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POINT DERIVATIONS ON THE L^1-ALGEBRA OF POLYNOMIAL HYPERGROUPS

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Abstract. We investigate whether the L^1-algebra of polynomial hypergroups has non-zero bounded point derivations. We show that the existence of such point derivations heavily depends on growth properties of the Haar weights. Many examples are studied in detail. We can thus demonstrate that the L^1-algebras of hypergroups have properties (connected with amenability) that are very different from those of groups.

1. Introduction. Polynomial hypergroups are a very interesting class of hypergroups with a great variety of examples which are quite different from groups. Hence the L^1-algebras of hypergroups have properties that are very different from those of L^1-algebras of groups, in particular as regards amenability and related conditions. Being weakly amenable, the L^1-algebra of a locally compact group has no non-zero bounded point derivation (see e.g. [7, p. 214]). We will show that for the L^1-algebra of hypergroups, where in fact we restrict ourselves to polynomial hypergroups, the situation is rather different. To have a good reference and for the sake of completeness we recall briefly the basic facts on polynomial hypergroups. For more details and proofs we refer to [16] and [17].

Let $(R_n)_{n\in\mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation

$$R_1(x)R_n(x) = a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x) \tag{1}$$

for $n \in \mathbb{N}$, and

$$R_0(x) = 1, \qquad R_1(x) = \frac{1}{a_0}(x - b_0),$$

where $a_n > 0$, $b_n \geq 0$ for all $n \in \mathbb{N}_0$, $c_n > 0$ for $n \in \mathbb{N}$. We assume that $a_n + b_n + c_n = 1$ for $n \in \mathbb{N}$ and $a_0 + b_0 = 1$. It follows from this assumption that $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. By a theorem of Favard there is a (unique) probability measure π on $\mathbb{R}$ with bounded support such that $(R_n)_{n\in\mathbb{N}_0}$ is orthogonal with respect to π, i.e. $\int_{\mathbb{R}} R_n(x)R_m(x)\,d\pi(x) = (1/h(n))\delta_{n,m}$.

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The recurrence relation (1) is a special case of the linearization formula

$$R_m(x)R_n(x) = \sum_{k=|n-m|}^{n+m} g(m,n;k)R_k(x) \tag{2}$$

for $m, n \in \mathbb{N}_0$. We shall suppose throughout this paper that the coefficients $g(m,n;k)$ are non-negative. There are many orthogonal polynomial systems which have this property (see [5, 16, 17]). We can define convolution multiplication on $\mathbb{N}_0$ by the formula

$$\delta_m * \delta_n = \sum_{k=|n-m|}^{n+m} g(m,n;k)\delta_k, \tag{3}$$

where δ_k is the point measure at $k \in \mathbb{N}_0$. With this convolution, the involution $\tilde{n} = n$ and the discrete topology the set of natural numbers $\mathbb{N}_0$ is a commutative hypergroup. Such a hypergroup is called the polynomial hypergroup induced by $(R_n)_{n\in\mathbb{N}_0}$ (see [16]).

The Haar measure on the polynomial hypergroup $\mathbb{N}_0$ is the counting measure with weights $h(n) = g(n,n;0)^{-1}$ of the points $n \in \mathbb{N}_0$. They satisfy the conditions $h(0) = 1$, $h(n+1) = (a_n/c_{n+1})h(n)$, $n \in \mathbb{N}_0$. The translation of a sequence $\beta = (\beta(n))_{n\in\mathbb{N}_0}$ reads

$$T_n\beta(m) = \sum_{k=|n-m|}^{n+m} g(m,n;k)\beta(k),$$

and the convolution of two sequences $f, g \in l^1(h)$ is given by

$$f * g(n) = \sum_{k=0}^{\infty} T_n f(k) g(k) h(k) \qquad (n \in \mathbb{N}_0).$$

(Here, $l^1(h) = \{f = (f(n))_{n\in\mathbb{N}_0} : \sum_{n=0}^{\infty} |f(n)|h(n) < \infty\}$). With this operation as multiplication, and $f^*(n) = \overline{f(n)}$ as involution, the Banach space $l^1(h)$ is a commutative Banach $*$-algebra with unit δ_0. The Hermitean dual space $\hat{\mathbb{N}}_0$ of $\mathbb{N}_0$ (i.e. the Hermitean structure space of $l^1(h)$) can be identified with the set

$$\{x \in \mathbb{R} : |R_n(x)| \le 1 \text{ for all } n \in \mathbb{N}_0\} \tag{4}$$

via the mapping $x \mapsto \alpha_x$, $\alpha_x(n) := R_n(x)$ (see [16]). Hence we consider $\hat{\mathbb{N}}_0$ as a compact subset of $\mathbb{R}$ which contains $1 \in \mathbb{R}$ (since $R_n(1) = 1$). (We note that in general there exist homomorphisms on $l^1(h)$ which are not Hermitean.) The support of the orthogonalization measure π is contained in $\hat{\mathbb{N}}_0$. The Fourier transform of an element $f \in l^1(h)$ is defined by

$$\hat{f}(x) = \sum_{k=0}^{\infty} f(k)R_k(x)h(k), \qquad x \in \hat{\mathbb{N}}_0.$$

$\hat{f}$ is a continuous bounded function on $\hat{\mathbb{N}}_0$ and satisfies $\widehat{f * g} = \hat{f}\hat{g}$.

2. Point derivations on $l^1(h)$. Fix any complex number $x \in \hat{\mathbb{N}}_0$. Denote by D_x any linear functional $D_x : l^1(h) \to \mathbb{C}$ such that for $f, g \in l^1(h)$,

$$D_x(f * g) = \hat{f}(x)D_x(g) + \hat{g}(x)D_x(f). \tag{5}$$

D_x is called the *point derivation* on $l^1(h)$ at $x \in \hat{\mathbb{N}}_0$.

Obviously $D_x = 0$ is a point derivation. It is the objective of this paper to characterize those $x \in \hat{\mathbb{N}}_0$ (given one of the plenty of polynomial hypergroups) for which there exist non-zero **bounded** point derivations D_x. We put $\varepsilon_n(m) = (1/h(n))\delta_{n,m}$. Then $\|\varepsilon_n\|_1 = 1$ and

$$\varepsilon_1 * \varepsilon_n = a_n\varepsilon_{n+1} + b_n\varepsilon_n + c_n\varepsilon_{n-1} \tag{6}$$

for all $n \in \mathbb{N}$.

PROPOSITION 1. *The following identity is valid for all $n \in \mathbb{N}_0$ and $x \in \hat{\mathbb{N}}_0$:*

$$D_x(\varepsilon_n) = a_0R_n'(x)D_x(\varepsilon_1). \tag{7}$$

(The prime denotes the usual derivative.) In particular, each bounded point derivation at x is given up to a constant factor by $D_x(\varepsilon_n) = R_n'(x)$.

Proof. Since $D_x(\varepsilon_0) = D_x(\varepsilon_0 * \varepsilon_0) = 2D_x(\varepsilon_0)$ it follows that $D_x(\varepsilon_0) = 0$. Further, $a_0R_1'(x) = 1$, and so (7) is true for $n = 1$. Now assume that (7) is valid for $k = n-1, n$. Then

$$\begin{aligned} D_x(\varepsilon_1 * \varepsilon_n) &= R_1(x)D_x(\varepsilon_n) + R_n(x)D_x(\varepsilon_1) \\ &= a_0R_n'(x)R_1(x)D_x(\varepsilon_1) + R_n(x)D_x(\varepsilon_1). \end{aligned}$$

On the other hand, from (6) we obtain

$$D_x(\varepsilon_1 * \varepsilon_n) = a_nD_x(\varepsilon_{n+1}) + b_nD_x(\varepsilon_n) + c_nD_x(\varepsilon_{n-1}),$$

and it follows that

$$\begin{aligned} D_x(\varepsilon_{n+1}) &= \frac{1}{a_n}\,(a_0R_n'(x)R_1(x) + R_n(x) - b_na_0R_n'(x) - c_na_0R_{n-1}'(x))D_x(\varepsilon_1) \\ &= a_0R_{n+1}'(x)D_x(\varepsilon_1), \end{aligned}$$

where the latter equality follows directly by differentiation of the three-term recurrence relation for $R_n(x)$. ∎

Another important identity for $D_x(\varepsilon_n)$ is obtained from the Christoffel–Darboux formula for $R_n(x)$. In fact, we have (see e.g. [25] or [19])

$$\frac{1}{a_nh(n)}\sum_{k=0}^{n} R_k^2(x)h(k) = a_0R_{n+1}'(x)R_n(x) - a_0R_n'(x)R_{n+1}(x) \tag{8}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Applying (7) we obtain

$$\begin{aligned} &(a_0R_{n+1}'(x)R_n(x) - a_0R_n'(x)R_{n+1}(x))D_x(\varepsilon_1) \\ &\qquad = R_n(x)D_x(\varepsilon_{n+1}) - R_{n+1}(x)D_x(\varepsilon_n). \end{aligned}$$

Since $\sum_{k=0}^{n} R_k^2(x)h(k)$ is strictly positive we have

PROPOSITION 2. *The following identity is valid for all $n \in \mathbb{N}$ and $x \in \hat{\mathbb{N}}_0$:*

$$D_x(\varepsilon_1) = a_n h(n) \frac{1}{\sum_{k=0}^{n} R_k^2(x)h(k)} (R_n(x)D_x(\varepsilon_{n+1}) - R_{n+1}(x)D_x(\varepsilon_n)). \tag{9}$$

If we know the sequence $(R_n'(x))_{n\in\mathbb{N}_0}$ we can decide whether there exists some bounded $D_x \neq 0$. If not we can apply Proposition 2 to give a sufficient condition for D_x being equal to zero.

THEOREM 1. *Assume $x \in \hat{\mathbb{N}}_0$.*

(i) *There exists a non-zero bounded point derivation D_x at x if and only if $\{R_n'(x) : n \in \mathbb{N}_0\}$ is bounded.*

(ii) *If*

$$\inf\left\{\frac{a_n h(n)}{\sum_{k=0}^{n} R_k^2(x)h(k)} (|R_n(x)| + |R_{n+1}(x)|) : n \in \mathbb{N}_0\right\} = 0, \tag{10}$$

then $D_x = 0$.

Proof. (i) If $\{R_n'(x) : n \in \mathbb{N}_0\}$ is unbounded and $D_x(\varepsilon_1) \neq 0$, then, by Proposition 1, D_x cannot be bounded. Hence $D_x(\varepsilon_1) = 0$, and then $D_x(\varepsilon_n) = 0$ for all $n \in \mathbb{N}_0$, which means $D_x = 0$. Conversely, if $\{R_n'(x) : n \in \mathbb{N}_0\}$ is bounded, the linear extension of equation (7) is a bounded map on the linear span of $\{\varepsilon_n : n \in \mathbb{N}_0\}$. If we select $D_x(\varepsilon_1) \neq 0$ we finally get a bounded non-zero derivation on $l^1(h)$.

(ii) If (10) holds true and D_x is bounded, then $D_x(\varepsilon_1) = 0$, and hence $D_x = 0$. ∎

EXAMPLE 1 (Ultraspherical polynomials $R_n^{(\alpha)}(x)$, $\alpha \geq -1/2$). For each $\alpha \geq -1/2$ we get a polynomial hypergroup on $\mathbb{N}_0$ induced by $R_n^{(\alpha)}(x)$ (see [5]). The three-term recurrence coefficients are $a_0 = 1$, $b_0 = 0$ and

$$a_n = \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad b_n = 0, \quad c_n = \frac{n}{2n+2\alpha+1} \quad \text{for } n \in \mathbb{N}.$$

The dual space $\hat{\mathbb{N}}_0$ is identified with $[-1, 1]$. From [25, (4.7.14)] we obtain

$$(R_n^{(\alpha)})'(x) = \frac{n(n+2\alpha+1)}{2+2\alpha} R_{n-1}^{(\alpha+1)}(x). \tag{11}$$

Furthermore, $R_n^{(\alpha)}(x) = O(n^{-\alpha-1/2})$ for $x \in \,]-1, 1[$ as $n \to \infty$ (see [25, (7.32.5)]), and hence $(R_n^{(\alpha)})'(x) = O(n^{-\alpha+1/2})$ for $x \in \,]-1, 1[$ as $n \to \infty$. Moreover, the orders are sharp as regards their orders in n.

For $x = \pm 1$ we deduce from (11) immediately that $(R_n^{(\alpha)})'(\pm 1) = O(n^2)$ as $n \to \infty$. Theorem 1(i) implies:

Corollary 1. *For the polynomial hypergroups induced by the ultraspherical polynomials* $R_n^{(\alpha)}(x)$, $\alpha \geq -1/2$, *we have*:

(1) *If* $-1/2 \leq \alpha < 1/2$, *then* $D_x = 0$ *for all* $x \in [-1, 1]$.
(2) *If* $\alpha \geq 1/2$, *then* $D_{\pm 1} = 0$, *and* $D_x \neq 0$ *exists for* $x \in \,]-1, 1[$.

Further examples are studied in Section 4. It is immediate to observe that D_1 is always equal to zero. In fact, putting $x = 1$ in the Christoffel–Darboux formula (8) we obtain

$$a_0 R'_{n+1}(1) - a_0 R'_n(1) = \frac{1}{a_n}\left(1 + \frac{h(n-1)}{h(n)} + \cdots + \frac{h(0)}{h(n)}\right) \geq 1,$$

which implies $a_0 R'_n(1) \geq n$. Hence by (7) we have $D_1 = 0$.

Proposition 3. *For each polynomial hypergroup on* $\mathbb{N}_0$ *we have* $D_1 = 0$. *If the polynomial hypergroup on* $\mathbb{N}_0$ *is symmetric* (*i.e.* $a_0 = 1$ *and* $b_n = 0$ *for all* $n \in \mathbb{N}_0$), *then also* $D_{-1} = 0$. *More generally*, *in the symmetric case*, *the existence of* $D_x \neq 0$ *is equivalent to the existence of* $D_{-x} \neq 0$.

Proof. The first statement has just been proven. If the polynomial hypergroup is symmetric we have $-1 \in \hat{\mathbb{N}}_0$ and $R_n(-x) = (-1)^n R_n(x)$, and so $R'_n(-x) = (-1)^{n+1} R'_n(x)$. In particular, $|R'_n(-1)| = R'_n(1) \geq n$, and the second statement follows by (7). ∎

Next we want to use criterion (10) to derive general conditions that guarantee that $D_x = 0$ for all $x \in \hat{\mathbb{N}}_0$. For $x \in \hat{\mathbb{N}}_0$ put

$$m(x) := \limsup_{n\to\infty} \frac{\sum_{k=0}^n R_k^2(x) h(k)}{\sum_{k=0}^n h(k)}. \tag{12}$$

Obviously, $0 \leq m(x) \leq 1$. We say the polynomial hypergroup has *property* (H) if the corresponding Haar weights $h(n)$ satisfy

$$\lim_{n\to\infty} \frac{h(n)}{\sum_{k=0}^n h(k)} = 0.$$

Next, let

$$\mathcal{T} := \{x \in \hat{\mathbb{N}}_0 : m(x) > 0\}.$$

The subset $\mathcal{T}$ of $\hat{\mathbb{N}}_0$ is non-empty, since $1 \in \mathcal{T}$. A direct consequence of (10) in Theorem 1 is:

Corollary 2. *Suppose the polynomial hypergroup has property* (H). *If* $x \in \mathcal{T}$, *then* $D_x = 0$.

Proof. Write

$$\frac{h(n)}{\sum_{k=0}^n R_k^2(x) h(k)} = \frac{h(n)}{\sum_{k=0}^n h(k)} \frac{\sum_{k=0}^n h(k)}{\sum_{k=0}^n R_k^2(x) h(k)}.$$

Since (H) holds true and $x \in \mathcal{T}$, we have

$$\lim_{n\to\infty} \frac{h(n)}{\sum_{k=0}^{n} R_k^2(x)h(k)} = 0.$$

Now apply Theorem 1(ii). ∎

In view of Corollary 2 we will investigate property (H) and the size of $\mathcal{T}$ in the next section.

We continue to derive general results on the existence of $D_x \neq 0$. In view of a result on spectral synthesis of polynomial hypergroups (see [26, Corollary 3.12]), the following characterization is valid. We recall briefly the definition of spectral sets. A closed subset E of $\widehat{\mathbb{N}}_0$ is called a *spectral set* if $I_E = \{f \in l^1(h) : \hat{f}(x) = 0 \text{ for all } x \in E\}$ is the only closed ideal in $l^1(h)$ with hull E. The *hull* $h(I)$ of a closed ideal I in $l^1(h)$ is given by $h(I) = \{x \in \widehat{\mathbb{N}}_0 : \hat{f}(x) = 0 \text{ for all } f \in I\}$.

Theorem 2. *Suppose that the polynomial hypergroup has polynomial growth, i.e. $h(n) = O(n^a)$ for some $a \geq 0$. Assume $x \in \hat{\mathbb{N}}_0$. Then there exists a non-zero bounded point derivation D_x at x if and only if $\{x\}$ is a non-spectral set.*

For general polynomial hypergroups (and even for general commutative hypergroups) we can adopt a method of [12]. For $x \in \hat{\mathbb{N}}_0$ we denote by $I(x) = \{f \in l^1(h) : \hat{f}(x) = 0\}$ the maximal ideal in $l^1(h)$ with hull $\{x\}$.

Proposition 4. *Let $x \in \hat{\mathbb{N}}_0$ and assume that $\{x\}$ is a spectral set or $I(x)$ has a (not necessarily bounded) approximate identity. Then $D_x = 0$.*

Proof. For $f, g \in I(x)$ we have $D_x(f * g) = 0$. Since $I(x) * I(x)$ is an ideal in $l^1(h)$ with hull $\{x\}$, $I(x) * I(x)$ is dense in $I(x)$, provided $\{x\}$ is a spectral set. Obviously, $I(x) * I(x)$ is also dense in $I(x)$ whenever $I(x)$ has an approximate identity. Hence $D_x(f) = 0$ for all $f \in I(x)$. Now select some $g \in l^1(h)$ such that $\hat{g}(x) = 1$ and $\|g\|_1 \leq 1$. Then $D_x(g^n) = nD_x(g)$, and hence $D_x(g) = 0$. Given any $f \in l^1(h)$ write $f = \hat{f}(x)g + (f - \hat{f}(x)g)$, which implies $D_x(f) = 0$. ∎

Another related problem, the existence of bounded approximate identities in the maximal ideals $I(x) = \{f \in l^1(h) : \hat{f}(x) = 0\}$, is investigated in [10]. The existence of bounded approximate identities is equivalent to the existence of a continuous linear functional $m_x \in l^\infty(\mathbb{N}_0)^*$ with $m_x(\alpha_x) = 1$ and $m_x(T_n f) = R_n(x)m_x(f)$ for all $f \in l^\infty(\mathbb{N}_0)$ (see [10, Theorem 3.4]). We now consider a weaker assumption where we do not require the boundedness of such linear functionals.

PROPOSITION 5. *Let $x \in \hat{\mathbb{N}}_0$ and assume that there exists a (not necessarily bounded) linear functional $m_x : l^\infty(\mathbb{N}_0) \to \mathbb{C}$ such that $m_x(\alpha_x) = 1$ and $m_x(T_n f) = R_n(x) m_x(f)$ for all $f \in l^\infty(\mathbb{N}_0)$ and $n \in \mathbb{N}_0$. Then $D_x = 0$.*

Proof. Let D_x be a bounded point derivation on $l^1(h)$ at $x \in \hat{\mathbb{N}}_0$. Consider the bounded sequence $\beta = (\beta(n))_{n\in\mathbb{N}_0}$, where $\beta(n) = D_x(\varepsilon_n)$. Then by Proposition 1 we have $\beta(n) = a_0\beta(1)R_n'(x)$, and hence

$$\begin{aligned} T_1\beta(n) &= a_n\beta(n+1) + b_n\beta(n) + c_n\beta(n-1) \\ &= a_0\beta(1)(a_n R_{n+1}'(x) + b_n R_n'(x) + c_n R_{n-1}'(x)) \\ &= a_0\beta(1)(R_1'(x)R_n(x) + R_1(x)R_n'(x)) = \beta(1)R_n(x) + \beta(n)R_1(x). \end{aligned}$$

Applying m_x we obtain

$$\begin{aligned} R_1(x)m_x(\beta) = m_x(T_1\beta) &= \beta(1)m_x((R_n(x))_{n\in\mathbb{N}_0}) + R_1(x)m_x(\beta) \\ &= \beta(1) + R_1(x)m_x(\beta). \end{aligned}$$

Therefore $\beta(1) = D_x(\varepsilon_1) = 0$, and so $D_x = 0$. ∎

Considering the examples of the ultraspherical polynomials $R_n^{(\alpha)}(x)$, we can see that for $-1/2 < \alpha < 1/2$ and $x \in \left]-1, 1\right[$ the sets $\{x\}$ are spectral sets, but the ideals $I(x)$ do not have bounded approximate identities. The latter fact is shown in [10, Example 4.6]. Thus we observe that these one-point sets $\{x\}$ are spectral sets, but not strong Wiener–Ditkin sets, whereas they are Wiener–Ditkin sets. In fact, $I(x)$ has an unbounded approximate identity in that case (see [8]).

Let $(P_n(x))_{n\in\mathbb{N}_0}$ be another orthogonal polynomial sequence that induces a polynomial hypergroup on $\mathbb{N}_0$. Then one can write

$$P_n(x) = \sum_{k=0}^{n} c(n,k) R_k(x). \tag{13}$$

The unique coefficients are called the *connection coefficients* (cf. [1]). If all connection coefficients $c(n,k)$ are non-negative, then setting $x = 1$ we see that $\sum_{k=0}^{n} c(n,k) = 1$. If $x \in \mathbb{R}$ determines a character α_x with respect to R_n, i.e. $|R_n(x)| \le 1$ for all $n \in \mathbb{N}_0$, then $|P_n(x)| \le 1$ for all $n \in \mathbb{N}_0$. Hence, if x is an element of the character space of the polynomial hypergroup with respect to R_n, then x is also an element of the character space of the hypergroup with respect to P_n. Moreover, $P_n'(x) = \sum_{k=0}^{n} c(n,k) R_k'(x)$. In particular, if $\{R_n'(x) : n \in \mathbb{N}_0\}$ is bounded, then so is $\{P_n'(x) : n \in \mathbb{N}_0\}$.

PROPOSITION 6. *Assume $(R_n)_{n\in\mathbb{N}_0}$ and $(P_n)_{n\in\mathbb{N}_0}$ each induce a polynomial hypergroup on $\mathbb{N}_0$, and suppose that the connection coefficients $c(n,k)$ in (13) are non-negative. If $x \in \hat{\mathbb{N}}_0$ and $D_x \ne 0$ is a non-zero bounded derivation with respect to R_n, then there also exists a non-zero bounded derivation D_x with respect to P_n.*

Proof. Follows immediately from Theorem 1(i) and the observations above. ∎

The non-existence of a non-zero bounded derivation D_x can be seen as a very weak and local condition of amenability. We now consider the global condition of weak amenability of $l^1(h)$. The Banach algebra $l^1(h)$ is weakly amenable if every bounded derivation $D : l^1(h) \to l^1(\mathbb{N}_0)^* \cong l^\infty(\mathbb{N}_0)$ is equal to zero (see [4]). The $l^1(h)$-bimodule action on $l^\infty(\mathbb{N}_0)$ is given by the convolution of $l^1(h)$ on $l^\infty(\mathbb{N}_0)$. The following proposition holds true for general Banach algebras (see [6, Theorem 2.8.63]).

PROPOSITION 7. *Suppose that $l^1(h)$ is weakly amenable. Then for every $x \in \hat{\mathbb{N}}_0$ each bounded point derivation D_x is equal to zero.*

A direct consequence of Proposition 7 is that the Banach algebra $l^1(h)$ induced by the ultraspherical polynomials $R_n^{(\alpha)}(x)$ is not weakly amenable whenever $\alpha \geq 1/2$. Weak amenability (and also amenability) of $l^1(h)$ is studied in [18]. Another recent contribution to amenability on hypergroups is [3].

3. Growth conditions. We begin by investigating property (H). Put

$$\sigma_0 = 0, \qquad \sigma_n = \frac{h(n)}{\sum_{k=0}^{n-1} h(k)} \qquad \text{for } n \in \mathbb{N}.$$

Part of the following lemma is already shown in [20].

LEMMA 1. *Suppose $(R_n(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup. Then*

(i) *$\sigma_n \to \varrho$ with $\varrho > 0$ if and only if $a_{n-1}/c_n \to 1 + \varrho$ as $n \to \infty$,*

(ii) *$\sigma_n \to 0$ and $\sigma_n/\sigma_{n-1} \to 1$ if and only if $a_{n-1}/c_n \to 1$ as $n \to \infty$.*

Proof. By induction we obtain

$$\sum_{k=0}^{n} h(k) = \prod_{k=0}^{n} (1 + \sigma_k). \tag{14}$$

In particular, it follows that $h(n) = (\prod_{k=0}^{n-1}(1 + \sigma_k))\sigma_n$, and hence

$$\frac{a_{n-1}}{c_n} = \frac{h(n)}{h(n-1)} = \frac{\sigma_n}{\sigma_{n-1}}\,(1 + \sigma_{n-1}). \tag{15}$$

Now $\sigma_n \to \varrho$ with $\varrho > 0$ implies $a_{n-1}/c_n \to 1+\varrho$. If $\sigma_n \to 0$ and $\sigma_n/\sigma_{n-1} \to 1$ we obtain $a_{n-1}/c_n \to 1$. To show the converse implications in (i) and (ii), write $q_n = c_n/a_{n-1}$, and let $\varepsilon > 0$. Assuming $a_{n-1}/c_n \to 1 + \varrho$ with $\varrho \geq 0$, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{1+\varrho} - \varepsilon \leq q_{m+n} \leq \frac{1}{1+\varrho} + \varepsilon \qquad \text{for all } n \in \mathbb{N}_0.$$

We consider first the case $\varrho = 0$ and suppose $0 < \varepsilon < 1$. Since

$$\begin{aligned}
0 < q_{m+n}\sigma_{m+n} &= \frac{q_{m+n}h(m+n)}{\sum_{k=0}^{m+n-1} h(k)} \\
&= \frac{h(m+n-1)}{\sum_{k=0}^{m+n-1} h(k)} \le \frac{h(m+n-1)}{\sum_{k=m}^{m+n-1} h(k)} \\
&= \frac{h(m+n-1)}{h(m+n-1)[1 + q_{m+n-1} + q_{m+n-1}q_{m+n-2} + \dots + q_{m+n-1}\cdots q_{m+1}]} \\
&\le \frac{1}{1 + (1-\varepsilon) + \dots + (1-\varepsilon)^{n-1}} = \frac{\varepsilon}{1-(1-\varepsilon)^n},
\end{aligned}$$

and $0 < \varepsilon < 1$ was arbitrary, it follows that $q_n\sigma_n \to 0$, and so $\sigma_n \to 0$.

Now suppose $\varrho > 0$. Since $a_{n-1}/c_n \to 1 + \varrho > 1$, and since $h(n) = h(n-1)a_{n-1}/c_n$, it follows that $h(n) \to \infty$ as $n \to \infty$. Moreover,

$$\begin{aligned}
\frac{1}{q_{m+n}\sigma_{m+n}} &= \frac{\sum_{k=0}^{m-1} h(k)}{h(m+n-1)} + \frac{\sum_{k=m}^{m+n-1} h(k)}{h(m+n-1)} \\
&\ge \frac{\sum_{k=0}^{m-1} h(k)}{h(m+n-1)} + 1 + \left(\frac{1}{1+\varrho} - \varepsilon\right) + \dots + \left(\frac{1}{1+\varrho} - \varepsilon\right)^{n-1}.
\end{aligned}$$

Hence we get

$$\liminf_{n\to\infty} \frac{1}{q_n\sigma_n} \ge \frac{1}{1 - \frac{1}{1+\varrho}} = \frac{1+\varrho}{\varrho},$$

and so

$$\liminf_{n\to\infty} \frac{1}{\sigma_n} = \liminf_{n\to\infty} \frac{1}{\sigma_n q_n} \lim_{n\to\infty} q_n \ge \frac{1}{\varrho},$$

which implies $\limsup_{n\to\infty} \sigma_n \le \varrho$. Similarly we obtain

$$\frac{1}{q_{m+n}\sigma_{m+n}} \le \frac{\sum_{k=0}^{m-1} h(k)}{h(m+n-1)} + 1 + \left(\frac{1}{1+\varrho} + \varepsilon\right) + \dots + \left(\frac{1}{1+\varrho} + \varepsilon\right)^{n-1},$$

which implies

$$\limsup_{n\to\infty} \frac{1}{q_n\sigma_n} \le \frac{1}{1 - \frac{1}{1+\varrho}} = \frac{1+\varrho}{\varrho},$$

and so $\liminf_{n\to\infty} \sigma_n \ge \varrho$. Thus we have shown that $\lim_{n\to\infty} \sigma_n = \varrho$. ■

Corollary 3. *Suppose* $(R_n(x))_{n\in\mathbb{N}_0}$ *induces a polynomial hypergroup. If* $a_{n-1}/c_n \to 1$ *as* $n \to \infty$, *then condition* (H) *is satisfied.*

Now we deal with $m(x)$ (see (12)). The following result of Nevai (see [24, Theorem 4.5.2]) is essential for further considerations. We reformulate it for our purposes.

Theorem 3 (Nevai). *Suppose that* $\operatorname{supp}\pi \subseteq [-1,1] = \hat{\mathbb{N}}_0$, *and that the Radon–Nikodym derivative* π' *satisfies Szegö's condition*

$$\int_{-1}^{1} \frac{\ln(\pi'(x))}{\sqrt{1-x^2}}\, dx > -\infty.$$

Given $x \in \,]-1,1[$ *suppose that* π *is absolutely continuous in a neighbourhood of* x, π' *is continuous at* x, *and* $\pi'(x) > 0$. *Then*

$$\lim_{n\to\infty} \frac{n+1}{\sum_{k=0}^{n} R_k^2(x)h(k)} = C\pi'(x)\sqrt{1-x^2} > 0. \tag{16}$$

COROLLARY 4. *Suppose that the assumptions of Theorem* 3 *on* π *and* $x \in \hat{\mathbb{N}}_0$ *are valid. Then*

(i) *if* $\liminf\limits_{n\to\infty} \dfrac{\sum_{k=0}^{n} h(k)}{n+1} < \infty,$ *then* $x \in \mathcal{T}$,

(ii) *if* $\lim\limits_{n\to\infty} \dfrac{\sum_{k=0}^{n} h(k)}{n+1} = \infty,$ *then* $x \notin \mathcal{T}$.

Proof. From Theorem 3 it follows that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} R_k^2(x)h(k)}{\sum_{k=0}^{n} h(k)} = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{n+1}{\sum_{k=0}^{n} h(k)} = 0. \ ■$$

COROLLARY 5. *Suppose that* $\hat{\mathbb{N}}_0 = [-1,1]$ *and assume that* $d\pi(x) = w(x)dx$, $w(x)$ *a positive continuous function on* $]-1,1[$. *If* $\{h(n) : n \in \mathbb{N}_0\}$ *is bounded then* $]-1,1[\, \subseteq \mathcal{T}$. *In particular*, $D_x = 0$ *for all* $x \in \hat{\mathbb{N}}_0$.

Proof. As $h(n)$ is bounded, Theorem 2(6) of [22] shows that the function $(w(x)\sqrt{1-x^2})^{-1}$ is essentially bounded. That means π' satisfies Szegö's condition. By Corollary 4 we have to show that $\liminf_{n\to\infty}(n+1)^{-1}\sum_{k=0}^{n} h(k) < \infty$, which is obviously true. The additional statement follows by Corollary 2. ■

EXAMPLE 2 (Bernstein–Szegö polynomials, see [25]). We consider the polynomials $Q_n^{(\nu,\kappa)}(x)$ that are orthogonal with respect to the measure

$$d\pi(x) = c_{\nu,\kappa}\, \frac{dx}{g(x)\sqrt{1-x^2}}$$

on $[-1,1]$, where $g(x) = |\nu e^{2it} + \kappa e^{it} + 1|^2$, $x = \cos t$, is a polynomial with $g(x) > 0$ for all $x \in [-1,1]$. By explicit representation by Chebyshev polynomials of the first kind (see [13]), it can be easily shown that the $Q_n^{(\nu,\kappa)}$ induce a polynomial hypergroup on $\mathbb{N}_0$ provided $\nu,\kappa \geq 0$ and $\kappa - 1 < \nu < 1$. The Haar weights $h(n)$ are bounded, and hence Corollary 5 can be applied.

To apply Corollaries 2 and 4 we have to check that the assumptions of Theorem 3 on the orthogonalization measure π are satisfied. We now concentrate on conditions depending directly on the recurrence coefficients a_n, b_n, c_n. For the next considerations it is more convenient to use the orthonormal polynomials $p_n(x) = \sqrt{h(n)}\, R_n(x)$, which satisfy the recurrence relation

$$xp_n(x) = \lambda_{n+1}p_{n+1}(x) + \beta_n p_n(x) + \lambda_n p_{n-1}(x), \qquad n \in \mathbb{N}_0. \tag{17}$$

with $p_0(x) = 1$ and $\lambda_n = a_0\sqrt{c_n a_{n-1}}$ for $n \geq 2$, $\lambda_1 = a_0\sqrt{c_1}$, $\lambda_0 = 0$ and $\beta_n = a_0 b_n + b_0$ for $n \geq 1$, $\beta_0 = b_0$. The polynomial sequence $(p_n(x))_{n\in\mathbb{N}_0}$ is an element of the *Nevai class* $M(0,1)$ if $\lim_{n\to\infty}\lambda_n = 1/2$ and $\lim_{n\to\infty}\beta_n = 0$. It is called of *bounded variation type* if

$$\sum_{n=1}^{\infty}(|\lambda_{n+1} - \lambda_n| + |\beta_{n+1} - \beta_n|) < \infty. \tag{18}$$

THEOREM 4. *Suppose $(p_n(x))_{n\in\mathbb{N}_0}$ is of Nevai class $M(0,1)$ and is of bounded variation type. Further, assume that $h(n) = O(n^a)$, $0 \leq a < 2$. Then $D_x = 0$ for every $x \in\,]-1,1[\, \subseteq \hat{\mathbb{N}}_0$.*

Proof. Since $(p_n(x))_{n\in\mathbb{N}_0}$ is a member of $M(0,1)$ we have $[-1,1] \subseteq \operatorname{supp}\pi \subseteq \hat{\mathbb{N}}_0$ (see [23, Theorem 7, p. 23]). In [10, Theorem 5.1] or [21] it is shown that the polynomials $p_n(x)$ are uniformly bounded on each closed subinterval of $]-1,1[$. By [21] we have

$$\lim_{n\to\infty}\left(p_n^2(x) - \frac{\lambda_{n+1}}{\lambda_n}\,p_{n+1}(x)p_{n-1}(x)\right) = f(x) > 0$$

for all $x \in\,]-1,1[$. Since $\lim_{n\to\infty}\lambda_{n+1}/\lambda_n = 1$ there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\max\{|p_{n-1}(x)|, |p_n(x)|, |p_{n+1}(x)|\} > f(x)/2.$$

Hence $\sum_{k=0}^{n} R_k^2(x)h(k) = \sum_{k=0}^{n} p_k^2(x)$ is growing exactly with order n. Furthermore, $a_n h(n)|R_n(x)| = a_n\sqrt{h(n)}|p_n(x)| = O(n^{a/2})$ and $a_n h(n)|R_{n+1}(x)| = c_{n+1}h(n+1)|R_{n+1}(x)| = O(n^{a/2})$ as $n \to \infty$. As $0 \leq a < 2$ we conclude from Theorem 1(ii) that $D_x = 0$ for $x \in\,]-1,1[$. ∎

If $\sum_{k=0}^{\infty}|R_k(x)|h(k)$ is convergent, then $\sum_{k=0}^{\infty}R_k^2(x)h(k)$ is convergent and $|R_n(x)|h(n)$ tends to zero. Theorem 1(ii) yields:

COROLLARY 6. *If $\sum_{k=0}^{\infty}|R_k(x)|h(k) < \infty$, then $D_x = 0$.*

In the next section we will show that $\sum_{k=0}^{\infty}R_k^2(x)h(k) < \infty$ is not sufficient for $D_x = 0$ (see Example 6 below).

4. Examples. We have already studied ultraspherical polynomials (see Corollary 1), and Bernstein–Szegö polynomials. In [26] there are examples of polynomial growth in order to determine when points are spectral sets. Appealing to Theorem 2 we can transfer the examples of [26] to our problem getting characterizations of $D_x = 0$ in the case of Jacobi polynomials, generalized Chebyshev polynomials, Geronimus polynomials, Grinspun polynomials and q-ultraspherical polynomials.

Example 3 (Little q-Legendre polynomials). We consider the orthogonal polynomials $R_n(x) = R_n(x;q)$, $0 < q < 1$, defined by $a_0 = 1/(q+1)$, $b_0 = q/(q+1)$ and for $n \in \mathbb{N}$,

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})},$$
$$b_n = \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})},$$
$$c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}.$$

These polynomials define a polynomial hypergroup on $\mathbb{N}_0$. Their Haar weights are

$$h(n) = \frac{1}{1-q}(q^{-n} - q^{n+1}),$$

and

$$\hat{\mathbb{N}}_0 = \{1\} \cup \{1 - q^k : k \in \mathbb{N}_0\}.$$

The hypergroup is of exponential growth. In [11] it is shown that it is of strong compact type. That is, the translation operator T_n−id is compact on $l^1(h)$ for each $n \in \mathbb{N}$. By Theorem 3 of [11] and Theorem 3.3 of [10] every maximal ideal $I(x), x \in \hat{\mathbb{N}}_0$, has a bounded approximate identity. Hence by Proposition 4 all D_x are zero.

Example 4 (Associated ultraspherical polynomials). In [17, §3] the associated ultraspherical polynomials $R_n(x) = R_n^{(\nu)}(x;\alpha)$, $\alpha > -1/2$, $\nu \geq 0$, are investigated. It is shown that each of these polynomial systems defines a polynomial hypergroup on $\mathbb{N}_0$. The recurrence coefficients are $a_n = 1 - c_n$, $b_n = 0$ for $n \in \mathbb{N}_0$, $c_0 = 0$ and

$$c_n = \frac{(\nu+n)(2\alpha+\nu)_{n+1} - (n+2\alpha+\nu)(\nu)_{n+1}}{(2n+2\alpha+2\nu+1)[(2\alpha+\nu)_{n+1} - (\nu)_{n+1}]} \quad \text{for } n \in \mathbb{N}.$$

For $\nu = 0$ we get the ultraspherical polynomials. The coefficients γ_n and β_n for the orthonormal versions are

$$\beta_n = 0 \quad \text{and} \quad \gamma_n = \left(\frac{(n+\nu+2\alpha)(n+\nu)}{(2n+2\nu+2\alpha)(2n+2\nu+2\alpha-1)}\right)^{1/2}, \qquad n \in \mathbb{N}.$$

Obviously these polynomials belong to the Nevai class $M(0,1)$. To check that this sequence is of bounded variation type it is sufficient to show that $\sum_{n=1}^{\infty} |\lambda_{n+1}^2 - \lambda_n^2| < \infty$. Since $\lambda_{n+1}^2 - \lambda_n^2 = (cn+d)/(8n^3 + \cdots)$, we see that the orthonormal polynomial sequence is of bounded variation type. In [17, (3.8)] we calculated the Haar weights $h(n)$ explicitly,

$$h(n) = \frac{(2n+2\alpha+2\nu+1)}{4\alpha^2(2\alpha+2\nu+1)(\nu+1)_n(2\alpha+\nu+1)_n}((2\alpha+\nu)_{n+1} - (\nu)_{n+1})^2.$$

Asymptotic properties of the Gamma function yield $h(n) = O(n^{2\alpha+1})$. Hence by Theorem 4 it follows that $D_x = 0$ for all $x \in [-1,1] = \hat{\mathbb{N}}_0$, provided $-1/2 < \alpha < 1/2$.

EXAMPLE 5 (Cartier–Dunau polynomials). These polynomials $R_n(x) = R_n(x;q)$, $q \geq 1$, are defined by $b_n = 0$ and $a_n = q/(q+1)$ and $c_n = 1/(q+1)$ for $n \in \mathbb{N}$. They are used to study homogeneous trees (see [13, 3.4]). We consider only $x = 0$ and determine $R_n'(0)$ explicitly. Since the polynomials are symmetric we have $R_{2k}'(0) = 0$ for $k \in \mathbb{N}_0$. To calculate $R_{2k+1}'(0)$ we determine $R_{2k}(0)$. The recurrence relation gives $R_{2k}(0) = (-1)^k/q^k$. Since

$$R_n(0) = \frac{q}{q+1}\, R_{n+1}'(0) + \frac{1}{q+1}\, R_{n-1}'(0),$$

a simple induction proof shows that

$$R_{2k+1}'(0) = (-1)^k\, \frac{(k+1)q+k}{q^{k+1}} \qquad \text{for } k \in \mathbb{N}_0.$$

For $q = 1$ we obtain $D_0 = 0$. (Note that $R_n(x;1)$ is the Chebyshev polynomial of first kind, and hence $D_x = 0$ for all $x \in [-1,1]$, as we already know.) For $q > 1$ there are $D_0 \neq 0$ by Theorem 1(i). Consequently, $l^1(h)$ is not weakly amenable for $q > 1$ (see Proposition 6).

EXAMPLE 6 (Karlin–McGregor polynomials). In [9] we studied the Reiter condition P_2 for orthogonal polynomials $R_n(x) = R_n(x;\alpha,\beta)$ defined by the recurrence coefficients $a_0 = 1$, $b_0 = 0$ and

$$a_n = \begin{cases} \dfrac{\alpha-1}{\alpha} & \text{for } n \text{ odd,} \\[2ex] \dfrac{\beta-1}{\beta} & \text{for } n \text{ even,} \end{cases}$$

and $c_n = 1 - a_n$, $b_n = 0$ for $n \in \mathbb{N}$. They were first considered by Karlin and McGregor in [15]. These polynomials induce a polynomial hypergroup on $\mathbb{N}_0$ whenever $\alpha \geq 2$ and $\beta \geq 2$. We consider only $x = 0$ (the dual space $\hat{\mathbb{N}}_0$ is equal to $[-1,1]$). Obviously $R_{2k+1}(0) = 0$, and as is easily shown, $R_{2k}(0) = (-1)^k/(\alpha-1)^k$ for $k \in \mathbb{N}$. Hence

$$\sum_{n=0}^{\infty} R_n^2(0)h(n) = 1 + \frac{\beta}{\beta-1}\sum_{k=1}^{\infty}\left(\frac{\beta-1}{\alpha-1}\right)^k,$$

and so $\sum_{n=0}^{\infty} R_n^2(0)h(n) = \alpha/(\alpha-\beta)$, provided $\alpha > \beta \geq 2$, which implies $\pi(\{0\}) = (\alpha-\beta)/\alpha$. Despite this fact we now show that there are bounded point derivations D_0 different from zero. Since the polynomials are symmetric, we have $R_{2k}'(0) = 0$. For the odd indices we have

$$(19)\qquad R'_{2k+1}(0) = \frac{(-1)^k\beta}{\beta-1}\sum_{j=0}^{k-1}\frac{1}{(\alpha-1)^{k-j}(\beta-1)^j} + \frac{(-1)^k}{(\beta-1)^k}$$

for $k \in \mathbb{N}_0$. To prove (19) we use induction and the equation $R'_{n+1}(0) = (1/a_n)R_n(0) - (c_n/a_n)R'_{n-1}(0)$. The calculation is straightforward.

If $\alpha > 2$ and $\beta > 2$ then $\lim_{k\to\infty}|R'_{2k+1}(0)| = 0$. In fact, let $\gamma = \min\{\alpha,\beta\} > 2$. Then

$$|R'_{2k+1}(0)| \le \frac{\beta}{\beta-1}\,\frac{k}{(\gamma-1)^k} + \frac{1}{(\beta-1)^k} \to 0 \quad \text{as } k\to\infty.$$

If $\alpha = 2$, $\beta > 2$ or $\alpha > 2$, $\beta = 2$, we see again by (19) that $\{R'_{2k+1}(0) : k \in \mathbb{N}_0\}$ is bounded. Applying Theorem 1(i) we find that $D_0 \neq 0$ exist whenever $\alpha,\beta > 2$, or $\alpha = 2$, $\beta > 2$, or $\alpha > 2$, $\beta = 2$.

Example 7 (Pollaczek polynomials). For symmetric Pollaczek polynomials $R_n(x) = R_n(x;\alpha,\mu)$ we have shown in [17, Theorem 4.1] that they define a polynomial hypergroup on $\mathbb{N}_0$ if $\alpha,\mu \ge 0$ or $-1/2 < \alpha < 0$, $0 \le \mu < \alpha + 1/2$. We use a result of Askey on the positivity of connection coefficients (see (13)) with respect to the Chebyshev polynomials of the second kind which are exactly the ultraspherical polynomials $R_n^{(\alpha)}(x)$ with $\alpha = 1/2$. By Corollary 1 we know that for these polynomials, $D_x \neq 0$ exist for each $x \in \,]-1,1[$. We consider the monic version $u_n(x)$ of the Chebyshev polynomials of the second kind, which satisfy

$$(20)\qquad xu_n(x) = u_{n+1}(x) + \frac{1}{4}\,u_{n-1}(x), \qquad n \in \mathbb{N},$$

and $u_0(x) = 1$, $u_1(x) = x$.

The monic version $\phi_n(x)$ of the Pollaczek polynomials satisfies

$$x\phi_n(x) = \phi_{n+1}(x) + \gamma_n\phi_{n-1}(x), \qquad n \in \mathbb{N},$$

$\phi_0(x) = 1$, $\phi_1(x) = x$, with

$$(21)\qquad \gamma_n = \frac{n(n+2\alpha)}{(2n+2\alpha+2\mu+1)(2n+2\alpha+2\mu-1)}$$

(cf. [17]). By Askey's result (see [2] or [14]) we get $\phi_n(x) = \sum_{k=0}^n d(n,k)u_k(x)$ with $d(n,k) \ge 0$ provided $1/4 \ge \gamma_n$ for all $n \in \mathbb{N}$. The inequality $1/4 \ge \gamma_n$ for all $n \in \mathbb{N}$ is satisfied whenever $\alpha + \mu \ge 1$. Since $u_n(1) > 0$ and $\phi_n(1) > 0$ for all $n \in \mathbb{N}_0$, we obtain

$$R_n(x;\alpha,\mu) = \sum_{k=0}^n c(n,k)R_k^{(1/2)}(x)$$

with $c(n,k) \ge 0$ whenever $\alpha + \mu \ge 1$. By Proposition 6 there exist non-zero derivations D_x for $x \in \,]-1,1[$ with respect to $R_n(x;\alpha,\mu)$ if $\alpha + \mu \ge 1$.

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