*,1}$ and since arbitrary row operations are allowed in each block row $A_{i,*}$, we may form the iterated Smith Normal Form of the first block column $A_{*,1}$, starting with $A_{n-1,1}$. So we already obtained the first block column of the coordinate matrix (4.1). Then we annihilate with all entries $p \in pA_{*,1}$ horizontally to the right in the rows of pA.

Arbitrary column transformations are allowed in the second block column $A_{*,2}$. If we leave unchanged the 0-rows that are forced by the p's in the first block column, then arbitrary row operations of the rest are allowed in each block row $A_{i,*}$. So excluding the rows that we leave unchanged we may form the iterated Smith Normal Form of the remaining block rows of the second block column $A_{*,2}$, starting with $A_{n,2}$. We thus obtain the second block column of the coordinate matrix (4.1). Again we annihilate with the entries p in the second block column horizontally to the right in the rows of pA. We continue to treat all block columns to the right successively in the same way, and we get the claimed coordinate matrix.

In particular, specializing to $n = 4$ and $k = 3$, we get the coordinate matrices of homocyclic $((1,4), p^3)$-groups in the form (4.2). ∎

A coordinate matrix as in Proposition 4.1 is called *standard*. The block format of a standard coordinate matrix of G and the number of entries p in each block $pA_{i,j}$ of pA are near-isomorphism invariants of G.

PROPOSITION 4.2 (cf. [7, Prop. 5]). *Let G be a homocyclic $((1,4), p^3)$-group with the standard coordinate matrix. Then the size of the I_i's, the format of the blocks $A_{i,j}$ and the numbers of entries p in a block $pA_{i,j}$ are near-isomorphism invariants of G for all i, j.*

5. Indecomposable groups in the class of homocyclic $((1,4),p^3)$-groups. $(1,4)$-groups are of rank ≥ 5 because the critical typeset consists of five types. But indecomposable $(1,4)$-groups have a regulator quotient of rank ≥ 2. So they are of rank ≥ 6. For indecomposables of rank 6 the part pA of the coordinate matrix is a 2×2 matrix and it is easy to list all near-isomorphism classes of indecomposable $((1,4),p^3)$-groups of rank 6 by their standard coordinate matrices.

List of indecomposable homocyclic $((1,4),p^3)$*-groups.*

$$\text{(i)}\quad \left[\begin{array}{cc||cc||cc} 1 & 0 & p & 0 & 1 & 0 \\ 0 & 1 & p^2 & p & 0 & 1 \\ \multicolumn{2}{c||}{\underbrace{\qquad}_{\tau_0}} & \underbrace{\ }_{\tau_1} & \underbrace{\ }_{\tau_3} & \underbrace{\ }_{\tau_2} & \underbrace{\ }_{\tau_4} \end{array}\right] \text{ of rank 6,}$$

$$\text{(ii)}\quad \left[\begin{array}{cc||cc||cc} 1 & 0 & p & 0 & 1 & 0 \\ 0 & 1 & p^2 & p^2 & 0 & 1 \\ \multicolumn{2}{c||}{\underbrace{\qquad}_{\tau_0}} & \underbrace{\ }_{\tau_1} & \underbrace{\ }_{\tau_2} & \underbrace{\ }_{\tau_3} & \underbrace{\ }_{\tau_4} \end{array}\right] \text{ of rank 6,}$$

$$\text{(iii)}\quad \left[\begin{array}{cc||cc||cc} 1 & 0 & p^2 & 0 & 1 & 0 \\ 0 & 1 & p^2 & p & 0 & 1 \\ \multicolumn{2}{c||}{\underbrace{\qquad}_{\tau_0}} & \underbrace{\ }_{\tau_1} & \underbrace{\ }_{\tau_2} & \underbrace{\ }_{\tau_3} & \underbrace{\ }_{\tau_4} \end{array}\right] \text{ of rank 6,}$$

$$\text{(iv)}\quad \left[\begin{array}{cc||cc||cc} 1 & 0 & p^2 & p & 1 & 0 \\ 0 & 1 & 0 & p^2 & 0 & 1 \\ \multicolumn{2}{c||}{\underbrace{\qquad}_{\tau_0}} & \underbrace{\ }_{\tau_1} & \underbrace{\ }_{\tau_2} & \underbrace{\ }_{\tau_3} & \underbrace{\ }_{\tau_4} \end{array}\right] \text{ of rank 6.}$$

PROPOSITION 5.1. *The homocyclic* $((1,4),p^3)$*-groups in the list above are indecomposable and pairwise not near-isomorphic.*

Proof. By Proposition 4.2 the four groups in the list above are pairwise not near-isomorphic.

For all groups G in the list, $pG+R$ and p^2G+R are again almost completely decomposable groups with regulator R. The decompositions of $pG+R$ and p^2G+R are refinements of the decompositions of G.

We show exemplarily that the group G of type (ii) is indecomposable. Let $(x_1,x_2,y_1,y_2,y_3,y_4)$ be the p-basis of the regulator R belonging to the given coordinate matrix. So x_1,x_2 are of type τ_0 and the y_i are of type τ_i. This p-basis is also a p-basis of R in $pG+R$ and p^2G+R.

From the given coordinate matrix we read off that $p^2G+R=\langle x_1,y_3\rangle_*\oplus\langle x_2,y_4\rangle_*\oplus\langle y_1\rangle_*\oplus\langle y_2\rangle_*$. Thus, if there are summands of G of rank 2, then they have critical typesets either (τ_0,τ_3) or (τ_0,τ_4). By straightforward calculations using the coordinate matrix it can be shown that the exponent of $\langle x,y\rangle_*/(R\cap\langle x,y\rangle_*)$ is $\leq p^2$ for all x,y of types τ_0,τ_4, respectively. For instance, from

$$r \cdot p^{-3}(x_1 + py_1 + y_3) + s \cdot p^{-3}(x_2 + p^2 y_1 + p^2 y_2 + y_4) \in \langle ax_1 + bx_2, y_4 \rangle_* + R$$

it can be deduced that $p^3 \,|\, r$, $p \,|\, s$ and $r/s = a/b$. Thus the exponent of $\langle ax_1 + bx_2, y_4 \rangle_* / (\langle ax_1 + bx_2, y_4 \rangle_* \cap R)$ is $\leq p^2$.

The same takes place for x, y of types τ_0, τ_3, respectively. This contradicts the fact that the regulator quotient is homocyclic of exponent p^3. So potential summands of G all are of rank 3, and as $pG + R = \langle x_1, y_1, y_3 \rangle_* \oplus \langle x_2, y_4 \rangle_* \oplus \langle y_2 \rangle_*$ we see that there are potential summands of rank 3 with critical typesets (τ_0, τ_1, τ_3) and (τ_0, τ_2, τ_4), respectively. Again we may calculate as above that the exponent of $\langle x, y, z \rangle_* / (R \cap \langle x, y, z \rangle_*)$ is $\leq p^2$ for all x, y, z of types τ_0, τ_2, τ_4, respectively. This contradicts the fact that the regulator quotient is homocyclic of exponent p^3. So G is indecomposable. ■

6. The class of homocyclic $((1, 4), p^3)$-groups is bounded

THEOREM 6.1. *There are precisely the four near-isomorphism types of homocyclic* $((1, 4), p^3)$*-groups of rank* 6 *in the list above that are indecomposable.*

Proof. Let G be an indecomposable $((1, 4), p^3)$-group with coordinate matrix in standard form (cf. (4.2); we omit the leading identity matrix). Recall that an indecomposable $(1, 4)$-group has rank ≥ 6. Thus summands of rank ≤ 5 are impossible.

First we show that quite a number of blocks in (4.2) can immediately be seen to be 0. An entry $p^2 \in p^2 C_9$ allows to annihilate in its complete row, and then in its complete column. This leads to a summand of rank 3. Thus $C_9 = 0$. In turn an entry $p^2 \in p^2 C_8$ allows to annihilate in its complete row, and then in its complete column, except for the pI above. This leads to a summand of rank 5. Thus $C_8 = 0$. Next, an entry $p^2 \in p^2 C_6$ allows to annihilate in its complete row, except for the pI to the right, and then it allows to annihilate in its complete column. This leads to a summand of rank 4. Thus $C_6 = 0$.

An entry $p^2 \in p^2 C_5$ allows to annihilate in its complete row, except for the pI to the right. Then we annihilate with this entry p^2 in its complete column, except for the pI above. So we get a summand of rank 6 of type (i) in the list. Assuming without loss of generality that our group is not near-isomorphic to a group of type (i) we must have $p^2 C_5 = 0$.

With the block pI in the first column we annihilate $p^2 C_1, p^2 C_4$, with the block pI in the second column we annihilate $p^2 B_1, p^2 C_2$, with the block pI in the fifth column we annihilate $p^2 E_1$. The fill-ins in the identity matrix to the right can be removed by the respective pI to the left. We replace in (4.2) all those blocks by 0 that we discussed to be 0, and get the coordinate matrix in the form

$$(6.1)\qquad \begin{array}{c}\left[\begin{array}{ccc|ccc|cc||ccc}
0 & pI & 0 & & & & & & \multicolumn{3}{l}{} \\
0 & 0 & p^2A & \multicolumn{3}{c|}{0} & \multicolumn{2}{c||}{0} & \multicolumn{3}{l}{I(\tau_2)} \\
\hline
pI & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & p^2B_2 & 0 & pI & 0 & \multicolumn{2}{c||}{0} & \multicolumn{3}{c}{I(\tau_3)} \\
0 & p^2B_3 & p^2B_4 & 0 & 0 & p^2D & & & & & \\
\hline
0 & 0 & p^2C_3 & pI & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & p^2E_2 & pI & 0 & \multicolumn{3}{r}{I(\tau_4)} \\
p^2C_7 & 0 & 0 & 0 & p^2E_3 & p^2E_4 & 0 & p^2F & & &
\end{array}\right] \\
\underbrace{\qquad\qquad\qquad}_{\tau_1}\ \underbrace{\qquad\qquad\qquad}_{\tau_2}\ \underbrace{\qquad\qquad}_{\tau_3}\ \underbrace{\quad}_{\tau_2}\ \underbrace{\quad}_{\tau_3}\ \underbrace{\quad}_{\tau_4}
\end{array}.$$

We show next that the p^2A-row of the matrix (6.1) is not present. There is no 0-row in p^2C_3 to avoid a summand of rank 3. So the Smith Normal Form of p^2C_3 is $[p^2I \mid 0]$. In the part of $\left[\begin{smallmatrix} p^2B_2 \\ p^2B_4 \end{smallmatrix}\right]$ above the 0-columns of p^2C_3 we form the iterated Smith Normal Form starting with the part in p^2B_4. Then we annihilate in sequence with the resulting p^2I's in p^2A, first with $p^2I \subset p^2B_2$, next with $p^2I \subset p^2B_4$ and finally with $p^2I \subset p^2C_3$. So nonzero entries of p^2A are above 0-columns and create summands of rank 3, while $p^2A = 0$ produces summands of rank 2. Hence the p^2A-row is not present.

Similarly, working with columns instead of rows, we show that the p^2F-column of (6.1) is not present. For p^2C_7 in the last row we produce the Smith Normal Form $\left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$. In the part of $[p^2E_3 \mid p^2E_4]$ to the right of the 0-rows of p^2C_7 we form the iterated Smith Normal Form starting with this part of p^2E_4. Then we annihilate in sequence with the resulting p^2I's in p^2F, first with $p^2I \subset p^2E_4$, next with $p^2I \subset p^2E_3$ and finally with $p^2I \subset p^2C_7$. So nonzero entries of p^2F are to the right of 0-rows and create summands of rank 3, while $p^2F = 0$ means the existence of summands of rank 2. Hence the p^2F-column is not present. Now we omit the p^2A-row and the p^2F-column of (6.1) and get the coordinate matrix

$$(6.2)\qquad \begin{array}{c}\left[\begin{array}{ccc|ccc|c||ccc}
0 & pI & 0 & 0 & 0 & 0 & 0 & \multicolumn{3}{l}{I(\tau_2)} \\
\hline
pI & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & p^2B_2 & 0 & pI & 0 & 0 & \multicolumn{3}{c}{I(\tau_3)} \\
0 & p^2B_3 & p^2B_4 & 0 & 0 & p^2D & 0 & & & \\
\hline
0 & 0 & p^2C_3 & pI & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & p^2E_2 & pI & \multicolumn{3}{r}{I(\tau_4)} \\
p^2C_7 & 0 & 0 & 0 & p^2E_3 & p^2E_4 & 0 & & &
\end{array}\right] \\
\underbrace{\qquad\qquad\qquad}_{\tau_1}\ \underbrace{\qquad\qquad\qquad}_{\tau_2}\ \underbrace{\quad}_{\tau_3}\ \underbrace{\quad}_{\tau_2}\ \underbrace{\quad}_{\tau_3}\ \underbrace{\quad}_{\tau_4}
\end{array}.$$

Next we show that the first row and the second column, crossing in pI, are not present. We form the iterated Smith Normal Form of $[p^2B_3 \mid p^2B_4]$ starting with p^2B_4. Then we annihilate with $p^2I \subset p^2B_3$ in p^2D. This displays a summand of rank 5. So $p^2B_3 = 0$ and this causes a summand of rank 3 by the pI in the first row above p^2B_3. Hence the row and the column crossing in this pI in the first row are not present.

Further we show that the last column and last but one row, crossing in pI, are not present. We form the iterated Smith Normal Form of $\left[\begin{smallmatrix} p^2E_2 \\ p^2E_4 \end{smallmatrix}\right]$ starting with p^2E_4. Then we annihilate with $p^2I \subset p^2E_2$ in p^2D. This displays a summand of rank 4. So $p^2E_2 = 0$ and this causes a summand of rank 3 by the pI in the seventh row to the right of p^2E_2. Hence the row and the column crossing in this pI in the seventh column are not present. We omit all block rows and block columns that we found to be absent and get the coordinate matrix

$$\begin{array}{c} \left[\begin{array}{cc|ccc||cc} pI & 0 & 0 & 0 & 0 & & \\ 0 & p^2B_2 & 0 & pI & 0 & I(\tau_3) & \\ 0 & p^2B_4 & 0 & 0 & p^2D & & \\ \hline 0 & p^2C_3 & pI & 0 & 0 & & \\ p^2C_7 & 0 & 0 & p^2E_3 & p^2E_4 & & I(\tau_4) \end{array}\right] \\ \underbrace{\qquad\qquad\quad}_{\tau_1}\quad\underbrace{\qquad\qquad\qquad\quad}_{\tau_2}\quad\underbrace{\quad}_{\tau_3}\quad\underbrace{\quad}_{\tau_4} \end{array} . \tag{6.3}$$

Next we collect some properties of $\begin{bmatrix} p^2B_2 \\ p^2B_4 \\ p^2C_3 \end{bmatrix}$. As we already know, the Smith Normal Form of p^2C_3 is $[p^2I \mid 0]$. Above the 0-columns of p^2C_3 there are 0-columns of p^2B_4 to avoid a summand of rank 3. Since an entry $p^2 \in C_3$ allows one to annihilate in p^2B_2, there are no 0-columns of p^2B_4 above the nonzero columns of p^2C_3 to avoid a summand of rank 3. Moreover, an entry $p^2 \in p^2B_4$ allows one to annihilate in p^2B_2, and furthermore $\left[\begin{smallmatrix} p^2B_2 \\ p^2B_4 \end{smallmatrix}\right]$ has no 0-columns to avoid summands of rank ≤ 4. Altogether we obtain the completely reduced form of

$$\left[\begin{array}{c} p^2B_2 \\ \hline p^2B_4 \\ \hline p^2C_3 \end{array}\right] = \left[\begin{array}{cc} 0 & p^2I \\ 0 & 0 \\ \hline p^2I & 0 \\ 0 & 0 \\ \hline p^2I & 0 \end{array}\right].$$

An entry $p^2 \in p^2B_4$ allows one to annihilate in p^2D and displays a summand of rank 6 of type (iii) in the list. Assuming without loss of generality that our group is not near-isomorphic to a group of type (iii) we must have $p^2B_4 = 0$. But then in turn $p^2C_3 = 0$. So the p^2C_3-row is not present, and in turn also the pI-column with pI in the p^2C_3-row is not present. We implement all of the above in the matrix (6.3) and obtain the coordinate matrix in the form

$$(6.4)\qquad \begin{array}{c}\left[\begin{array}{cc|ccc||cc} pI & 0 & 0 & 0 & 0 & & \\ 0 & p^2I & pI & 0 & 0 & I(\tau_3) & \\ 0 & 0 & 0 & pI & 0 & & \\ 0 & 0 & 0 & 0 & p^2D & & \\ \hline p^2C_7 & 0 & p^2E_3^{(1)} & p^2E_3^{(2)} & p^2E_4 & & I(\tau_4) \end{array}\right] \\ \underbrace{\qquad\qquad}_{\tau_1}\quad \underbrace{\qquad\qquad\qquad\qquad}_{\tau_2}\quad \underbrace{\qquad}_{\tau_3}\quad \underbrace{\qquad}_{\tau_4}\end{array}.$$

An entry $p^2 \in p^2E_4$ allows one to annihilate in p^2D, thus a remaining entry $p^2 \in p^2D$ is in a column that is 0 in p^2E_4. So it displays a summand of rank 3, which is impossible. Hence $p^2D = 0$ and the p^2D-row is not present.

The submatrix $[p^2E_3^{(1)} \mid p^2E_3^{(2)} \mid p^2E_4]$ has no 0-line to avoid a summand of rank ≤ 5 and there is no 0-column in p^2C_7 to avoid a summand of rank 3. Hence the Smith Normal Form of p^2C_7 is $\left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$.

We can annihilate with an entry $p^2 \in [p^2C_7 \mid p^2E_4]$ in $[p^2E_3^{(1)} \mid p^2E_3^{(2)}]$, and we can annihilate with $p^2 \in p^2E_3^{(1)}$ in $p^2E_3^{(2)}$. Moreover, a row of p^2C_7 is nonzero if and only if the same row of p^2E_4 is nonzero. Otherwise there is a summand of rank 3. Hence we get the complete reduced form

$$[p^2C_7 \mid p^2E_3^{(1)} \mid p^2E_3^{(2)} \mid p^2E_4] = \left[\begin{array}{c|c|c|c} p^2I & 0 & 0 & p^2I \\ 0 & p^2I & 0 & 0 \\ 0 & 0 & p^2I & 0 \end{array}\right].$$

In the matrix (6.4) we replace the block p^2C_7 by its Smith Normal Form and p^2E by its complete reduced form, and get the coordinate matrix

$$\begin{array}{c}\left[\begin{array}{cc|ccc||cc} pI & 0 & 0 & 0 & 0 & & \\ 0 & p^2I & pI & 0 & 0 & I(\tau_3) & \\ 0 & 0 & 0 & pI & 0 & & \\ \hline p^2I & 0 & 0 & 0 & p^2I & & \\ 0 & 0 & p^2I & 0 & 0 & & I(\tau_4) \\ 0 & 0 & 0 & p^2I & 0 & & \end{array}\right] \\ \underbrace{\qquad\qquad}_{\tau_1}\quad \underbrace{\qquad\qquad\qquad}_{\tau_2}\quad \underbrace{\qquad}_{\tau_3}\quad \underbrace{\qquad}_{\tau_4}\end{array}.$$

Now the pI in the third block row displays summands of rank 5. Hence the row and the column crossing in this pI and the last row are not present. The rest displays the two remaining groups in the list, of type (ii) and (iv). ■

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REFERENCES

[1] D. M. Arnold, *Abelian Groups and Representations of Finite Partially Ordered Sets*, CMS Books Math. 2, Springer, New York, 2000.

[2] D. M. Arnold and M. Dugas, *Representation type of finite rank almost completely decomposable groups*, Forum Math. 10 (1998), 729–749.

[3] D. M. Arnold and M. Dugas, *Co-purely indecomposable modules over discrete valuation rings*, J. Pure Appl. Algebra 161 (2001), 1–12.

[4] D. M. Arnold, M. Dugas, and K. M. Rangaswamy, *Torsion-free modules of finite rank over a discrete valuation ring*, J. Algebra 272 (2004), 456–469.

[5] D. M. Arnold, A. Mader, O. Mutzbauer, and E. Solak, *Indecomposable* $(1,3)$*-groups and a matrix problem*, Czechoslovak Math. J. 63 (2013), 307–355.

[6] D. M. Arnold, A. Mader, O. Mutzbauer, and E. Solak, *Almost completely decomposable groups and unbounded representation type*, J. Algebra 349 (2012), 50–62.

[7] D. M. Arnold, A. Mader, O. Mutzbauer, and E. Solak, *The class of* $(1,3)$*-groups with homocyclic regulator quotient of exponent* p^4 *has bounded representation type*, J. Algebra 400 (2014), 43–55.

[8] D. M. Arnold, A. Mader, O. Mutzbauer, and E. Solak, *Representations of posets and indecomposable torsion-free abelian groups*, Comm. Algebra 42 (2014), 1287–1311.

[9] D. M. Arnold, A. Mader, O. Mutzbauer, and E. Solak, $\mathbb{Z}_{p^m}$*-representations of finite posets*, submitted (2014), 21 pp.

[10] D. M. Arnold and D. Simson, *Representations of finite partially ordered sets over commutative artinian uniserial rings*, J. Pure Appl. Algebra 205 (2006), 640–659.

[11] D. M. Arnold and D. Simson, *Representations of finite partially ordered sets over discrete valuation rings*, Comm. Algebra 35 (2007), 3128–3144.

[12] R. Burkhardt, *On a special class of almost completely decomposable torsion free Abelian groups I*, in: Abelian Groups and Modules (Udine, 1984), CISM Courses and Lectures 287, Springer, Vienna, 1984, 141–150.

[13] T. Faticoni and P. Schultz, *Direct decompositions of almost completely decomposable groups with primary regulating index*, in: Abelian Groups and Modules (Colorado Springs, CO, 1995), Dekker, New York, 1996, 233–241.

[14] E. L. Lady, *Almost completely decomposable torsion free abelian groups*, Proc. Amer. Math. Soc. 45 (1974), 41–47.

[15] E. L. Lady, *Nearly isomorphic torsion free abelian groups*, J. Algebra 35 (1975), 235–238.

[16] A. Mader, *Almost Completely Decomposable Groups*, Gordon and Breach, Amsterdam, 2000.

[17] O. Mutzbauer and E. Solak, $(1,2)$*-groups with* p^3*-regulator quotient*, J. Algebra 320 (2008), 3821–3831.

[18] O. Mutzbauer and E. Solak, $(1,2)$*-groups for a regulator quotient of exponent* p^4, in: Groups and Model Theory, Contemp. Math. 576, Amer. Math. Soc., Providence, RI, 2012, 269–285.

[19] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, Amsterdam, 1992.

[20] E. Solak, *Almost completely decomposable groups of type* (1, 2), doctoral dissertation, Univ. Würzburg, 2007.

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