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SELFINJECTIVE ALGEBRAS OF STRICTLY CANONICAL TYPE

BY

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Abstract. We develop the representation theory of selfinjective algebras of strictly canonical type and prove that their Auslander–Reiten quivers admit quasi-tubes maximally saturated by simple and projective modules.

Introduction and the main result. Throughout the article, K will denote a fixed algebraically closed field. By an *algebra* is meant an associative, finite-dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules, by $\operatorname{ind} A$ its full subcategory formed by the indecomposable modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$. An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. By a classical result due to Nakayama [30], a basic algebra A is selfinjective if and only if A is a *Frobenius algebra*, that is, there exists a nondegenerate K-bilinear form $(-,-) : A \times A \to K$ satisfying the associativity condition $(ab, c) = (a, bc)$ for all elements $a, b, c \in A$. Moreover, an algebra A is said to be *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules, or equivalently, there exists an associative nondegenerate symmetric K-bilinear form $(-,-) : A \times A \to K$. An important class of selfinjective algebras is formed by the orbit algebras $\hat{B}/G$, where $\hat{B}$ is the repetitive algebra (locally finite-dimensional, without identity)

$$\hat{B} = \bigoplus_{m\in\mathbb{Z}} (B_m \oplus D(B)_m)$$

of an algebra B, where $B_m = B$ and $D(B)_m = D(B)$ for all $m \in \mathbb{Z}$, and the multiplication in $\hat{B}$ is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

for $a_m, b_m \in B_m$, $f_m, g_m \in D(B)_m$, and G is an admissible group of automorphisms of $\hat{B}$. For example, the identity maps $B_m \to B_{m+1}$ and

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$D(B)_m \to D(B)_{m+1}$ induce an algebra automorphism $\nu_{\hat{B}}$ of $\hat{B}$, called the *Nakayama automorphism* of $\hat{B}$, and the orbit algebra $\hat{B}/(\nu_{\hat{B}})$ is the *trivial extension* $T(B) = B \ltimes D(B)$ of B by $D(B)$, which is a symmetric algebra. We note that if B is of finite global dimension then the stable module category $\underline{\mathrm{mod}}\,\hat{B}$ of $\mathrm{mod}\,\hat{B}$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\,B)$ of bounded complexes over $\mathrm{mod}\,B$ [21].

In the representation theory of selfinjective algebras a prominent role is played by the *selfinjective algebras of canonical type*, which are the orbit algebras $\hat{\Lambda}/G$ given by (finite-dimensional) algebras Λ whose derived category $D^b(\mathrm{mod}\,\Lambda)$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\,C)$ of a canonical algebra C (in the sense of [33]) and torsion-free admissible automorphism groups G of $\hat{\Lambda}$. For example, the class of representation-infinite tame selfinjective algebras of polynomial growth coincides with the class of socle deformations of tame selfinjective algebras of canonical type, as described in [38] (see also [9]–[11], [13]–[16], [37]). By general theory (see [1], [3], [25], [26], [31], [37]), every selfinjective algebra of canonical type is isomorphic to an algebra of the form $\hat{B}/G$, where B is a branch extension (equivalently, branch coextension) of a concealed canonical algebra Λ (a tilt of a canonical algebra C), and G is an infinite cyclic group generated by a strictly positive automorphism of $\hat{B}$. A selfinjective algebra A of the form $\hat{B}/G$, where B is a branch extension (equivalently, branch coextension) of a canonical algebra C and G is an infinite cyclic group generated by a strictly positive automorphism of $\hat{B}$, is said to be a *selfinjective algebra of strictly canonical type*.

An important combinatorial and homological invariant of the module category $\mathrm{mod}\,A$ of an algebra A is its *Auslander–Reiten quiver* Γ_A. The vertices of Γ_A are the isoclasses $[X]$ of modules X in $\mathrm{ind}\,A$, and the number of arrows from $[X]$ to $[Y]$ in Γ_A is the number of linearly independent irreducible morphisms in $\mathrm{mod}\,A$ starting at X and ending at Y. Moreover, we have the Auslander–Reiten translations $\tau_A = D\,\mathrm{Tr}$ and $\tau_A^- = \mathrm{Tr}\,D$. We shall identify a vertex $[X]$ of Γ_A with the module X. By a *component* of Γ_A we mean a connected component of Γ_A. A component $\mathcal{C}$ of Γ_A is said to be *standard* if the full subcategory of $\mathrm{mod}\,A$ formed by the indecomposable modules of $\mathcal{C}$ is equivalent to the mesh category $K(\mathcal{C})$ of $\mathcal{C}$ (the quotient category $K\mathcal{C}/I_{\mathcal{C}}$ of the path category $K\mathcal{C}$ of $\mathcal{C}$ modulo the ideal $I_{\mathcal{C}}$ generated by the meshes of $\mathcal{C}$). Two components $\mathcal{C}$ and $\mathcal{D}$ of Γ_A are said to be *orthogonal* if $\mathrm{Hom}_A(X,Y) = 0$ and $\mathrm{Hom}_A(Y,X) = 0$ for modules X in $\mathcal{C}$ and Y in $\mathcal{D}$. For a component $\mathcal{C}$ of Γ_A, we denote by $s(\mathcal{C})$ the number of simple modules in $\mathcal{C}$, by $p(\mathcal{C})$ the number of projective modules in $\mathcal{C}$, and by $i(\mathcal{C})$ the number of injective modules in $\mathcal{C}$.

A general shape of the Auslander–Reiten quiver of a selfinjective algebra of canonical type has been described (see [1], [31], [38], [39]), and its characteristic property is the presence of families of quasi-tubes indexed by the projective line $\mathbb{P}_1(K)$. We are interested in distribution of simple and projective modules in the Auslander–Reiten quivers of selfinjective algebras of canonical type (see [8], [12], [26], [31] for some results in this direction). It is known that the Auslander–Reiten quiver Γ_A of an arbitrary orbit algebra $A = \hat{C}/G$ of a canonical algebra C admits a $\mathbb{P}_1(K)$-family of stable tubes containing simple modules. On the other hand, for all orbit algebras $A = \hat{B}/G$ of the concealed canonical algebras B constructed in [24, Theorem 3], all quasi-tubes of Γ_A are stable tubes and do not contain simple modules. We will show in this paper that the quasi-tubes of the Auslander–Reiten quivers Γ_A of selfinjective algebras of strictly canonical type are maximally saturated by simple and projective modules.

Let A be a selfinjective algebra. We denote by Γ_A^s the *stable Auslander–Reiten quiver* of A, obtained from Γ_A by removing the projective-injective modules and the arrows attached to them. For a component $\mathcal{C}$ of Γ_A, we denote by $\mathcal{C}^s$ its *stable part*. It is well-known that, for any indecomposable projective module P in $\operatorname{mod} A$, there is a canonical Auslander–Reiten sequence in $\operatorname{mod} A$ of the form

$$0 \to \operatorname{rad} P \to (\operatorname{rad} P/\operatorname{soc} P) \oplus P \to P/\operatorname{soc} P \to 0.$$

Hence, we may recover Γ_A from Γ_A^s if we know the positions of socle factors $P/\operatorname{soc} P$ of indecomposable projective modules P in Γ_A^s. The Auslander–Reiten translation τ_A is an automorphism of the quiver Γ_A^s and τ_A^- its inverse. The stable Auslander–Reiten quiver Γ_A^s of a selfinjective algebra A also admits the action of the *syzygy operator* Ω_A which assigns to a module X in Γ_A^s the kernel $\Omega_A(X)$ of a minimal projective cover $P_A(X) \to X$ of X in $\operatorname{mod} A$. The inverse Ω_A^- of Ω_A on Γ_A^s assigns to a module Y in Γ_A^s the cokernel $\Omega_A^-(Y)$ of a minimal injective envelope $Y \to I_A(Y)$ of Y in $\operatorname{mod} A$. The Auslander–Reiten and syzygy operators are related by $\tau_A = \Omega_A^2 \mathcal{N}_A = \mathcal{N}_A \Omega_A^2$ and $\tau_A^- = \Omega_A^{-2} \mathcal{N}_A^{-1} = \mathcal{N}_A^{-1} \Omega_A^{-2}$, where $\mathcal{N}_A = D\mathrm{Hom}_A(-, A)$ is the *Nakayama functor* and $\mathcal{N}_A^{-1} = \mathrm{Hom}_{A^{\mathrm{op}}}(-, A)D$ its inverse. In particular, $\tau_A = \Omega_A^2$ and $\tau_A^- = \Omega_A^{-2}$ if A is symmetric. We also note that the position of a simple module S in Γ_A^s determines the position of its projective cover $P_A(S)$ in Γ_A because $\Omega_A(S) = \operatorname{rad} P_A(S)$.

Recall that if $\mathbb{A}_\infty$ is the infinite linear quiver $0 \to 1 \to 2 \to \cdots$ then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form

$$\begin{tikzcd}[column sep=small]
& (i-1,0) \arrow[dr] & & (i,0) \arrow[dr] & & (i+1,0) \arrow[dr] & & (i+2,0) \arrow[dr] & \\
\cdot^{\cdot^{\cdot}} \arrow[ur] & & (i-1,1) \arrow[ur] \arrow[dr] & & (i,1) \arrow[ur] \arrow[dr] & & (i+1,1) \arrow[ur] \arrow[dr] & & \ddots \\
& \cdot^{\cdot^{\cdot}} \arrow[ur] & & (i-1,2) \arrow[ur] \arrow[dr] & & (i,2) \arrow[ur] \arrow[dr] & & \ddots & \\
& & \cdot^{\cdot^{\cdot}} \arrow[ur] & & \vdots \arrow[ur] & & \ddots & &
\end{tikzcd}$$

with the translation τ defined by $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex x of $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r x$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r x \to \tau^r y$. Then $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is a translation quiver consisting of τ-periodic vertices of period r, called a *stable tube of rank* r. The set of all vertices of a stable tube $\mathcal{T}$ having exactly one immediate predecessor (equivalently, exactly one immediate successor) is called the *mouth* of $\mathcal{T}$. We refer to [35, Chapter X] for more information concerning stable tubes.

Let A be a selfinjective algebra. A component $\mathcal{C}$ of Γ_A is said to be a *quasi-tube* if its stable part $\mathcal{C}^s$ is a stable tube of Γ_A^s. By general theory (see [27], [41]) an infinite component $\mathcal{C}$ of Γ_A is a quasi-tube if and only if $\mathcal{C}$ contains an oriented cycle. Clearly, every stable tube of Γ_A is a quasi-tube. For a quasi-tube $\mathcal{C}$ of Γ_A, we denote by $r(\mathcal{C})$ the rank of the stable tube $\mathcal{C}^s$. Then $s(\mathcal{C})+p(\mathcal{C}) \leq r(\mathcal{C})-1$ (see [28, Theorem A]). Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are quasi-tubes of Γ_A such that $\mathcal{D}^s = \Omega_A(\mathcal{C}^s)$ then $s(\mathcal{C}) = p(\mathcal{D})$, $p(\mathcal{C}) = s(\mathcal{D})$, and $r(\mathcal{C}) = r(\mathcal{D})$.

The following main result of the paper describes the structure and homological properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type.

Main Theorem. *Let A be a selfinjective algebra of strictly canonical type. The Auslander–Reiten quiver Γ_A of A has a decomposition*

$$\Gamma_A = \bigvee_{q\in\mathbb{Z}/n\mathbb{Z}} (\mathcal{X}_q^A \vee \mathcal{C}_q^A),$$

for some positive integer n, and the following statements hold:

(i) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathcal{C}_q^A = (\mathcal{C}_q^A(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes with $s(\mathcal{C}_q^A(\lambda)) + p(\mathcal{C}_q^A(\lambda)) = r(\mathcal{C}_q^A(\lambda)) - 1$ for each $\lambda \in \mathbb{P}_1(K)$.*

(ii) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathcal{X}_q^A$ is a family of components containing exactly one simple module S_q.*

(iii) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, we have* $\mathrm{Hom}_A(S_q, \mathcal{C}_q^A(\lambda)) \neq 0$ *for all $\lambda \in \mathbb{P}_1(K)$, and* $\mathrm{Hom}_A(S_p, \mathcal{C}_q^A) = 0$ *for $p \neq q$ in $\mathbb{Z}/n\mathbb{Z}$.*

(iv) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, we have* $\mathrm{Hom}_A(\mathcal{C}_q^A(\lambda), S_{q+1}) \neq 0$ *for all $\lambda \in \mathbb{P}_1(K)$, and* $\mathrm{Hom}_A(\mathcal{C}_q^A, S_p) = 0$ *for $p \neq q+1$ in $\mathbb{Z}/n\mathbb{Z}$.*

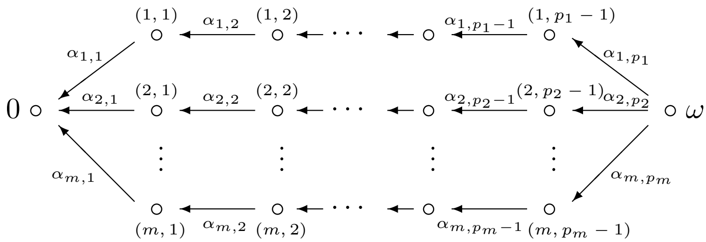
(v) *For each $q\in\mathbb{Z}/n\mathbb{Z}$, $\Omega_A((\mathcal{C}_{q+1}^A)^s) = (\mathcal{C}_q^A)^s$ and $\Omega_A((\mathcal{X}_{q+1}^A)^s) = (\mathcal{X}_q^A)^s$.*

The paper is organized as follows. In Section 1 we introduce the canonical algebras and describe their canonical $\mathbb{P}_1(K)$-family of stable tubes. Section 2 is devoted to the branch extensions and coextensions of canonical algebras, and Section 3 to the quasi-tube enlargements of canonical algebras, playing the fundamental role in the proof of the Main Theorem. In Section 4 we recall the needed facts on repetitive algebras and their orbit algebras. Section 5 contains the proof of the Main Theorem.

For background on the representation theory of algebras we refer to the books [2], [7], [33], [35], [36], and to the survey articles [38]–[40].

1. Canonical algebras. The aim of this section is to introduce the canonical algebras and describe their canonical family of stable tubes.

Let $m \geq 2$ be a natural number, $\boldsymbol{p} = (p_1, \ldots, p_m)$ a sequence of positive natural numbers and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ a sequence of pairwise different elements of the projective line $\mathbb{P}_1(K) = K \cup \{\infty\}$ normalized so that $\lambda_1 = \infty$ and $\lambda_2 = 0$. Consider the quiver $\Delta(\boldsymbol{p})$ of the form



For $m = 2$, $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is defined to be the path algebra $K\Delta(\boldsymbol{p})$ of the quiver $\Delta(\boldsymbol{p})$ over K. For $m \geq 3$, $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is defined to be the quotient algebra $K\Delta(\boldsymbol{p})/I(\boldsymbol{p}, \boldsymbol{\lambda})$ of the path algebra $K\Delta(\boldsymbol{p})$ by the ideal $I(\boldsymbol{p}, \boldsymbol{\lambda})$ of $K\Delta(\boldsymbol{p})$ generated by the elements

$$\alpha_{j,p_j} \ldots \alpha_{j,1} + \alpha_{1,p_1} \ldots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \ldots, \alpha_{2,1}, \qquad j \in \{3, \ldots, m\}.$$

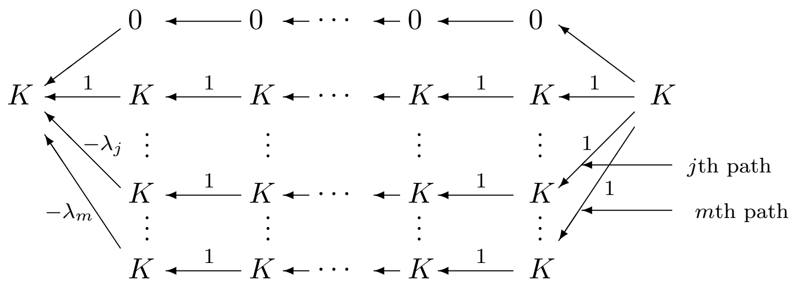
Following [33], $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is said to be a *canonical algebra* of type $(\boldsymbol{p}, \boldsymbol{\lambda})$, $\boldsymbol{p}$ the *weight sequence* of $C(\boldsymbol{p}, \boldsymbol{\lambda})$, and $\boldsymbol{\lambda}$ the (*normalized*) *parameter sequence* of $C(\boldsymbol{p}, \boldsymbol{\lambda})$. It follows from [33, (3.7)] that, for a canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$, the Auslander–Reiten quiver Γ_C of C is of the form

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C,$$

where $\mathcal{P}^C$ is a family of components containing all indecomposable projective C-modules (hence the unique simple projective C-module $S(0)$ associated to the vertex 0 of $\Delta(\boldsymbol{p})$), $\mathcal{Q}^C$ is a family of components containing all indecomposable injective C-modules (hence the unique simple injective C-module $S(\omega)$ associated to the vertex ω of $\Delta(\boldsymbol{p})$), and $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a *canonical* $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes separating $\mathcal{P}^C$ from $\mathcal{Q}^C$ and containing all simple C-modules except $S(0)$ and $S(\omega)$. Moreover, if r^C_λ denotes the rank of the stable tube $\mathcal{T}^C_\lambda$, then $r^C_{\lambda_i} = p_i$ for $i \in \{1, \ldots, m\}$, and $r^C_\lambda = 1$ for $\lambda \in \mathbb{P}_1(K) \setminus \{\lambda_1, \ldots, \lambda_m\}$.

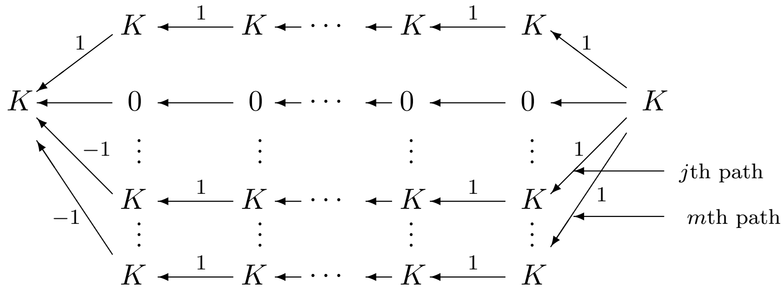
Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra. We recall a description of modules lying on the mouth of stable tubes of the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of Γ_C:

(a) For $\lambda = \lambda_1 = \infty$, the mouth of $\mathcal{T}_\lambda^C = \mathcal{T}_\infty^C$ consists of the simple C-modules $S(1,1), \dots, S(1,p_1-1)$ at the vertices $(1,1), \dots, (1,p_1-1)$ of $\Delta(\boldsymbol{p})$ if $p_1 \ge 2$, and the nonsimple C-module $E^{(\infty)}$ of the form



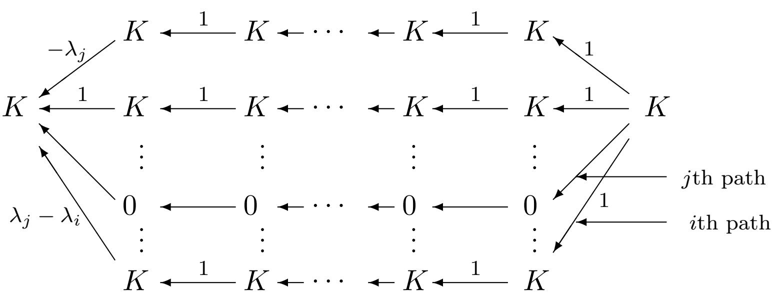
with $j \in \{3, \dots, m\}$.

(b) For $\lambda = \lambda_2 = 0$, the mouth of $\mathcal{T}_\lambda^C = \mathcal{T}_0^C$ consists of the simple C-modules $S(2,1), \dots, S(2,p_2-1)$ at the vertices $(2,1), \dots, (2,p_2-1)$ of $\Delta(\boldsymbol{p})$ if $p_2 \ge 2$, and the nonsimple C-module $E^{(0)}$ of the form



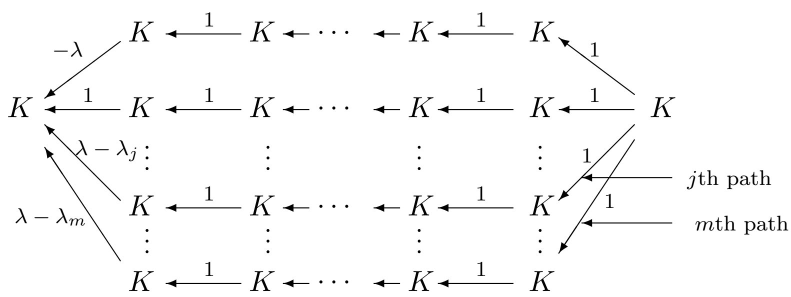
with $j \in \{3, \dots, m\}$.

(c) For $\lambda = \lambda_j$ with $j \in \{3, \dots, m\}$, the mouth of $\mathcal{T}_\lambda^C$ consists of the simple C-modules $S(j,1), \dots, S(j,p_j-1)$ at the vertices $(j,1), \dots, (j,p_j-1)$ of $\Delta(\boldsymbol{p})$ if $p_j \ge 2$, and the nonsimple C-module $E^{(\lambda_j)}$ of the form



for $i \in \{3, \dots, m\} \setminus \{j\}$.

(d) For $\lambda \in \mathbb{P}_1(K) \setminus \{\lambda_1, \ldots, \lambda_m\}$, the mouth of $\mathcal{T}_\lambda^C$ consists of one nonsimple C-module $E^{(\lambda)}$ of the form



with $j \in \{3, \ldots, m\}$.

The following lemma will be useful in further considerations.

LEMMA 1.1. *Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra, with $\boldsymbol{p} = (p_1, \ldots, p_m)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Let μ be an element in $\mathbb{P}_1(K) \setminus \{\lambda_1, \ldots, \lambda_m\}$. Take $\boldsymbol{p}_\mu = (p_1, \ldots, p_m, 1)$ and $\boldsymbol{\lambda}_\mu = (\lambda_1, \ldots, \lambda_m, \mu)$. Then the canonical algebras $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ and $C_\mu = C(\boldsymbol{p}_\mu, \boldsymbol{\lambda}_\mu)$ are isomorphic.*

Proof. We have $C = K\Delta(\boldsymbol{p})/I(\boldsymbol{p}, \boldsymbol{\lambda})$, with $I(\boldsymbol{p}, \boldsymbol{\lambda}) = 0$ for $m = 2$, $C_\mu = K\Delta(\boldsymbol{p}_\mu)/I(\boldsymbol{p}_\mu, \boldsymbol{\lambda}_\mu)$, where the quiver $\Delta(\boldsymbol{p}_\mu)$ is obtained from the quiver $\Delta(\boldsymbol{p})$ by adding the single arrow $0 \xleftarrow{\alpha_{m+1,1}} \omega$, and $I(\boldsymbol{p}_\mu, \boldsymbol{\lambda}_\mu)$ is the ideal of the path algebra $K\Delta(\boldsymbol{p}_\mu)$ generated by the elements generating $I(\boldsymbol{p}, \boldsymbol{\lambda})$ in $K\Delta(\boldsymbol{p})$ and the additional element

$$\alpha_{m+1,1} + \alpha_{1,p_1} \ldots \alpha_{1,1} + \mu\alpha_{2,p_2} \ldots \alpha_{2,1}.$$

Then the canonical embedding of the quivers $\Delta(\boldsymbol{p}) \hookrightarrow \Delta(\boldsymbol{p}_\mu)$ induces an isomorphism $C \xrightarrow{\sim} C_\mu$ of algebras. ∎

2. Branch extensions and coextensions of canonical algebras. The aim of this section is to introduce the branch extensions and coextensions of canonical algebras, playing a fundamental role in the paper.

Let B be an algebra and X a module in $\operatorname{mod} B$. The *one-point extension* of B by X is the 2×2-matrix algebra

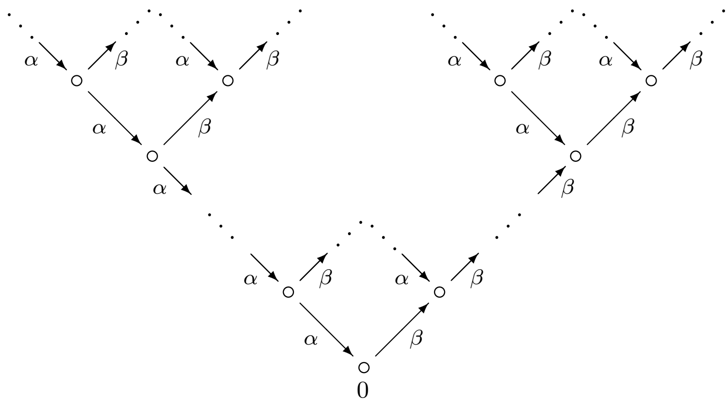
$$B[X] = \begin{bmatrix} B & 0 \\ {}_K X_B & K \end{bmatrix} = \left\{ \begin{bmatrix} b & 0 \\ x & \lambda \end{bmatrix} \,\middle|\, b \in B,\ \lambda \in K,\ x \in X \right\}$$

with the usual addition of matrices and the multiplication induced from the canonical K-B-bimodule structure ${}_K X_B$ of X. The quiver $Q_{B[X]}$ of $B[X]$ contains the quiver Q_B of B as a full convex subquiver, and there is a single additional vertex in $Q_{B[X]}$, which is a source. Dually, the *one-point coextension* of B by X is the 2×2-matrix algebra

$$[X]B = \begin{bmatrix} K & 0 \\ D(X) & B \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & 0 \\ f & b \end{bmatrix} \middle| \, b \in B,\, \lambda \in K,\, f \in D(X) \right\}$$

with the usual addition of matrices and the multiplication induced from the canonical B-K-bimodule structure of $D(X) = \mathrm{Hom}_K({}_KX_B, K)$. The quiver $Q_{[X]B}$ of $[X]B$ contains the quiver Q_B of B as a full convex subquiver, and there is a single additional vertex in $Q_{[X]B}$, which is a sink.

A *branch* is a finite connected full bounded subquiver $\mathcal{L} = (Q_{\mathcal{L}}, I_{\mathcal{L}})$, containing the lowest vertex 0, of the following infinite tree:



with $I_{\mathcal{L}}$ generated by all paths $\alpha\beta$ contained in $Q_{\mathcal{L}}$. The lowest vertex 0 of $\mathcal{L}$ is called the *germ* of $\mathcal{L}$, the number of vertices of $\mathcal{L}$ is called the *capacity* of $\mathcal{L}$, and the bound quiver algebra $K\mathcal{L} = KQ_{\mathcal{L}}/I_{\mathcal{L}}$ is called the *branch algebra* of $\mathcal{L}$. It is known that the class of branch algebras $K\mathcal{L}$ of branches $\mathcal{L}$ of capacity $n \geq 1$ coincides with the class of tilted algebras of the Dynkin equioriented linear quiver type

$$\Delta(\mathbb{A}_n): \quad \overset{1}{\circ} \longleftarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ} \longleftarrow \cdots \longleftarrow \overset{n-1}{\circ} \longleftarrow \overset{n}{\circ}$$

(see [33, (4.4)] or [36, (XVI.2.2)]).

Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra of type $(\boldsymbol{p}, \boldsymbol{\lambda})$ and $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ its canonical $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes. Let $E_1, \dots, E_s$ be a set of pairwise different modules lying on the mouth of the tubes of $\mathcal{T}^C$. Consider the multiple one-point extension of C,

$$C[E_1, \dots, E_s] = \begin{bmatrix} C & 0 \\ E_1 \oplus \cdots \oplus E_s & K_1 \times \cdots \times K_s \end{bmatrix},$$

and the multiple one-point coextension of C,

$$[E_1, \dots, E_s]C = \begin{bmatrix} K_1 \times \cdots \times K_s & 0 \\ D(E_1 \oplus \cdots \oplus E_s) & C \end{bmatrix},$$

where $K_1 = \cdots = K_s = K$ and the left module structure of $E_1 \oplus \cdots \oplus E_s$ over $K_1 \times \cdots \times K_s$ is given by $(\lambda_1, \dots, \lambda_s)(u_1, \dots, u_s) = (\lambda_1 u_1, \dots, \lambda_s u_s)$ for $\lambda_1, \dots, \lambda_s \in K$, $u_1 \in E_1, \dots, u_s \in E_s$. Observe that $C[E_1, \dots, E_s]$ is an iterated one-point extension $C[E_1][E_2] \dots [E_s]$, and $[E_1, \dots, E_s]C$ is an iterated one-point coextension $[E_1][E_2] \dots [E_s]C$. Moreover, let $C[E_1, \dots, E_s] = KQ_{C[E_1,\dots,E_s]}/I_{C[E_1,\dots,E_s]}$ and $[E_1, \dots, E_s]C = KQ_{[E_1,\dots,E_s]C}/I_{[E_1,\dots,E_s]C}$ be canonical bound quiver presentations of $C[E_1, \dots, E_s]$ and $[E_1, \dots, E_s]C$.

Denote by $0_1^+, \dots, 0_s^+$ (respectively, $0_1^-, \dots, 0_s^-$) the extension vertices of $Q_{C[E_1,\dots,E_s]}$ (respectively, coextension vertices of $Q_{[E_1,\dots,E_s]C}$) corresponding to the extensions (respectively, coextensions) by the modules $E_1, \dots, E_s$. Choose now branches $\mathcal{L}_1 = (Q_{\mathcal{L}_1}, I_{\mathcal{L}_1}), \dots, \mathcal{L}_s = (Q_{\mathcal{L}_s}, I_{\mathcal{L}_s})$ with the germs $0_1^*, \dots, 0_s^*$, respectively. The *branch extension* of C (*branch $\mathcal{T}^C$-extension* of C in the sense of [36, (XV.3)]), with respect to the mouth modules $E_1, \dots, E_s$ and the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$, is the bound quiver algebra

$$C[E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s] = KQ_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]}/I_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]},$$

where the bound quiver $(Q_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]}, I_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]})$ is obtained from the bound quiver $(Q_{C[E_1,\dots,E_s]}, I_{C[E_1,\dots,E_s]})$ of $C[E_1, \dots, E_s]$ by adding the bound quivers of the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$ and making the identification of the vertices 0_i^+ with 0_i^* for $i \in \{1, \dots, s\}$. Dually, the *branch coextension* of C (*branch $\mathcal{T}^C$-coextension* of C in the sense of [36, (XV.3)]), with respect to the mouth modules $E_1, \dots, E_s$ and the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$, is the bound quiver algebra

$$[E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s]C = KQ_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C}/I_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C},$$

where the bound quiver $(Q_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C}, I_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C})$ is obtained from the bound quiver $(Q_{[E_1,\dots,E_s]C}, I_{[E_1,\dots,E_s]C})$ of $[E_1, \dots, E_s]C$ by adding the bound quivers of the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$ and making the identification of the vertices 0_i^- with 0_i^* for $i \in \{1, \dots, s\}$.

We now describe the bound quivers $(Q_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]}, I_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]})$ and $(Q_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C}, I_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C})$ explicitly. Observe first that, by Lemma 1.1, we may assume that the mouth modules $E_1, \dots, E_s$ belong to the tubes $\mathcal{T}_\lambda^C$ with $\lambda \in \{\lambda_1, \dots, \lambda_m\}$. Moreover, the top $E = E/\operatorname{rad} E$ and the socle $\operatorname{soc} E$ of any module E lying on the mouth of a tube of $\mathcal{T}^C$ are one-dimensional. Hence, each extension vertex 0_i^+ is connected to Q_C by a single arrow γ_i^+ with source 0_i^+ and sink at the vertex x_i of Q_C corresponding to the simple top $S(x_i)$ of E_i for any $i \in \{1, \dots, s\}$. Similarly, each coextension vertex 0_i^- is connected to Q_C by a single arrow γ_i^- with sink 0_i^- and source at the vertex y_i of Q_C corresponding to the simple socle $S(y_i)$ of E_i for any $i \in \{1, \dots, s\}$. Therefore, we obtain the following description of the bound quivers $(Q_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]}, I_{C[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]})$ and $(Q_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C}, I_{[E_1,\mathcal{L}_1,\dots,E_s,\mathcal{L}_s]C})$.

PROPOSITION 2.1. *Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra of type $(\boldsymbol{p}, \boldsymbol{\lambda})$.*

(i) *The quiver $Q_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}$ is obtained from the quivers $Q_C, Q_{\mathcal{L}_1}, \ldots, Q_{\mathcal{L}_s}$ by identifying $0_i^+ = 0_i^*$ and adding the arrows*

$$0_i^+ = 0_i^* \xrightarrow{\gamma_i^+} x_i, \quad i \in \{1, \ldots, s\},$$

and the ideal $I_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}$ is generated by the elements generating the ideals $I_C, I_{\mathcal{L}_1}, \ldots, I_{\mathcal{L}_s}$ and the paths of length 2

$$\gamma_i^+ \alpha_{j_i,t_i}, \quad i \in \{1, \ldots, s\},$$

where E_i is a mouth module of $\mathcal{T}^C_{\lambda_{j_i}}$ with $\lambda_{j_i} \in \{\lambda_1, \ldots, \lambda_m\}$ and α_{j_i,t_i} is the unique arrow on the path $\alpha_{j_i,p_{j_i}} \ldots \alpha_{j_i,1}$ with source x_i.

(ii) *The quiver $Q_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}$ is obtained from the quivers $Q_C, Q_{\mathcal{L}_1}, \ldots, Q_{\mathcal{L}_s}$ by identifying $0_i^- = 0_i^*$ and adding the arrows*

$$y_i \xrightarrow{\gamma_i^-} 0_i^- = 0_i^*, \quad i \in \{1, \ldots, s\},$$

and the ideal $I_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}$ is generated by the elements generating the ideals $I_C, I_{\mathcal{L}_1}, \ldots, I_{\mathcal{L}_s}$ and the paths of length 2

$$\alpha_{j_i,r_i} \gamma_i^-, \quad i \in \{1, \ldots, s\},$$

where E_i is a mouth module of $\mathcal{T}^C_{\lambda_{j_i}}$ with $\lambda_{j_i} \in \{\lambda_1, \ldots, \lambda_m\}$ and α_{j_i,r_i} is the unique arrow on the path $\alpha_{j_i,p_{j_i}} \ldots \alpha_{j_i,1}$ with sink y_i.

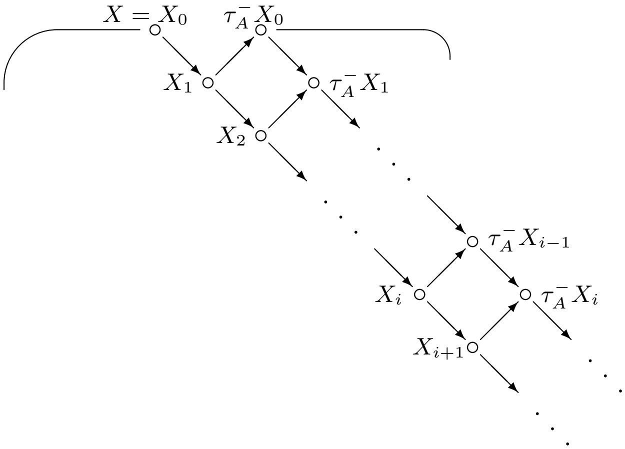
Observe that $\alpha_{j_i,t_i} = \alpha_{j_i,p_{j_i}}$ and $\alpha_{j_i,r_i} = \alpha_{j_i,1}$ if E_i is the unique nonsimple mouth module $E^{(\lambda_{j_i})}$ of $\mathcal{T}^C_{\lambda_{j_i}}$.

By the general theory (see [5, Section XV], [33, Chapter 4]), for a branch extension $C[E_1, \mathcal{L}_1, \ldots, E_s, \mathcal{L}_s]$ (respectively, branch coextension $[E_1, \mathcal{L}_1, \ldots, E_s, \mathcal{L}_s]C$) of a canonical algebra C, the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes of Γ_C is transformed into a canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]} = (\mathcal{T}_\lambda^{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]})_{\lambda \in \mathbb{P}_1(K)}$ of ray tubes of $\Gamma_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}$ (respectively, a canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C} = (\mathcal{T}_\lambda^{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C})_{\lambda \in \mathbb{P}_1(K)}$ of coray tubes of $\Gamma_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}$). In particular, the ray tubes of $\mathcal{T}^{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}$ may contain projective modules but not injective modules, while the coray tubes of $\mathcal{T}^{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}$ may contain injective modules but not projective modules. We will need precise information on the number of simple and projective modules in the ray tubes of $\mathcal{T}^{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}$ (respectively, simple and injective modules in the coray tubes of $\mathcal{T}^{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}$). According to [36, Theorem XV.3.9] the class of branch $\mathcal{T}^C$-extensions (respectively, branch $\mathcal{T}^C$-coextensions) of a canonical algebra C coincides with the class of $\mathcal{T}^C$-tubular extensions (respectively, $\mathcal{T}^C$-tubular coextensions) of C, as described below.

Let A be an algebra and $\mathcal{C}$ a standard component of Γ_A, that is, the full subcategory of $\operatorname{mod} A$ given by modules of $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$. Assume that X is an admissible ray module of $\mathcal{C}$, that is, a module X lying on an infinite sectional path (ray)

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_i \to \cdots$$

satysfying the conditions of [36, XV.2.1]. Then $\mathcal{C}$ looks as follows:

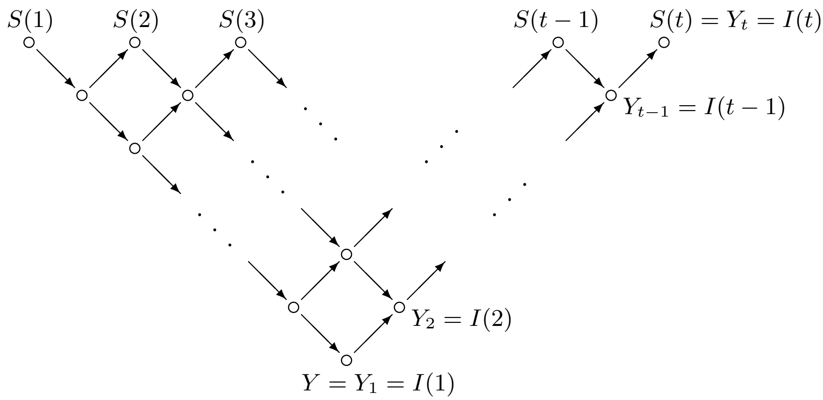


We note that if $\mathcal{C}$ is a standard stable tube then the admissible ray modules of $\mathcal{C}$ are exactly the modules lying on its mouth.

For a positive integer t, denote by H_t the path algebra of the quiver

$$\Delta(\mathbb{A}_t): \quad \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \longleftarrow \cdots \longleftarrow \underset{t-1}{\circ} \longleftarrow \underset{t}{\circ}\,.$$

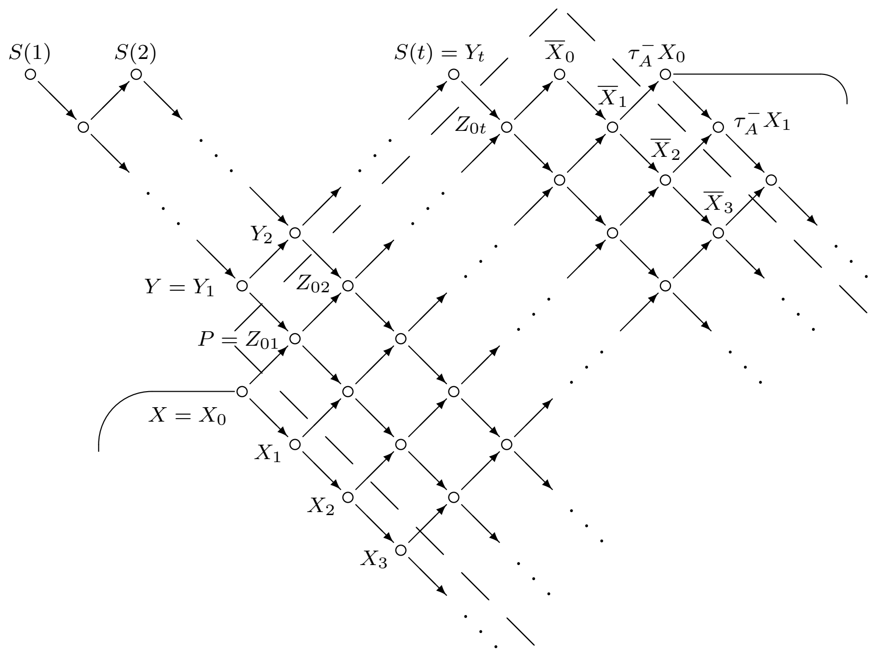
Recall that the Auslander–Reiten quiver Γ_{H_t} of H_t is of the form



where $S(1), \dots, S(t)$ and $I(1), \dots, I(t)$ are the simple and indecomposable injective H_t-modules at the vertices $1, , \dots, t$, respectively. If $t = 0$, we denote by H_0 the zero algebra and set $Y = 0$. Then the one-point extension algebra

$$A(X,t) = [A \times H_t][X \oplus Y] = \begin{bmatrix} A \times H_t & 0 \\ X \oplus Y & K \end{bmatrix}$$

is called the *t-linear extension* of A at X. It follows from [36, Proposition XV.2.7] that the component $\mathcal{C}'$ of $\Gamma_{A(X,t)}$ containing the module X is a standard component obtained from $\mathcal{C}$ and Γ_{H_t} by inserting an infinite rectangle as follows:



Observe that $\mathcal{C}'$ is obtained from $\mathcal{C}$ by inserting $t+1$ rays, among them t rays starting from the simple H_t-modules, and $P = Z_{01}$ is the new projective module, corresponding to the extension vertex of $A(X,t)$. Clearly, for $t = 0$, $\mathcal{C}'$ is obtained from $\mathcal{C}$ by inserting only one ray starting at $P = Z_{01}$.

Dually, for an admissible coray module X of $\mathcal{C}$, the one-point coextension

$$(X,t)A = [X \oplus Y][A \times H_t] = \begin{bmatrix} K & 0 \\ D(X \oplus Y) & A \times H_t \end{bmatrix}$$

is called the *t-linear coextension* of A at X. Then the connected component $\mathcal{C}''$ of $\Gamma_{(X,t)A}$ containing X is a standard component obtained from $\mathcal{C}$ by inserting $t+1$ corays, among them t corays ending at the simple H_t-modules, and $\mathcal{C}''$ contains the new indecomposable injective module corresponding to the coextension vertex of $(X,t)A$.

Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra. An algebra B is said to be a $\mathcal{T}^C$*-tubular extension* (respectively, $\mathcal{T}^C$*-tubular coextension*) of C if there exist a sequence of algebras $B_0 = C, B_1, \ldots, B_n = B$ such that, for each $i \in \{1, \ldots, n\}$, the algebra B_i is a t_i-linear extension $B_{i-1}(X_i, t_i)$ of B_{i-1} (respectively, t_i-linear coextension $(X_i, t_i)B_{i-1}$ of B_{i-1}), for some $t_i \geq 0$, with respect to an admissible ray module X_i (respectively, admissible coray module X_i) lying in a standard stable tube $\mathcal{T}^C$ or in a component of $\Gamma_{B_{i-1}}$, obtained from a stable tube of $\mathcal{T}^C$ by rectangle insertions (respectively, rectangle coinsertions) created by the linear extensions (respectively, linear coextensions) made so far. For a tubular extension (respectively, coextension) B of C, the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is transformed into a canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of standard ray tubes (respectively, standard coray tubes) $\mathcal{T}^B_\lambda$ obtained from the standard stable tubes $\mathcal{T}^C_\lambda$ by the corresponding iterated rectangle insertions (respectively, iterated rectangle coinsertions).

Let B be a $\mathcal{T}^C$-tubular extension of a canonical algebra C and $\lambda \in \mathbb{P}_1(K)$. Then every module M of the ray tube $\mathcal{T}^B_\lambda$ lies on exactly one ray $r(M)$ of $\mathcal{T}^B_\lambda$. We denote by $p^*(\mathcal{T}^B_\lambda)$ the number of projective B-modules P in $\mathcal{T}^B_\lambda$ which are not proper predecessors of a projective module lying on the ray $r(P)$.

Let B be a $\mathcal{T}^C$-tubular coextension of a canonical algebra C and $\lambda \in \mathbb{P}_1(K)$. Then every module N of the coray tube $\mathcal{T}^B_\lambda$ lies on exactly one coray $c(N)$ of $\mathcal{T}^B_\lambda$. We denote by $i^*(\mathcal{T}^B_\lambda)$ the number of injective B-modules I in $\mathcal{T}^B_\lambda$ which are not proper successors of an injective module lying on the coray $c(I)$.

PROPOSITION 2.2. *Let C be a canonical algebra and $\mathcal{T}^C$ the canonical $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes of Γ_C.*

(i) *Let B be a $\mathcal{T}^C$-tubular extension of C. Then the Auslander–Reiten quiver Γ_B of B is of the form*

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where $\mathcal{P}^B = \mathcal{P}^C$ is a family of components consisting of C-modules and containing all indecomposable projective C-modules, $\mathcal{Q}^B$ is a family of components containing all indecomposable injective B-modules but no projective B-module, and $\mathcal{T}^B$ is a $\mathbb{P}_1(K)$-family $(\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, the number of rays of $\mathcal{T}^B_\lambda$ is equal to $s(\mathcal{T}^B_\lambda) + p^*(\mathcal{T}^B_\lambda) + 1$.

(ii) *Let B be a $\mathcal{T}^C$-tubular coextension of C. Then the Auslander–Reiten quiver Γ_B of B is of the form*

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where $\mathcal{P}^B$ is a family of components containing all indecomposable projective B-modules but no injective B-modules, $\mathcal{Q}^B = \mathcal{Q}^C$ is a family of components consisting of C-modules and containing all indecomposable injective C-modules, and $\mathcal{T}^B$ is a $\mathbb{P}_1(K)$-family $(\mathcal{T}_\lambda^B)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, the number of corays of $\mathcal{T}_\lambda^B$ is equal to $s(\mathcal{T}_\lambda^B) + i^*(\mathcal{T}_\lambda^B) + 1$.

Proof. This follows from [4, Section 2], [6, Section 2], [33, Section 4], the above discussion, and the fact that, for any stable tube $\mathcal{T}_\lambda^C$ of $\mathcal{T}^C$, we have $p(\mathcal{T}_\lambda^C) = 0 = i(\mathcal{T}_\lambda^C)$ and $s(\mathcal{T}_\lambda^C) + 1$ is the rank r_λ^C of $\mathcal{T}_\lambda^C$, hence the number of rays (equivalently, corays) of $\mathcal{T}_\lambda^C$. ∎

We end this section with the following consequence of [36, Theorem XV.3.9].

PROPOSITION 2.3. *Let C be a canonical algebra and $\mathcal{T}^C$ the canonical $\mathbb{P}_1(K)$-family of standard stable tubes of Γ_C. For an algebra A the following equivalences hold*:

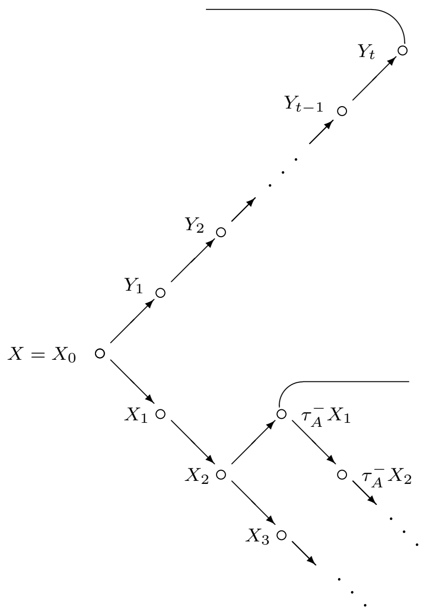
(i) *A is a $\mathcal{T}^C$-tubular extension of C if and only if A is a branch $\mathcal{T}^C$-extension of C.*
(ii) *A is a $\mathcal{T}^C$-tubular coextension of C if and only if A is a branch $\mathcal{T}^C$-coextension of C.*

3. Quasi-tube enlargements of canonical algebras. We know from Section 2 that, for a branch extension (respectively, branch coextension) B of a canonical algebra C, the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes is transformed into a canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda\in\mathbb{P}_1(K)}$ of ray tubes (respectively, coray tubes). Following [4]–[6], we describe here canonical enlargements of branch extensions (respectively, branch coextensions) B of canonical algebras C to algebras B^* such that the $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda\in\mathbb{P}_1(K)}$ is transformed into a canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B^*} = (\mathcal{T}_\lambda^{B^*})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise othogonal standard quasi-tubes. In general, a component $\mathcal{C}$ of an Auslander–Reiten quiver Γ_A is called a *quasi-tube* if the projective and injective modules in $\mathcal{C}$ coincide, and the stable part $\mathcal{C}^s$ of $\mathcal{C}$ is a stable tube.

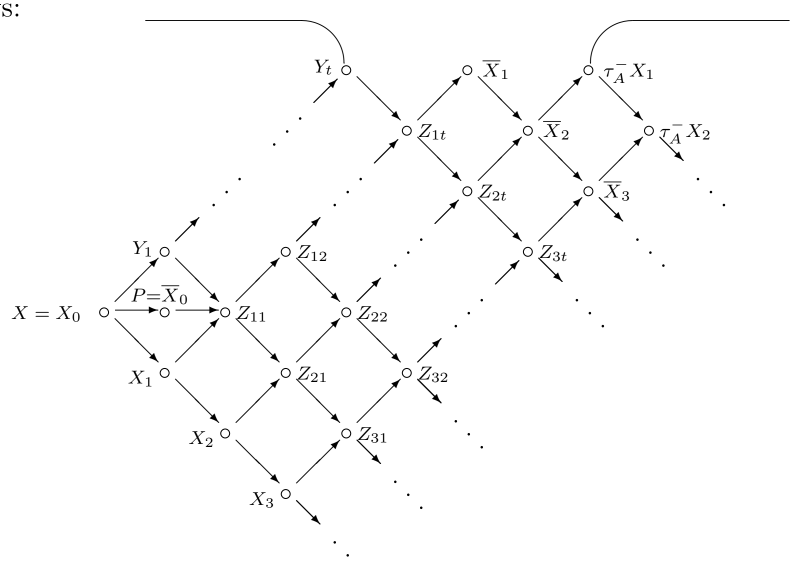
Let A be an algebra and $\mathcal{C}$ a standard component of Γ_A. Assume that X is an indecomposable injective module in $\mathcal{C}$ and source of exactly two sectional paths

$$Y_t \leftarrow Y_{t-1} \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

with $t \geq 1$. The first left hand one is finite and consists of injective modules $Y_1, \ldots, Y_t$, and the second one is infinite. Hence $\mathcal{C}$ looks as follows:



Let $A' = A[X]$ be the one-point extension of A by X. It follows from [6, Section 2] that the component $\mathcal{C}'$ of $\Gamma_{A'}$ containing the module X is a standard component obtained from $\mathcal{C}$ by inserting an infinite rectangle as follows:



We note that the new projective A'-module corresponding to the extension vertex of $A' = A[X]$ is injective, and the injective A-modules $Y_1, \dots, Y_t$ are not injective A'-modules.

Let B be an algebra and Q_B its ordinary quiver. For a vertex i of Q_B, we denote by e_i the idempotent of B corresponding to i, by $P_B(i)$ the associated indecomposable projective B-module e_iB, and by $I_B(i)$ the associated indecomposable injective B-module $D(Be_i)$. Moreover, we denote by T_i^+B the one-point extension $B[I_B(i)]$ of B by $I_B(i)$, and by T_i^-B the one-point coextension $[P_B(i)]B$ of B by $P_B(i)$. More generally, for a sequence $i_1,\dots,i_t$ of vertices of Q_B, we denote by $T^+_{i_1,\dots,i_t}B$ the iterated extension $B[I_B(i_1)][I_{T^+_{i_1}B}(i_2)]\dots[I_{T^+_{i_1,\dots,i_{t-1}}B}(i_t)]$, and by $T^-_{i_1,\dots,i_t}B$ the iterated coextension $[P_{T^-_{i_1,\dots,i_{t-1}}B}(i_t)]\dots[P_{T^-_{i_1}B}(i_2)][P_B(i_1)]B$.

Assume that B is a triangular algebra, that is, the quiver Q_B of B is acyclic. For a sink i of Q_B, the *reflection* S_i^+B of B at i is the quotient of T_i^+B by the two-sided ideal generated by e_i (see [23]). The quiver $\sigma_i^+Q_B$ of S_i^+B is called the reflection of Q_B at i. Observe that the sink i of Q_B is replaced in $\sigma_i^+Q_B$ by a source $\nu(i)$. Dually, for a source j of Q_B, we define the *reflection* S_j^-B of B at j as the quotient of T_j^-B by the two-sided ideal generated by e_j. The quiver $\sigma_j^-Q_B$ of S_j^-B is called the reflection of Q_B at j. The source j of Q_B is replaced in $\sigma_j^-Q_B$ by a sink $\nu^-(j)$. Clearly, for a sink i (respectively, source j) of Q_B, we have $S^-_{\nu(i)}S_i^+B\cong B$ (respectively, $S^+_{\nu^-(j)}S_j^-B\cong B$). A *reflection sequence of sinks* of Q_B is a sequence $i_1,\dots,i_t$ of vertices of Q_B such that i_s is a sink of $\sigma^+_{i_{s-1}}\dots\sigma^+_{i_1}Q_B$ for any $s\in\{1,\dots,t\}$. Dually, a *reflection sequence of sources* of Q_B is a sequence $j_1,\dots,j_t$ of vertices of Q_B such that j_s is a source of $\sigma^-_{j_{s-1}}\dots\sigma^+_{j_1}Q_B$ for any $s\in\{1,\dots,t\}$.

Theorem 3.1. *Let C be a canonical algebra and $\mathcal{T}^C$ the canonical $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes of Γ_C.*

(i) *Let B be a branch $\mathcal{T}^C$-coextension of C. Then there exists a reflection sequence of sinks $i_1,\dots,i_t$ of Q_B such that the iterated reflection $B^+=S^+_{i_t}\dots S^+_{i_1}B$ of B is a branch $\mathcal{T}^C$-extension of C and the Auslander–Reiten quiver Γ_{B^*} of the iterated extension $B^*=T^+_{i_1,\dots,i_t}B$ of B is of the form*

$$\Gamma_{B^*}=\mathcal{P}^{B^*}\vee\mathcal{C}^{B^*}\vee\mathcal{Q}^{B^*},$$

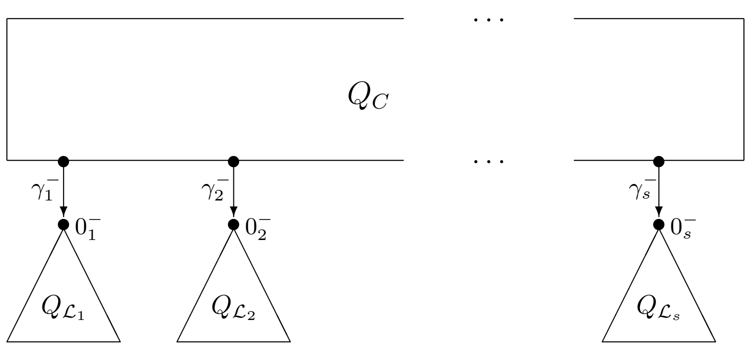
where $\mathcal{P}^{B^*}=\mathcal{P}^B$ is a family of components containing all indecomposable projective B-modules, $\mathcal{Q}^{B^*}=\mathcal{Q}^{B^+}$ is a family of components containing all indecomposable injective B^+-modules, and $\mathcal{C}^{B^*}$ is a $\mathbb{P}_1(K)$-family $(\mathcal{C}^{B^*}_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes separating $\mathcal{P}^{B^*}$ from $\mathcal{Q}^{B^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B=(\mathcal{T}^B_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes of Γ_B by iterated infinite rectangle insertions. Moreover, for each $\lambda\in\mathbb{P}_1(K)$, we have $s(\mathcal{C}^{B^*}_\lambda)+p(\mathcal{C}^{B^*}_\lambda)=r(\mathcal{C}^{B^*}_\lambda)-1$, where $r(\mathcal{C}^{B^*}_\lambda)$ is the rank of the stable part of the quasi-tube $\mathcal{C}^{B^*}_\lambda$.

(ii) *Let B be a branch $\mathcal{T}^C$-extension of C. Then there exists a reflection sequence of sources $j_1, \dots, j_t$ of Q_B such that the iterated reflection $B^- = S^-_{j_t} \dots S^-_{j_1} B$ of B is a branch $\mathcal{T}^C$-coextension of C and the Auslander–Reiten quiver Γ_{B^*} of the iterated coextension $B^* = T^-_{j_1,\dots,j_t} B$ of B is of the form*

$$\Gamma_{B^*} = \mathcal{P}^{B^*} \vee \mathcal{C}^{B^*} \vee \mathcal{Q}^{B^*},$$

where $\mathcal{P}^{B^*} = \mathcal{P}^{B^-}$ is a family of components containing all indecomposable projective B^--modules, $\mathcal{Q}^{B^*} = \mathcal{Q}^B$ is a family of components containing all indecomposable injective B-modules, and $\mathcal{C}^{B^*}$ is a $\mathbb{P}_1(K)$-family $(\mathcal{C}^{B^*}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes separating $\mathcal{P}^{B^*}$ from $\mathcal{Q}^{B^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes of Γ_B by iterated infinite rectangle coinsertions. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, we have $s(\mathcal{C}^{B^*}_\lambda) + p(\mathcal{C}^{B^*}_\lambda) = r(\mathcal{C}^{B^*}_\lambda) - 1$, where $r(\mathcal{C}^{B^*}_\lambda)$ is the rank of the stable part of the quasi-tube $\mathcal{C}^{B^*}_\lambda$.

Proof. (i) Let $B = [E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s]C$ be a branch $\mathcal{T}^C$-coextension of C with respect to mouth modules $E_1, \dots, E_s$ of $\mathcal{T}^C$ and branches $\mathcal{L}_1, \dots, \mathcal{L}_s$. Then B is a triangular algebra, because C and the branch algebras $K\mathcal{L}_1, \dots, K\mathcal{L}_s$ are triangular algebras. Applying Proposition 2.3, we conclude that B is a tubular $\mathcal{T}^C$-coextension of C, and hence the Auslander–Reiten quiver Γ_B of B is of the form $\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B$, where $\mathcal{P}^B$ is a family of components containing all indecomposable projective B-modules but no injective module, $\mathcal{Q}^B = \mathcal{Q}^C$ is a family of components consisting of C-modules and containing all indecomposable injective C-modules, and $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a family of pairwise orthogonal coray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, the number of corays of $\mathcal{T}^B_\lambda$ is equal to $s(\mathcal{T}^B_\lambda) + i^*(\mathcal{T}^B_\lambda) + 1$. Further, the family of indecomposable injective B-modules located in the family $\mathcal{T}^B$ coincides with the family of indecomposable injective B-modules $I_B(i)$ given by the vertices i of the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$. Observe also that the quiver Q_B of B is of the form



Since the family $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ contains only a finite number of injective B-modules, for all but finitely many $\lambda \in \mathbb{P}_1(K)$, we have $\mathcal{T}^B_\lambda = \mathcal{T}^C_\lambda$, and then $s(\mathcal{T}^B_\lambda) = r(\mathcal{T}^B_\lambda) - 1$ holds. In fact, we have $\mathcal{T}^B_\lambda \neq \mathcal{T}^C_\lambda$ if and only if $\mathcal{T}^C_\lambda$ contains a module E_i for some $i \in \{1,\dots,s\}$. Let Λ_B be the set of all $\lambda \in \mathbb{P}_1(K)$ such that $\mathcal{T}^C_\lambda$ contains at least one module E_i. For each $\lambda \in \Lambda_B$, we denote by $\Sigma_B(\lambda)$ the set of all vertices j of Q_B (in fact of $Q_{\mathcal{L}_1} \cup \dots \cup Q_{\mathcal{L}_s}$) such that the injective B-module $I_B(j)$ lies in $\mathcal{T}^B_\lambda$ and is not a proper successor of an injective B-module on the coray $c(I_B(j))$ of $\mathcal{T}^B_\lambda$ containing $I_B(j)$. Observe that $|\Sigma_B(\lambda)| = i^*(\mathcal{T}^B_\lambda)$. Moreover, for different $\lambda,\mu \in \Lambda_B$, the sets $\Sigma_B(\lambda)$ and $\Sigma_B(\mu)$ are disjoint, because they belong to different branches. Finally, let Σ_B be the union of the sets $\Sigma_B(\lambda)$, $\lambda \in \Lambda_B$, and let $t = |\Sigma_B|$. We will show that a reflection sequence of sinks $i_1,\dots,i_t$, satisfying the conditions of (i), is formed by properly ordered vertices of the set Σ_B.

Fix $\lambda \in \Lambda_B$. We will show that there exists a reflection sequence of sinks $i_1,\dots,i_r$ of Q_B, formed by the elements of $\Sigma_B(\lambda)$, hence $r = i^*(\mathcal{T}^B_\lambda) = |\Sigma_B(\lambda)|$, such that after the iterated extension $B(\lambda) = T^+_{i_1,\dots,i_r}B$ of B, the coray tube $\mathcal{T}^B_\lambda$ is transformed into a standard quasi-tube $\mathcal{C}^{B(\lambda)}_\lambda$ of $\Gamma_{B(\lambda)}$. Since $\lambda \in \Lambda_B$, the stable tube $\mathcal{T}^C_\lambda$ of Γ_C contains a mouth module E_i involved in the branch $\mathcal{T}^C$-coextension B of C. Let $0^*_i = b_1 \to \dots \to b_k$ be the maximal path of the branch $\mathcal{L}_i$ starting at its germ 0^*_i, which is also the coextension vertex 0^-_i of the one-point coextension $[E_i]C$. Then the coray tube $\mathcal{T}^B_\lambda$ admits a ray

$$\circ \to \cdots \to \underset{I_B(b_k)}{\circ} \to \cdots \to \underset{I_B(b_2)}{\circ} \to \cdots \to \underset{I_B(b_1)}{\circ} \longrightarrow \underset{E_i}{\circ} \longrightarrow \circ \to \cdots$$

containing the indecomposable injective B-modules $I_B(b_1),\dots,I_B(b_k)$ at the vertices $b_1,\dots,b_k$. Let $i_1 = b_k, i_2 = b_{k-1},\dots,i_k = b_1$. Observe that, for $l \in \{2,\dots,k\}$, b_l is the sink of a unique arrow of Q_B with source b_{l-1}, and consequently $i_1,\dots,i_k$ is a reflection sequence of sinks of Q_B. Applying the one-point extension $T^+_{i_1}B = B[I_B(i_1)]$, we modify the standard coray tube $\mathcal{T}^B_\lambda$ of Γ_B into a standard component $\mathcal{T}^{T^+_{i_1}B}_\lambda$ of $\Gamma_{T^+_{i_1}B}$, obtained from $\mathcal{T}^B_\lambda$ by the infinite rectangle insertion given by the extension $B[I_B(i_1)]$. Moreover, the indecomposable injective B-module $I_B(i_1)$ is extended to the indecomposable projective-injective $T^+_{i_1}B$-module $P_{T^+_{i_1}B}(\nu(i_1)) = \overline{I_B(i_1)}$, while the indecomposable injective B-modules $I_B(i_2),\dots,I_B(i_k)$ are extended to the indecomposable injective $T^+_{i_1}B$-modules $I_{T^+_{i_1}B}(i_2) = \overline{I_B(i_2)},\dots,I_{T^+_{i_1}B}(i_k) = \overline{I_B(i_k)}$. For $k \geq 2$, we consider the one-point extension $T^+_{i_1}B[I_{T^+_{i_1}B}(i_2)] = T^+_{i_1,i_2}B$. Then the standard component $\mathcal{T}^{T^+_{i_1}B}_\lambda$ of $\Gamma_{T^+_{i_1}B}$ is modified into a standard component $\mathcal{T}_\lambda^{T^+_{i_1,i_2}B}$ of $\Gamma_{T^+_{i_1,i_2}B}$, obtained from $\mathcal{T}_\lambda^{T^+_{i_1}B}$ by the infinite rectangle insertion given by the extension $T^+_{i_1}B[I_{T^+_{i_1}B}(i_2)]$. In this extension, the indecomposable injective $T^+_{i_1}B$-module $I_{T^+_{i_1}B}(i_2)$ is extended to the indecomposable projective-injective $T^+_{i_1,i_2}B$-module $P_{T^+_{i_1,i_2}}(\nu(i_2)) = \overline{I_{T^+_{i_1}B}(i_2)}$, the indecomposable injective $T^+_{i_1}B$-modules $I_{T^+_{i_1}B}(i_3), \dots, I_{T^+_{i_1}B}(i_k)$ (if $k \geq 3$) are extended to the indecomposable injective $T^+_{i_1,i_2}B$-modules $I_{T^+_{i_1,i_2}B}(i_3) = \overline{I_{T^+_{i_1}B}(i_3)}, \dots, I_{T^+_{i_1,i_2}B}(i_k) = \overline{I_{T^+_{i_1}B}(i_k)}$, and $P_{T^+_{i_1}B}(\nu(i_1))$ is the indecomposable projective-injective $T^+_{i_1,i_2}B$-module $P_{T^+_{i_1,i_2}B}(\nu(i_1))$ at the vertex $\nu(i_1)$. Applying the extension procedure to all vertices of the sequence $i_1, \dots, i_k$, we obtain the iterated extension

$$T^+_{i_1,\dots,i_k}B = B[I_B(i_1)][I_{T^+_{i_1}B}(i_2)]\dots[I_{T^+_{i_1,\dots,i_{k-1}}B}(i_k)]$$

of B such that the standard coray tube $\mathcal{T}_\lambda^B$ of Γ_B is modified into a standard component $\mathcal{T}_\lambda^{T^+_{i_1,\dots,i_k}B}$ of $\Gamma_{T^+_{i_1,\dots,i_k}B}$, obtained from $\mathcal{T}_\lambda^B$ by k infinite rectangle insertions, and the indecomposable injective B-modules $I_B(i_1), \dots, I_B(i_k)$ of $\mathcal{T}_\lambda^B$ are extended to the indecomposable projective-injective $T^+_{i_1,\dots,i_k}B$-modules

$$I_{T^+_{i_1,\dots,i_k}B}(i_1) = P_{T^+_{i_1,\dots,i_k}B}(\nu(i_1)), \dots, I_{T^+_{i_1,\dots,i_k}B}(i_k) = P_{T^+_{i_1,\dots,i_k}B}(\nu(i_k)).$$

We also note that the indecomposable injective B-modules $I_B(j)$ with $j \in \Sigma_B(\lambda) \setminus \{i_1, \dots, i_k\}$ remain indecomposable injective $T^+_{i_1,\dots,i_k}B$-modules of the component $\mathcal{T}_\lambda^{T^+_{i_1,\dots,i_k}B}$. On the other hand, if the branch $\mathcal{L}_i$ admits a path

$$b_j \leftarrow a_{j_1} \leftarrow \dots \leftarrow a_{j_{q_j}} \quad \text{with } a_{j_1} \neq b_{j-1} \text{ and } j \in \{1, \dots, k\},$$

then the indecomposable injective B-modules $I_B(a_{j_1}), \dots, I_B(a_{j_{q_j}})$ are extended to the indecomposable injective $T^+_{i_1,\dots,i_k}B$-modules

$$I_{T^+_{i_1,\dots,i_k}B}(a_{j_1}), \dots, I_{T^+_{i_1,\dots,i_k}B}(a_{j_{q_j}}),$$

which are no longer located in $\mathcal{T}_\lambda^{T^+_{i_1,\dots,i_k}B}$. Assume that the branch $\mathcal{L}_i$ admits a subquiver of the form

$$b_j \leftarrow a_{j1} \leftarrow \dots \leftarrow a_{jl} \rightarrow c_{jl1} \rightarrow c_{jl2} \rightarrow \dots \rightarrow c_{jlm},$$

and let the path passing through $a_{jl}, c_{jl1}, \dots, c_{jlm}$ be the maximal path of $Q_{\mathcal{L}_1}$ with source a_{jl}. Then the coray tube $\mathcal{T}_\lambda^B$ admits a maximal finite sectional path of the form

$$\circ \to \cdots \to \underset{I_B(c_{jlm})}{\circ} \to \cdots \to \underset{I_B(c_{jl2})}{\circ} \to \cdots \to \underset{I_B(c_{jl1})}{\circ} \to \cdots \to \underset{I_B(a_{jl})}{\circ}$$

and the subpath

$$\underset{I_B(b_j)}{\circ} \longrightarrow \underset{I_B(a_{j1})}{\circ} \longrightarrow \underset{I_B(a_{j2})}{\circ} \to \cdots \to \underset{I_B(a_{jl})}{\circ}$$

of the unique coray $c(I_B(b_j))$ of $\mathcal{T}_\lambda^B$ passing through $I_B(b_j)$. In the component $\mathcal{T}_\lambda^{T^+_{i_1,\ldots,i_u}B}$ of $\Gamma_{T^+_{i_1,\ldots,i_u}B}$ the first (finite) sectional path is completed to an (infinite) ray by the infinite sectional path with source $I_B(a_{jl})$ in the infinite rectangle insertion created by the one-point extension $T^+_{i_1,\ldots,i_{u-1}}B[I_{T^+_{i_1,\ldots,i_{u-1}}}(b_j)]$ leading from $T^+_{i_1,\ldots,i_{u-1}}B$ to $T^+_{i_1,\ldots,i_u}B$, where $u = k+1-j$. Note that in $\mathcal{T}_\lambda^{T^+_{i_1,\ldots,i_{u-1}}B}$ we have the sectional path

$$\underset{I_{T^+_{i_1,\ldots,i_{u-1}}}(b_j)}{\circ} \longrightarrow \underset{I_B(a_{j1})}{\circ} \longrightarrow \underset{I_B(a_{j2})}{\circ} \to \cdots \to \underset{I_B(a_{jl})}{\circ}$$

because $I_B(a_{j1}),\ldots,I_B(a_{jl})$ are still the indecomposable injective $T^+_{i_1,\ldots,i_{u-1}}B$-modules. Therefore, the vertices $c_{jlm},\ldots,c_{jl1}$ form a reflection sequence of sinks of Q_B and $Q_{T^+_{i_1,\ldots,i_k}B}$, and we may consider the iterated extension

$$T^+_{c_{jlm},\ldots,c_{jl1}}T^+_{i_1,\ldots,i_k}B = T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B.$$

Moreover, $i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}$ is a reflection sequence of sinks of Q_B. In the extension process leading from $T^+_{i_1,\ldots,i_k}B$ to $T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B$ the standard component $\mathcal{T}_\lambda^{T^+_{i_1,\ldots,i_k}B}$ of $\Gamma_{T^+_{i_1,\ldots,i_k}B}$ is modified to a standard component $\mathcal{T}_\lambda^{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}$ of $\Gamma_{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}$, by m infinite rectangle insertions, and the indecomposable injective B-modules

$$I_B(c_{jlm}) = I_{T^+_{i_1,\ldots,i_k}B}(c_{jlm}),\ldots,I_B(c_{jl1}) = I_{T^+_{i_1,\ldots,i_k}B}(c_{jl1})$$

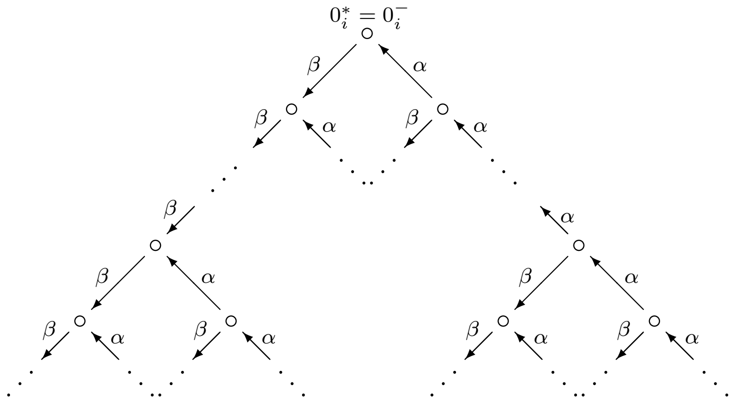
are extended to the indecomposable projective-injective $T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B$-modules

$$I_{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}(c_{jlm}) = P_{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}(\nu(c_{jlm})),\ldots$$
$$\ldots,I_{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}(c_{jl1}) = P_{T^+_{i_1,\ldots,i_k,c_{jlm},\ldots,c_{jl1}}B}(\nu(c_{jl1})).$$

We will now define a reflection sequence of sinks $i_1,\ldots,i_p$ of Q_B, consisting of the common vertices of $\Sigma_B(\lambda)$ and $Q_{\mathcal{L}_i}$, such that after the iterated extension $T^+_{i_1,\ldots,i_p}B$ of B the standard coray tube $\mathcal{T}_\lambda^B$ of Γ_B is extended to a standard component $\mathcal{T}_\lambda^{T^+_{i_1,\dots,i_p}B}$ of $\Gamma_{T^+_{i_1,\dots,i_p}B}$, by p infinite rectangle insertions, and the indecomposable injective B-modules $I_B(i_1),\dots,I_B(i_p)$ of $\mathcal{T}_\lambda^B$ are extended to the indecomposable projective-injective $T^+_{i_1,\dots,i_p}B$-modules

$$I_{T^+_{i_1,\dots,i_p}B}(i_1) = P_{T^+_{i_1,\dots,i_p}B}(\nu(i_1)),\dots,I_{T^+_{i_1,\dots,i_p}B}(i_p) = P_{T^+_{i_1,\dots,i_p}B}(\nu(i_p)).$$

Recall that the branch $\mathcal{L}_i = (Q_{\mathcal{L}_i}, I_{\mathcal{L}_i})$ is a finite connected full bound subquiver of the infinite tree



containing the germ $0_i^* = 0_i^-$, with $I_{\mathcal{L}_i}$ generated by all paths $\alpha\beta$ contained in $Q_{\mathcal{L}_i}$. Denote by $Q^-_{\mathcal{L}_i}$ the quiver obtained from $Q_{\mathcal{L}_i}$ by adding the arrow $y_i \xrightarrow{\beta=\beta_i^-} 0_i^- = 0_i^*$ connecting Q_C with $Q_{\mathcal{L}_i}$ (see Proposition 2.1). By a *β-path* of $Q^-_{\mathcal{L}_i}$ we mean a subpath $\circ \xrightarrow{\beta} \circ \xrightarrow{\beta} \cdots \to \circ \xrightarrow{\beta} \circ$ consisting of consecutive arrows β, and by an *α-path* of $Q^-_{\mathcal{L}_i}$ we mean a subpath $\circ \xrightarrow{\alpha} \circ \xrightarrow{\alpha} \dots \to \circ \xrightarrow{\alpha} \circ$ consisting of consecutive arrows α. Denote by $M_\beta^{(i)}$ the set of all maximal β-paths of $Q^-_{\mathcal{L}_i}$. Observe that different paths in $M_\beta^{(i)}$ have no common vertices. Moreover, if p is a β-path $j_1 \xleftarrow{\beta} \cdots \xleftarrow{\beta} j_r \xleftarrow{\beta} j_{r+1}$ in $M_\beta^{(i)}$ then $j_1,\dots,j_r$ is a reflection sequence of sinks of $Q_{\mathcal{L}_i}$, and hence of Q_B, called the reflection sequence of sinks of p.

We assign to each β-path p in $M_\beta^{(i)}$ a natural number $d(p)$, called the *degree* of p, as follows. The unique maximal β-path of $Q^-_{\mathcal{L}_i}$

$$y_i \xrightarrow{\beta=\beta_i^-} b_1 = i_k \xrightarrow{\beta} b_2 = i_{k-1} \xrightarrow{\beta} \cdots \xrightarrow{\beta} b_{k-1} = i_2 \xrightarrow{\beta} b_k = i_1$$

passing through the germ $0_i^* = 0_i^-$ of $\mathcal{L}_i$ is said to be the β-path of degree 0. This is the unique β-path of $M_\beta^{(i)}$ of degree 0. We say that a β-path

$$c_1 \xrightarrow{\beta} \cdots \xrightarrow{\beta} c_m$$

in $M_\beta^{(i)}$ is of degree 1 if its source c_1 is connected to the unique β-path of degree 0 by an α-path $b_j \xleftarrow{\alpha} \circ \xleftarrow{\alpha} \cdots \xleftarrow{\alpha} \circ \xleftarrow{\alpha} c_1$ for some $j \in \{1, \dots, k\}$. Inductively, we define a β-path of $M_\beta^{(i)}$ to be of degree $d(p) = d+1$ if the source of p is also the source of an α-path of $Q_{\mathcal{L}_i}$ with sink on a β-path q of $M_\beta^{(i)}$ with degree $d(q) = d$. Observe that we may have in $M_\beta^{(i)}$ several paths of the same nonzero degree.

We define the required reflection sequence of sinks $i_1, \dots, i_p$ of Q_B related with the branch $\mathcal{L}_i$ as follows. We start with the reflection sequence of sinks $i_1, \dots, i_k$ given by the unique β-path in $M_\beta^{(i)}$ of degree 0. Consider next all β-paths $p_1, \dots, p_r$ in $M_\beta^{(i)}$ of degree 1 (if such paths exist), in an arbitrary order. For each $j \in \{1, \dots, r\}$, let $i_1^{(j)}, \dots, i_{l_j}^{(j)}$ be the reflection sequence of sinks associated to the β-path p_j. Then we complete $i_1, \dots, i_k$ to a reflection sequence of sinks of Q_B as follows:

$$i_1, \dots, i_k, i_1^{(1)}, \dots, i_{l_1}^{(1)}, i_1^{(2)}, \dots, i_{l_2}^{(2)}, \dots, i_1^{(r)}, \dots, i_{l_r}^{(r)}.$$

Next we complete this reflection sequence of sinks by the segments of reflection sequences given by all β-paths in $M_\beta^{(i)}$ of degree 2, in an arbitrary order (if $M_\beta^{(i)}$ admits paths of degree 2). Inductively, for $d \geq 2$, if a reflection sequence of sinks given by the segments of reflection sequences of β-paths in $M_\beta^{(i)}$ of degree at most d is defined, we complete it by the segments of reflection sequences in $M_\beta^{(i)}$ of degree $d+1$ (if $M_\beta^{(i)}$ admits paths of degree $d+1$). Summing up, we obtain a reflection sequence of sinks $i_1, \dots, i_p$ of Q_B given by the reflection sequence of sinks of all β-paths in $M_\beta^{(i)}$. Hence p is the number of common vertices of $\Sigma_B(\lambda)$ and $Q_{\mathcal{L}_i}$, and the iterated extension $T^+_{i_1,\dots,i_p} B$ has the required property. We also note that the iterated reflection $S^+_{i_p} \dots S^+_{i_1} B$ of B is of the form

$$[E_1, \mathcal{L}_1, \dots, E_{i-1}, \mathcal{L}_{i-1}, E_{i+1}, \mathcal{L}_{i+1}, \dots E_s, \mathcal{L}_s] C [E_i, S^+_{i_p} \dots S^+_{i_1} \mathcal{L}_i],$$

hence is obtained from the branch $\mathcal{T}^C$-coextension $B = [E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s]C$ of C by replacing the branch $\mathcal{T}^C$-coextension part $[E_i, \mathcal{L}_i]C$ by a branch $\mathcal{T}^C$-extension part $C[E_i, S^+_{i_p} \dots S^+_{i_1} \mathcal{L}_i]$, where $S^+_{i_p} \dots S^+_{i_1} \mathcal{L}_i$ is the branch obtained from $\mathcal{L}_i$ by the reflections at the vertices $i_1, \dots, i_p$, and hence $\nu(i_1)$ is the extension vertex of the one-point extension $C[E_i]$ inside $S^+_{i_p} \dots S^+_{i_1} B$.

In general, the tube $\mathcal{T}_\lambda^C$ may contain several mouth modules E_i involved in the branch $\mathcal{T}^C$-coextension $B = [E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s]C$. Applying the above procedures to all modules E_i belonging to $\mathcal{T}_\lambda^C$ and the connected branches $\mathcal{L}_i$, we obtain segments of independent reflection sequences of sinks, which collected together form a reflection sequence of sinks $i_1, \dots, i_p, \dots, i_r$ such that, after the iteration extension $T^+_{i_1,\dots,i_r}B$ of B, the standard coray tube $\mathcal{T}^B_\lambda$ is transformed into a standard quasi-tube $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$ of $\Gamma_{T^+_{i_1,\dots,i_r}B}$ whose indecomposable projective-injective $T^+_{i_1,\dots,i_r}B$-modules are the modules

$$I_{T^+_{i_1,\dots,i_r}B}(i_1) = P_{T^+_{i_1,\dots,i_r}B}(\nu(i_1)), \dots, I_{T^+_{i_1,\dots,i_r}B}(i_r) = P_{T^+_{i_1,\dots,i_r}B}(\nu(i_r)),$$

that is, the modules $I_{T^+_{i_1,\dots,i_r}B}(j) = P_{T^+_{i_1,\dots,i_r}B}(\nu(j))$ for all vertices $j \in \Sigma_B(\lambda)$. Moreover, the quasi-tube $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$ is obtained from the coray tube $\mathcal{T}^B_\lambda$ by r infinite rectangle insertions, corresponding to the r one-point extensions leading from B to $T^+_{i_1,\dots,i_r}B$. In particular, we conclude that all modules of the coray tube $\mathcal{T}^B_\lambda$ lie in the quasi-tube $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$. Observe also that the number of corays of the coray tube $\mathcal{T}^B_\lambda$ equals the number of corays of the stable part of the quasi-tube $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$. Hence, applying Proposition 2.2(ii), we infer that $s(\mathcal{T}^B_\lambda) + i^*(\mathcal{T}^B_\lambda) + 1$ is the rank $r(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B})$ of the stable tube associated to $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$. Further, in the iterated transformation of the coray tube $\mathcal{T}^B_\lambda$ into the quasi-tube $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$ no new simple modules are created, and so $s(\mathcal{T}^B_\lambda) = s(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B})$. Finally, observe that $i^*(\mathcal{T}^B_\lambda)$ is exactly the number of indecomposable projective-injective $T^+_{i_1,\dots,i_r}B$-modules in $\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}$. Therefore,

$$s(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}) + p(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}) = r(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}) - 1.$$

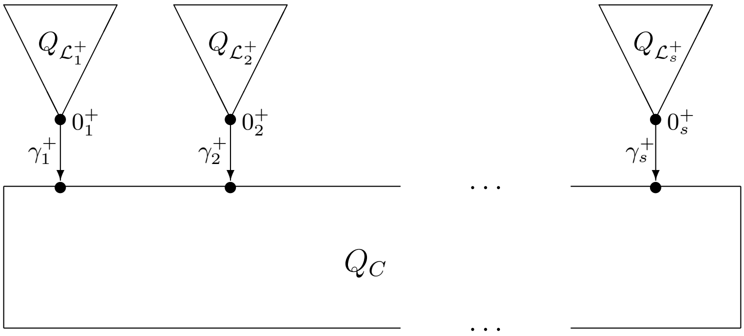
We also note that

$$r(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B}) = r(\mathcal{T}^C_\lambda) + \sum_{E_i \in \mathcal{T}^C_\lambda} |\mathcal{L}_i|,$$

where $|\mathcal{L}_i|$ denotes the capacity of the branch $\mathcal{L}_i$. Indeed, $i^*(\mathcal{T}^B_\lambda) = p(\mathcal{C}_\lambda^{T^+_{i_1,\dots,i_r}B})$ is the number of vertices of all branches $\mathcal{L}_i$ with $E_i \in \mathcal{T}^C_\lambda$ which are sinks of arrows β, including the coextension vertices of $[E_i]C$, while $s(\mathcal{T}^B_\lambda) - s(\mathcal{T}^C_\lambda) = s(\mathcal{T}^B_\lambda) - r(\mathcal{T}^C_\lambda) + 1$ is the number of vertices of all branches $\mathcal{L}_i$ with $E_i \in \mathcal{T}^C_\lambda$ which are sources of arrows α.

Applying the above considerations to all standard coray tubes $\mathcal{T}^B_\lambda$ with $\lambda \in \Lambda_B$, we find a reflection sequence of sinks $i_1, \dots, i_t$ of Q_B such that after the iterated extension $B^* = T^+_{i_1,\dots,i_t}B$ of B, the standard coray tubes $\mathcal{T}^B_\lambda$ of Γ_B, $\lambda \in \Lambda_B$, are transformed into standard quasi-tubes $\mathcal{C}^{B^*}_\lambda$ of Γ_{B^*}, $\lambda \in \Lambda$, while the standard stable tubes $\mathcal{T}^B_\lambda = \mathcal{T}^C_\lambda$, $\lambda \in \mathbb{P}_1(K) \setminus \Lambda_B$, remain standard stable tubes of Γ_{B^*}. In particular, we have $s(\mathcal{C}^{B^*}_\lambda) + p(\mathcal{C}^{B^*}_\lambda) = r(\mathcal{C}^{B^*}_\lambda) - 1$ for any $\lambda \in \mathbb{P}_1(K)$. Moreover, the iterated reflection algebra $B^+ = S_{i_t}^+ \dots S_{i_1}^+ B$ of B is the branch $\mathcal{T}^C$-extension

$$S_{i_t}^+ \dots S_{i_1}^+ B = C[E_1, \mathcal{L}_1^+, \dots, E_s, \mathcal{L}_s^+]$$



where the branches $\mathcal{L}_1^+, \dots, \mathcal{L}_s^+$ are obtained from the branches $\mathcal{L}_1, \dots, \mathcal{L}_s$ by the corresponding reflections at some vertices $i_1, \dots, i_t$, as described above. Finally, applying [6, Theorem 4.1], we conclude that the Auslander–Reiten quiver $\mathit{\Gamma}_{B^*}$ of B^* is of the form

$$\mathit{\Gamma}_{B^*} = \mathcal{P}^{B^*} \vee \mathcal{C}^{B^*} \vee \mathcal{Q}^{B^*},$$

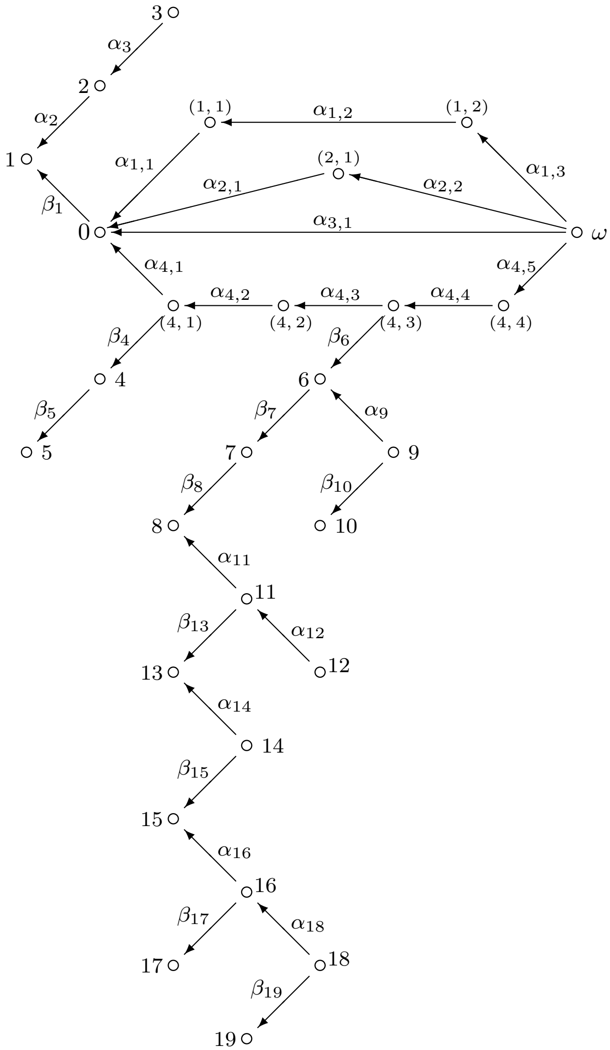
where $\mathcal{P}^{B^*} = \mathcal{P}^B$ is a family of components containing all indecomposable projective B-modules, $\mathcal{Q}^{B^*} = \mathcal{Q}^{B^+}$ is a family of components containing all indecomposable injective B^+-modules, and $\mathcal{C}^{B^*} = (\mathcal{C}_\lambda^{B^*})_{\lambda \in \mathbb{P}_1(K)}$ is a family of pairwise orthogonal standard quasi-tubes separating $\mathcal{P}^{B^*}$ from $\mathcal{Q}^{B^*}$ (in the sense of [6, (2.1)]).

The proof of (ii) is dual. ∎

Remark 3.2. In the terminology of [6] the algebra B^* associated (in Theorem 3.1(i)) to a branch $\mathcal{T}^C$-coextension B of a canonical algebra C is a *quasi-tube enlargement* of C, $B = B^-$ is a unique maximal branch coextension of C inside B^*, with Q_B a convex subquiver of Q_{B^*}, and $B^+ = S_{i_t}^+ \dots S_{i_1}^+ B$ is a unique maximal branch extension of C inside B^*, with Q_{B^+} a convex subquiver of Q_{B^*}. Dually, the algebra B^* associated (in Theorem 3.1(ii)) to a branch $\mathcal{T}^C$-extension B of a canonical algebra C is a *quasi-tube enlargement* of C, $B = B^+$ is a unique maximal branch extension of C inside B^+, with Q_B a convex subquiver of Q_{B^*}, and $B^- = S_{j_t}^- \dots S_{j_1}^- B$ is a unique maximal branch coextension of C inside B^*, with Q_{B^-} a convex subquiver of Q_{B^*}.

We end this section with an example illustrating the above considerations.

Example 3.3. Let $B = KQ/I$ be the bound quiver algebra given by the quiver Q of the form



and I is the ideal of the path algebra KQ of Q generated by the elements

$$\alpha_{3,1}+\alpha_{1,3}\alpha_{1,2}\alpha_{1,1}+\alpha_{2,2}\alpha_{2,1},\quad \alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}\alpha_{4,1}+\alpha_{1,3}\alpha_{1,2}\alpha_{1,1}+\mu\alpha_{2,2}\alpha_{2,1},$$

for a fixed $\mu \in K \setminus \{0,1\}$, and

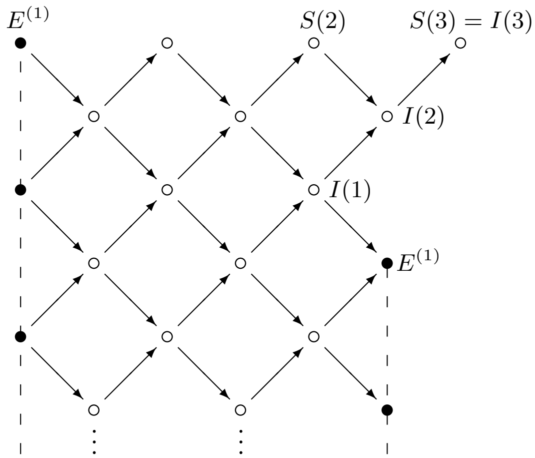
$$\alpha_{3,1}\beta_1,\ \alpha_{4,2}\beta_4,\ \alpha_{4,4}\beta_6,\ \alpha_9\beta_7,\ \alpha_{12}\beta_{13},\ \alpha_{18}\beta_{17}.$$

Let $C = KQ_C/I_C$ be the bound quiver algebra, where Q_C is the full subquiver of Q given by the vertices 0, ω, $(1,1)$, $(1,2)$, $(2,1)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$ and I_C is generated only by the first two generators of I, that is, the generators of I involving only the arrows of Q_C. Then C is a canonical algebra $C(\boldsymbol{p},\boldsymbol{\lambda})$ of type $(\boldsymbol{p},\boldsymbol{\lambda})$ with the weight sequence $\boldsymbol{p} = (3,2,1,4)$ and the parameter sequence $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, with $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$,

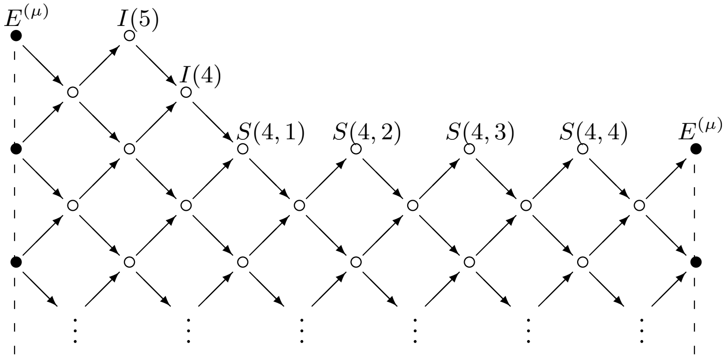
$\lambda_4 = \mu$. Then B is the branch $\mathcal{T}^C$-coextension $B = [E_1, \mathcal{L}_1, E_2, \mathcal{L}_2, E_3, \mathcal{L}_3]C$ of C, where

- $E_1 = E^{(1)}$ is the unique module lying on the mouth of the stable tube $\mathcal{T}_1^C$ of rank 1, $\mathcal{L}_1 = (Q_{\mathcal{L}_1}, I_{\mathcal{L}_1})$ is the branch with $Q_{\mathcal{L}_1}$ the full subquiver of Q given by the vertices 1, 2, 3, $I_{\mathcal{L}_1} = 0$, and $\beta_1 = \gamma_1^-$ is the arrow connecting Q_C with $Q_{\mathcal{L}_1}$;
- $E_2 = S(4,1)$ is the simple C-module at the vertex $(4,1)$, lying on the mouth of the stable tube $\mathcal{T}_\mu^C$ of rank 5, $\mathcal{L}_2 = (Q_{\mathcal{L}_2}, I_{\mathcal{L}_2})$ is the branch with $Q_{\mathcal{L}_2}$ the full subquiver of Q given by the vertices 4, 5, and $I_{\mathcal{L}_2} = 0$, and $\beta_4 = \gamma_2^-$ is the arrow connecting Q_C with $Q_{\mathcal{L}_2}$;
- $E_3 = S(4,3)$ is the simple C-module at the vertex $(4,3)$, lying on the mouth of the stable tube $\mathcal{T}_\mu^C$ of rank 5, $\mathcal{L}_3 = (Q_{\mathcal{L}_3}, I_{\mathcal{L}_3})$ is the branch with $Q_{\mathcal{L}_3}$ the full subquiver of Q given by the vertices $6, 7, 8, \dots, 18, 19$, and $I_{\mathcal{L}_3}$ is the ideal of $KQ_{\mathcal{L}_3}$ generated by the paths $\alpha_9\beta_7$, $\alpha_{12}\beta_{13}$, $\alpha_{18}\beta_{17}$, and $\beta_6 = \gamma_3^-$ is the arrow connecting Q_C with $Q_{\mathcal{L}_3}$.

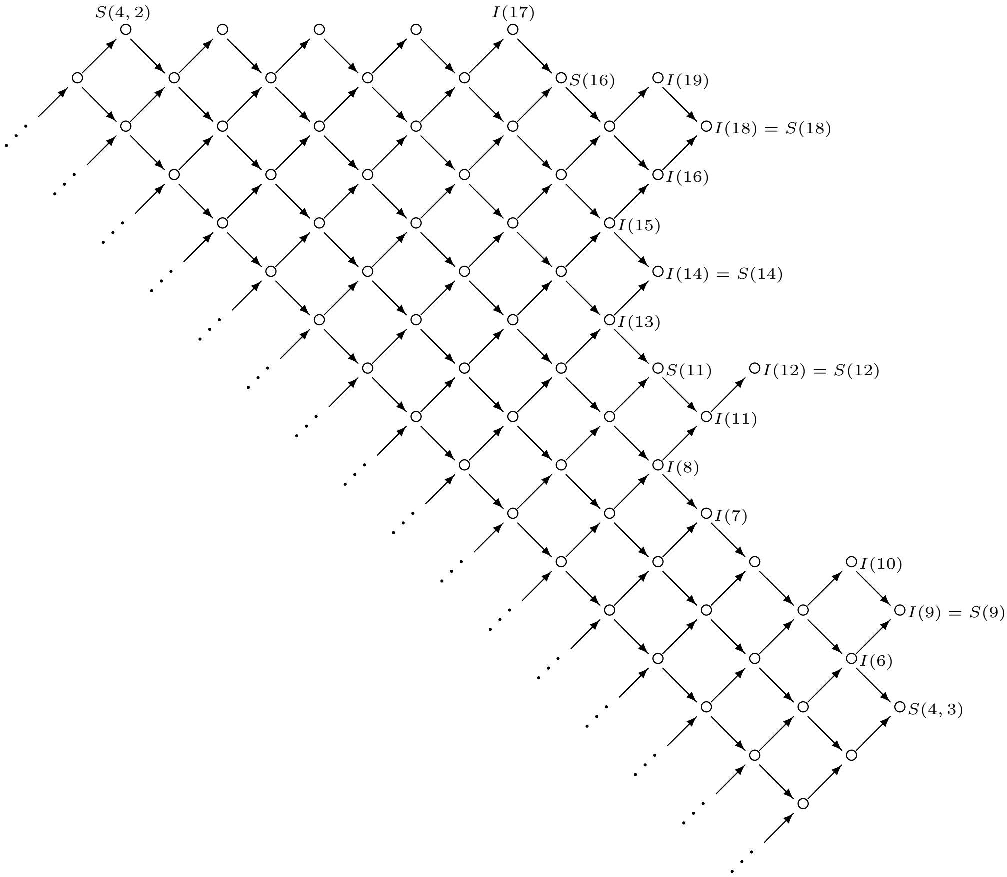
Then the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes of Γ_B is described as follows. Since E_1 lies in $\mathcal{T}_1^C$ and E_2, E_3 lie in $\mathcal{T}_\mu^C$, we have $\mathcal{T}_\lambda^B = \mathcal{T}_\lambda^C$ (hence it is a stable tube) for $\lambda \in \mathbb{P}_1(K) \setminus \{1, \mu\}$. The coray tube $\mathcal{T}_1^B$ is obtained from the stable tube $\mathcal{T}_1^C$ (of rank 1) by insertion of three corays and looks as follows:



where the corresponding vertices along the dashed lines have to be identified, and $S(2) = S_B(2)$, $S(3) = S_B(3)$, $I(1) = I_B(1)$, $I(2) = I_B(2)$, $I(3) = I_B(3)$. The coray tube $\mathcal{T}_\mu^B$ is obtained from the stable tube $\mathcal{T}_\mu^C$ (of rank 5), by insertion of 16 corays, obtained from the coray tube $\mathcal{T}_\mu^{[E_2, \mathcal{L}_2]C}$ of the branch $\mathcal{T}^C$-coextension $[E_2, \mathcal{L}_2]C$



by removing the arrows connecting the vertices on the two corays ending at the vertices $S(4,2)$ and $S(4,3)$, and inserting between these two corays the translation quiver of the form



We now indicate a reflection sequence of sinks $i_1, \dots, i_t$ of Q_B leading to the quasi-tube enlargement $B^* = T^+_{i_1,\dots,i_t} B$ of C and the branch $\mathcal{T}^C$-extension $B^+ = S^+_{i_t} \dots S^+_{i_1} B$ of C, according to the proof of Theorem 3.1(i).

(1) For the branch $\mathcal{L}_1$, the set $M^{(1)}_\beta$ of all maximal β-paths of $Q_{\mathcal{L}_1}$ consists of one arrow β_1, and hence the reflection sequence of sinks given by $M^{(1)}_\beta$ reduces to $i_1 = 1$.

(2) For the branch $\mathcal{L}_2$, the set $M^{(2)}_\beta$ consists of the path $(4,1) \xrightarrow{\beta_4} 4 \xrightarrow{\beta_5} 5$ of degree 0, and hence we have a unique reflection sequence of sinks $i_2 = 5$, $i_3 = 4$, associated to $M^{(2)}_\beta$.

(3) For the branch $\mathcal{L}_3$, the set $M^{(3)}_\beta$ consists of the paths

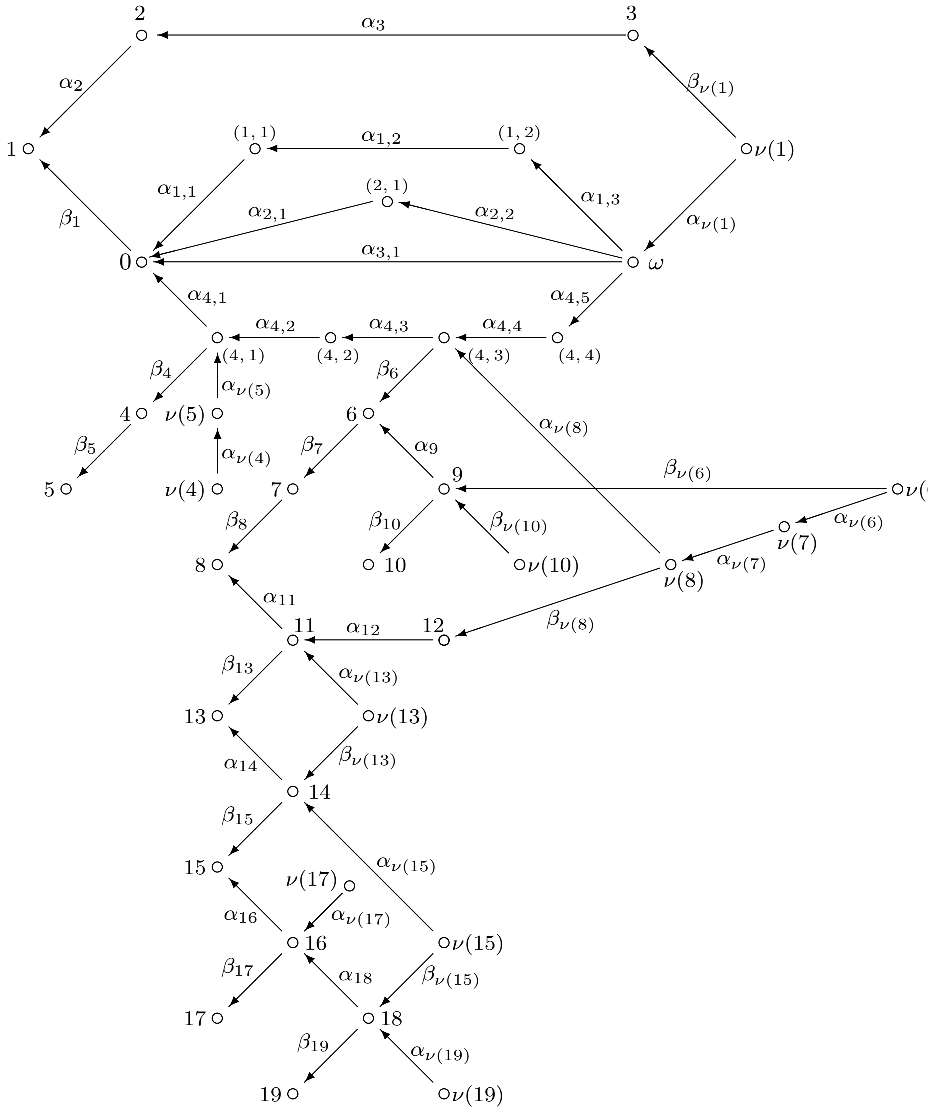
$$
\begin{array}{rl}
(4,3) \xrightarrow{\beta_6} 6 \xrightarrow{\beta_7} 7 \xrightarrow{\beta_8} 8 & \quad \text{(of degree 0);} \\
9 \xrightarrow{\beta_{10}} 10, \quad 11 \xrightarrow{\beta_{13}} 13 & \quad \text{(of degree 1);} \\
14 \xrightarrow{\beta_{15}} 15 \qquad\qquad\quad & \quad \text{(of degree 2);} \\
16 \xrightarrow{\beta_{17}} 17, \quad 18 \xrightarrow{\beta_{19}} 19 & \quad \text{(of degree 3).}
\end{array}
$$

Then as a reflection sequence of sinks associated to $M^{(3)}_\beta$ we may take $i_4 = 8$, $i_5 = 7$, $i_6 = 6$, $i_7 = 10$, $i_8 = 13$, $i_9 = 15$, $i_{10} = 17$, $i_{11} = 19$. (We note that interchanging 10 with 13, or 17 with 19, gives another admissible sequence of sinks associated to $M^{(3)}_\beta$.)

Therefore, $i_1 = 1$, $i_2 = 5$, $i_3 = 4$, $i_4 = 8$, $i_5 = 7$, $i_6 = 6$, $i_7 = 10$, $i_8 = 13$, $i_9 = 15$, $i_{10} = 17$, $i_{11} = 19$ is a required reflection sequence of sinks of Q_B, and so $t = 11$.

The iterated extension $B^* = T^+_{i_1,\dots,i_{11}} B$ is the bound quiver algebra $B^* = KQ_{B^*}/I_{B^*}$, where Q_{B^*} is the quiver on the page opposite and I_{B^*} is the ideal in the path algebra KQ_{B^*} of Q_{B^*} generated by the elements

$$
\begin{aligned}
&\alpha_{3,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1}, \\
&\alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}\alpha_{4,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \mu\alpha_{2,2}\alpha_{2,1}, \\
&\alpha_{3,1}\beta_1,\ \alpha_{4,2}\beta_4,\ \alpha_{4,4}\beta_6,\ \alpha_9\beta_7,\ \alpha_{12}\beta_{13},\ \alpha_{18}\beta_{17}, \\
&\alpha_{\nu(1)}\alpha_{1,3}\alpha_{1,2}\alpha_{1,1}\beta_1 - \beta_{\nu(1)}\alpha_3\alpha_2,\ \alpha_{\nu(5)}\alpha_{4,1},\ \alpha_{\nu(4)}\alpha_{\nu(5)}\beta_4\beta_5, \\
&\beta_{\nu(8)}\alpha_{12}\alpha_{11} - \alpha_{\nu(8)}\beta_6\beta_7\beta_8,\ \alpha_{\nu(8)}\alpha_{4,3},\ \alpha_{\nu(7)}\beta_{\nu(8)}, \\
&\beta_{\nu(6)}\alpha_9 - \alpha_{\nu(6)}\alpha_{\nu(7)}\alpha_{\nu(8)}\beta_6,\ \beta_{\nu(13)}\alpha_{14} - \alpha_{\nu(13)}\beta_{13}, \\
&\alpha_{\nu(13)}\beta_{15},\ \beta_{\nu(13)}\beta_{15}, \\
&\beta_{\nu(15)}\alpha_{18}\alpha_{16} - \alpha_{\nu(15)}\beta_{15},\ \beta_{\nu(15)}\beta_{19},\ \alpha_{\nu(17)}\alpha_{16},\ \alpha_{\nu(19)}\alpha_{18}.
\end{aligned}
$$

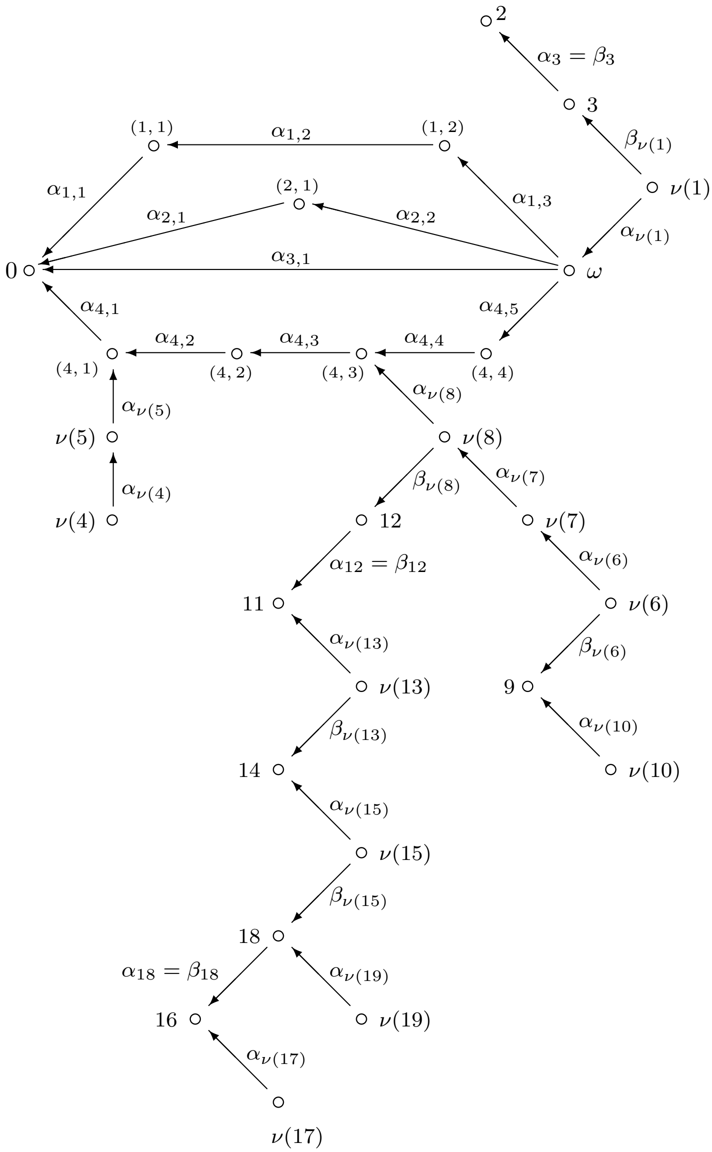


The iterated reflection $B^+ = S^+_{i_{11}} \dots S^+_{i_1} B$ is the bound quiver algebra $B^+ = KQ_{B^+}/I_{B^+}$, where Q_{B^+} is the quiver on the next page and I_{B^+} is the ideal in the path algebra KQ_{B^+} of Q_{B^+} generated by the elements

$$\alpha_{3,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1},$$
$$\alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}\alpha_{4,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \mu\alpha_{2,2}\alpha_{2,1},$$
$$\alpha_{\nu(1)}\alpha_{3,1},\ \alpha_{\nu(5)}\alpha_{4,1},\ \alpha_{\nu(8)}\alpha_{4,3},\ \alpha_{\nu(7)}\beta_{\nu(8)},\ \alpha_{\nu(19)}\beta_{18}.$$

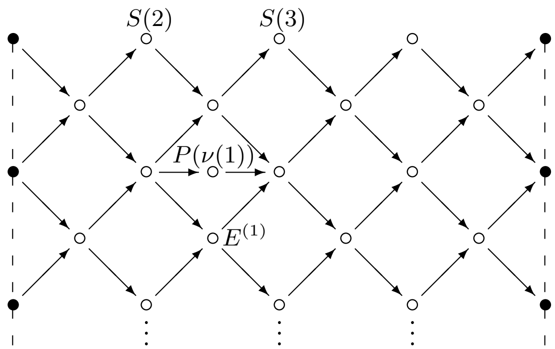
Therefore, B^+ is the branch $\mathcal{T}^C$-extension

$$B^+ = C[E_1, \mathcal{L}_1^+, E_2, \mathcal{L}_2^+, E_3, \mathcal{L}_3^+]$$



of C, where $\mathcal{L}_1^+ = (Q_{\mathcal{L}_1^+}, I_{\mathcal{L}_1^+})$ is the branch with $Q_{\mathcal{L}_1^+}$ the full subquiver of Q_{B^+} given by the vertices $\nu(1), 3, 2$, and $I_{\mathcal{L}_1^+} = 0$; $\mathcal{L}_2^+ = (Q_{\mathcal{L}_2^+}, I_{\mathcal{L}_2^+})$ is the branch with $Q_{\mathcal{L}_2^+}$ the full subquiver of Q_{B^+} given by the vertices $\nu(4), \nu(5)$, and $I_{\mathcal{L}_2^+} = 0$; $\mathcal{L}_3^+ = (Q_{\mathcal{L}_3^+}, I_{\mathcal{L}_3^+})$ is the branch with $Q_{\mathcal{L}_3^+}$ the full translation subquiver of Q_{B^+} given by the vertices $\nu(8)$, $\nu(7)$, $\nu(6)$, 9, $\nu(10)$, 12, 11, $\nu(13)$, 14, $\nu(15)$, 18, $\nu(19)$, 16, $\nu(17)$, and $I_{\mathcal{L}_3^+}$ is the ideal of the path algebra $KQ_{\mathcal{L}_3^+}$ of $Q_{\mathcal{L}_3^+}$ generated by $\alpha_{\nu(7)}\beta_{\nu(8)}$, $\alpha_{\nu(19)}\beta_{18}$.

The $\mathbb{P}_1(K)$-family $\mathcal{C}^{B^*} = (\mathcal{C}_\lambda^{B^*})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes is as follows. For $\lambda \in \mathbb{P}_1(K) \setminus \{1, \mu\}$, we have $\mathcal{C}_\lambda^{B^*} = \mathcal{T}_\lambda^B = \mathcal{T}_\lambda^C$ (a stable tube). The coray tube $\mathcal{T}_1^B$ of Γ_B is transformed into a quasi-tube $\mathcal{C}_1^{B^*}$ of Γ_{B^*} of the form

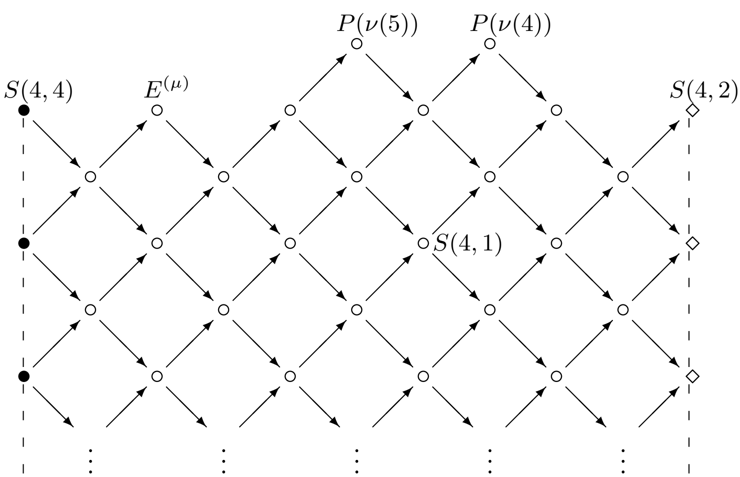


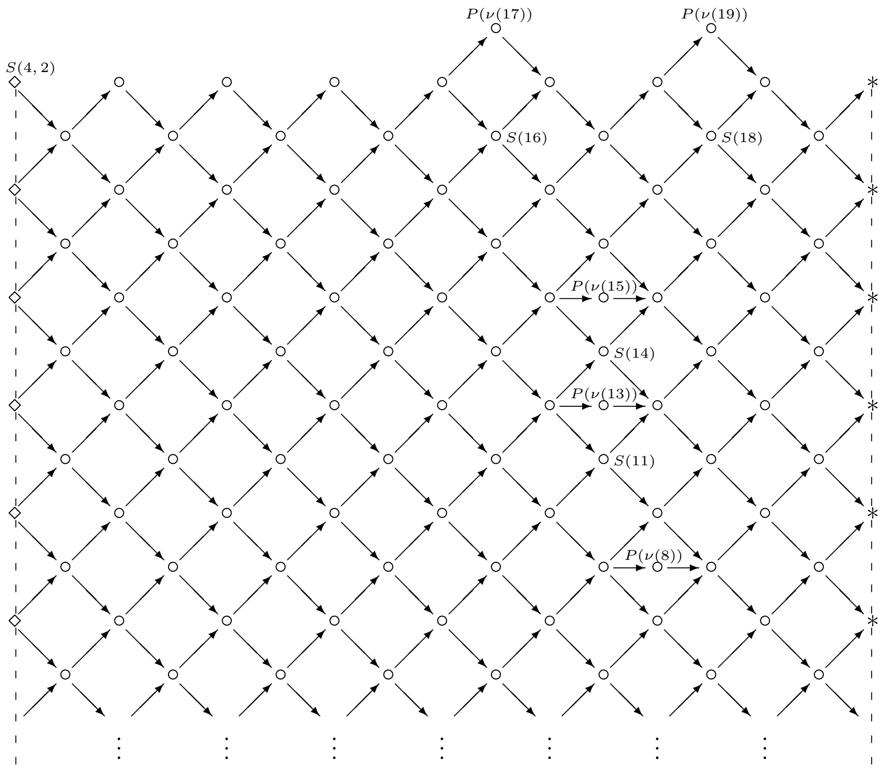
where the corresponding vertices (marked by •) along the dashed lines have to be identified. Observe that

$$s(\mathcal{C}_1^{B^*}) + p(\mathcal{C}_1^{B^*}) + 1 = 2 + 1 + 1 = 4$$

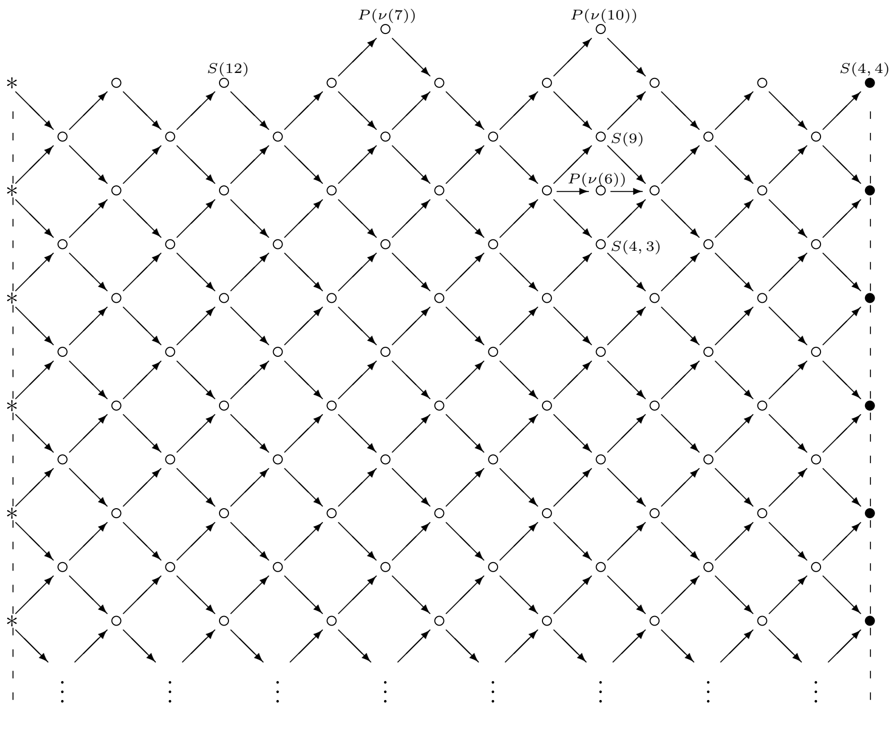
is the rank $r(\mathcal{C}_1^{B^*})$ of the stable tube $(\mathcal{C}_1^{B^*})^s$ associated to $\mathcal{C}_1^{B^*}$.

The coray tube $\mathcal{T}_\mu^B$ of Γ_B is transformed into a quasi-tube $\mathcal{C}_\mu^{B^*}$ of Γ_{B^*}, which is obtained by glueing the following translation quivers along the dashed lines passing through vertices marked by •, ⋄, ∗, respectively:





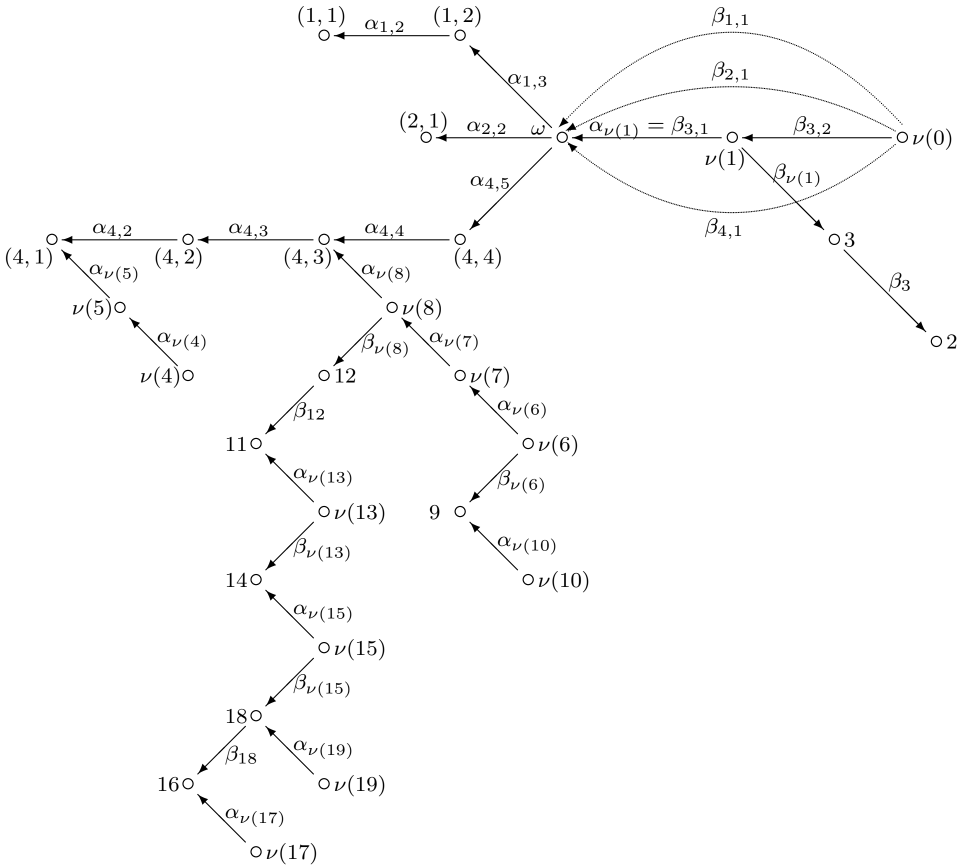
P(ν(17))
P(ν(19))
S(4, 2)
S(16)
S(18)
P(ν(15))
S(14)
P(ν(13))
S(11)
P(ν(8))



P(ν(7))
P(ν(10))
S(12)
S(4, 4)
S(9)
P(ν(6))
S(4, 3)

Observe that $s(\mathcal{C}_\mu^{B^*}) = 10$, $p(\mathcal{C}_\mu^{B^*}) = 10$ and $s(\mathcal{C}_\mu^{B^*}) + p(\mathcal{C}_\mu^{B^*}) + 1 = 21$ is the rank $r(\mathcal{C}_\mu^{B^*})$ of the stable tube $(\mathcal{C}_\mu^{B^*})^s$ associated to $\mathcal{C}_\mu^{B^*}$.

We now claim that the reflection $B_1 = S_0^+ B^+$ of B^+ at the sink 0 of Q_{B^+} is again a branch $\mathcal{T}^{C_1}$-coextension of a canonical algebra C_1. Indeed, $B_1 = KQ_{B_1}/I_{B_1}$, where Q_{B_1} is the quiver



and I_{B_1} is the ideal in the path algebra KQ_{B_1} of Q_{B_1} generated by the elements

$$\beta_{3,2}\beta_{3,1} + \beta_{1,1} + \beta_{2,1},\ \beta_{4,1} + \beta_{1,1} + \mu\beta_{2,1},$$
$$\beta_{1,1}\alpha_{1,3},\ \beta_{2,1}\alpha_{2,2},\ \beta_{3,2}\beta_{\nu(1)},\ \beta_{4,1}\alpha_{4,5},\ \alpha_{\nu(8)}\alpha_{4,3},\ \alpha_{\nu(7)}\beta_{\nu(8)},\ \alpha_{\nu(19)}\beta_{18}.$$

Then the bound quiver algebra $C_1 = KQ_{C_1}/I_{C_1}$, where Q_{C_1} is the full subquiver of Q_{B_1} given by the vertices ω, $\nu(0)$ and $\nu(1)$, and I_{C_1} is the ideal in KQ_{C_1} generated by the elements $\beta_{3,2}\beta_{3,1} + \beta_{1,1} + \beta_{2,1}$, $\beta_{4,1} + \beta_{1,1} + \mu\beta_{2,1}$, is a canonical algebra of type $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{\lambda}})$ with the weight sequence $\overline{\boldsymbol{p}} = (1, 1, 2, 1)$ and the parameter sequence $\overline{\boldsymbol{\lambda}} = (\infty, 0, 1, \mu)$. Moreover, B_1 is the branch $\mathcal{T}^{C_1}$-coextension $[\overline{E}_1, \overline{\mathcal{L}}_1, \overline{E}_2, \overline{\mathcal{L}}_2, \overline{E}_3, \overline{\mathcal{L}}_3, \overline{E}_4, \overline{\mathcal{L}}_4]C_1$, where

- $\overline{E}_1 = E^{\infty}$ is the unique module on the mouth of the stable tube $\mathcal{T}_{\infty}^{C_1}$ of rank 1, $\overline{\mathcal{L}}_1 = (Q_{\overline{\mathcal{L}}_1}, I_{\overline{\mathcal{L}}_1})$ is the branch with $Q_{\overline{\mathcal{L}}_1}$ the full subquiver of Q_{B_1} given by the vertices $(1,1)$, $(1,2)$, and $I_{\overline{\mathcal{L}}_1} = 0$;
- $\overline{E}_2 = E^{0}$ is the unique module on the mouth of the stable tube $\mathcal{T}_{0}^{C_1}$ of rank 1, $\overline{\mathcal{L}}_2 = (Q_{\overline{\mathcal{L}}_2}, I_{\overline{\mathcal{L}}_2})$ is the branch with $Q_{\overline{\mathcal{L}}_2}$ given by the vertex $(2,1)$, and hence $I_{\overline{\mathcal{L}}_2} = 0$;
- $\overline{E}_3 = S(\nu(1))$ is the simple C_1-module lying on the mouth of the stable tube $\mathcal{T}_{1}^{C_1}$ of rank 2, $\overline{\mathcal{L}}_3 = (Q_{\overline{\mathcal{L}}_3}, I_{\overline{\mathcal{L}}_3})$ is the branch with $Q_{\overline{\mathcal{L}}_3}$ the full subquiver of Q_{B_1} given by the vertices 2, 3, and $I_{\overline{\mathcal{L}}_3} = 0$;
- $\overline{E}_4 = E^{(\mu)}$ is the unique module on the mouth of the stable tube $\mathcal{T}_{\mu}^{C_1}$ of rank 1, $\overline{\mathcal{L}}_4 = (Q_{\overline{\mathcal{L}}_4}, I_{\overline{\mathcal{L}}_4})$ is the branch with $Q_{\overline{\mathcal{L}}_4}$ the full subquiver of Q_{B_1} given by the vertices $(4,4)$, $(4,3)$, $(4,2)$, $(4,1)$, $\nu(5)$, $\nu(4)$, $\nu(8)$, $\nu(7)$, $\nu(6)$, 9, $\nu(10)$, 12, 11, $\nu(13)$, 14, $\nu(15)$, 18, $\nu(19)$, 16, $\nu(17)$, and $I_{\overline{\mathcal{L}}_4}$ is the ideal of $KQ_{\overline{\mathcal{L}}_4}$ generated by $\alpha_{\nu(8)}\alpha_{4,3}$, $\alpha_{\nu(7)}\beta_{\nu(8)}$, $\alpha_{\nu(19)}\beta_{18}$.

Consider the set of vertices of Q_{B_1}: $j_1 = (1,1)$, $j_2 = (1,2)$, $j_3 = (2,1)$, $j_4 = 2$, $j_5 = 3$, $j_6 = (4,1)$, $j_7 = (4,2)$, $j_8 = (4,3)$, $j_9 = (4,4)$, $j_{10} = 9$, $j_{11} = 11$, $j_{12} = 12$, $j_{13} = 14$, $j_{14} = 16$, $j_{15} = 18$. Then $j_1, \dots, j_{15}$ is a reflection sequence of sinks of Q_{B_1} associated to the branch $\mathcal{T}^{C_1}$-coextension B_1 of C_1, according to the rule presented in the proof of Theorem 3.1(i), and hence the iterated reflection $S_{j_{15}}^{+} \dots S_{j_1}^{+} B_1$ is a branch $\mathcal{T}^{C_1}$-extension B_1^{+} of C_1. Moreover, the reflection $S_{\omega}^{+} B_1^{+}$ at the sink ω of $Q_{B_1^{+}}$ is isomorphic to B.

Therefore, $i_1, \dots, i_{11}, 0, j_1, \dots, j_{15}, \omega$ is a reflection sequence of sinks of Q_B, exhausting all 28 vertices of Q_B, such that $S_{\omega}^{+} S_{j_{15}} \dots S_{j_1}^{+} S_0^{+} S_{i_{11}} \dots S_{i_1} B$ is isomorphic to B.

4. Selfinjective orbit algebras. In this section we recall the needed background on selfinjective orbit algebras.

Let B be an algebra and $\mathcal{E}_B = \{e_i \mid 1 \le i \le n\}$ be a fixed set of orthogonal primitive idempotents of B with $1_B = e_1 + \dots + e_n$. Then we have the associated canonical set $\hat{\mathcal{E}}_B = \{e_{m,i} \mid m \in \mathbb{Z}, 1 \le i \le n\}$ of orthogonal primitive idempotents of the repetitive algebra $\hat{B}$ of B such that $e_{m,1} + \dots + e_{m,n}$ is the identity of B_m, and $\nu_{\hat{B}}(e_{m,i}) = e_{m+1,i}$ for any $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$. By an *automorphism* of $\hat{B}$ we mean a K-linear algebra automorphism φ of $\hat{B}$ preserving the set $\hat{\mathcal{E}}_B$. An automorphism φ of $\hat{B}$ is said to be

- *positive* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \ge m$ and some $j \in \{1, \dots, n\}$;
- *rigid* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{m,j}$ for some $j \in \{1, \dots, n\}$;
- *strictly positive* if it is positive but not rigid.

Observe that the Nakayama automorphism $\nu_{\hat{B}}$ is a strictly positive automorphisms of $\hat{B}$. A group G of automorphisms of $\hat{B}$ is said to be *admissible* if it acts freely on the set $\hat{\mathcal{E}}_B$ and has finitely many orbits. We may identify the algebra B with a finite K-category B whose objects are elements of $\mathcal{E}_B$, the morphism spaces are defined by $B(e_i, e_j) = e_j B e_i$ for all $i, j \in \{1, \dots, n\}$, and the composition of morphisms is given by the multiplication in B. Similarly, we consider the repetitive algebra $\hat{B}$ of B as a K-category with the objects the set $\hat{\mathcal{E}}_B$, the morphism spaces defined by

$$\hat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i, & r = m, \\ D(e_i B e_j), & r = m+1, \\ 0, & \text{otherwise}, \end{cases}$$

and the composition of morphisms given by multiplication in B and the canonical B-B-bimodule structure of $D(B) = \mathrm{Hom}_K(B, K)$. Then an automorphism of the repetitive algebra $\hat{B}$ is just an automorphism of the K-category $\hat{B}$. Moreover, an admissible group of automorphisms of $\hat{B}$ is a group G of automorphisms of the K-category $\hat{B}$ acting freely on the set $\hat{\mathcal{E}}_B$ of objects of $\hat{B}$ and having finitely many orbits. We refer to [32] for more information on automorphisms of repetitive algebras (categories).

Let B be an algebra and G be an admissible group of automorphisms of $\hat{B}$. Following Gabriel [20] we may consider the finite orbit K-category $\hat{B}/G$ defined as follows. The objects of $\hat{B}/G$ are the elements $a = Gx$ of the set $\hat{\mathcal{E}}_B/G$ of G-orbits in $\hat{\mathcal{E}}_B$ and the morphism spaces are given by

$$(\hat{B}/G)(a,b) = \Big\{ (f_{y,x}) \in \prod_{(x,y) \in a \times b} \hat{B}(x,y) \,\Big|\, g \cdot f_{y,x} = f_{gy,gx} \text{ for all } g \in G,\ x \in a,\ y \in b \Big\},$$

for all objects a, b of $\hat{B}/G$. Then we have a canonical *Galois covering functor* $F : \hat{B} \to \hat{B}/G$ which assigns to each object x of $\hat{B}$ its G-orbit Gx, and, for any objects x of $\hat{B}$ and a of $\hat{B}/G$, F induces natural K-linear isomorphisms

$$\bigoplus_{y \in \hat{\mathcal{E}}_B,\, Fy = a} \hat{B}(x,y) \xrightarrow{\sim} (\hat{B}/G)(Fx, a), \qquad \bigoplus_{y \in \hat{\mathcal{E}}_B,\, Fy = a} \hat{B}(y,x) \xrightarrow{\sim} (\hat{B}/G)(a, Fx).$$

The finite-dimensional algebra $\bigoplus_{a,b \in \hat{\mathcal{E}}/G} (\hat{B}/G)(a,b)$ associated to the orbit category $\hat{B}/G$ is a selfinjective algebra, denoted by $\hat{B}/G$ and called an *orbit algebra* of $\hat{B}$, with respect to the admissible automorphism group G of $\hat{B}$. The group G also acts on the category $\mathrm{mod}\, \hat{B}$ of right $\hat{B}$-modules (identified with contravariant functors from $\hat{B}$ to $\mathrm{mod}\, K$ with finite support) by $gM = M \circ g^{-1}$ for any $M \in \mathrm{mod}\, \hat{B}$ and $g \in G$. Further, we have the *push-down functor* $F_\lambda : \mathrm{mod}\, \hat{B} \to \mathrm{mod}\, \hat{B}/G$ such that $F_\lambda(M)(a) = \bigoplus_{x \in a} M(x)$ for a module M in $\mathrm{mod}\, \hat{B}$ and an object a of $\hat{B}/G$.

The following theorem is a consequence of [20, Lemma 3.5, Theorem 3.6].

THEOREM 4.1. *Let B be an algebra and G a torsion-free admissible group of K-linear automorphisms of $\hat{B}$. Then*

(i) *The push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G$ induces an injection from the set of G-orbits of isomorphism classes of indecomposable modules in $\operatorname{mod} \hat{B}$ into the set of isomorphism classes of indecomposable modules in $\operatorname{mod} \hat{B}/G$.*
(ii) *The push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G$ preserves the Auslander–Reiten sequences.*

In general, the push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G$ associated to a Galois covering $F : \hat{B} \to \hat{B}/G$ is not dense (see [18], [19]). Following [18], a repetitive category $\hat{B}$ is said to be *locally support-finite* if for any object x of $\hat{B}$, the full subcategory of $\hat{B}$ given by the supports $\operatorname{supp} M$ of all indecomposable modules M in $\operatorname{mod} \hat{B}$ with $M(x) \neq 0$ is finite. Here, by the *support* of a module M in $\operatorname{mod} \hat{B}$ we mean the full subcategory of $\hat{B}$ given by all objects z with $M(z) \neq 0$.

The following consequence of [19, Proposition 2.5] (see also [18, Theorem]) will be essentially applied in the next section.

THEOREM 4.2. *Let B be an algebra with locally support-finite repetitive category $\hat{B}$, and G be a torsion-free admissible group of automorphisms of $\hat{B}$. Then the push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G$ is dense. In particular, F_λ induces an isomorphism of the orbit translation quiver $\Gamma_{\hat{B}}/G$ of the Auslander–Reiten quiver $\Gamma_{\hat{B}}$ of $\hat{B}$, with respect to the action of G, and the Auslander–Reiten quiver $\Gamma_{\hat{B}/G}$ of $\hat{B}/G$.*

We end this section with information on isomorphisms of repetitive categories (algebras) of algebras.

Let B be a triangular algebra, identified with the full subcategory of $\hat{B}$ given by the objects $e_{0,k}$, $k \in \{1, \dots, n\}$. Then for any sink i (respectively, source j) of Q_B, the full subcategory of $\hat{B}$ given by the objects $e_{0,k}$, $k \in \{1, \dots, n\} \setminus \{i\}$, and $e_{1,i} = \nu_B(e_{0,i})$ (respectively, the objects $e_{0,k}$, $k \in \{1, \dots, n\} \setminus \{j\}$, and $e_{-1,j} = \nu_{\hat{B}}^{-}(e_{0,j})$) is the reflection $S_i^+ B$ of B at i (respectively, the reflection $S_j^- B$ of B at j), and we have an isomorphism of K-categories (algebras) $\hat{B} \cong \widehat{S_i^+ B}$ (respectively, $\hat{B} \cong \widehat{S_j^- B}$). In fact, we have the following general theorem (see [23]).

THEOREM 4.3. *Let B and B' be triangular algebras. The following statements are equivalent.*

(i) $\hat{B} \cong \hat{B}$.
(ii) *$B' \cong S_{i_r}^+ \dots S_{i_1}^+ B$ for a reflection sequence of sinks $i_1, \dots, i_r$ of Q_B.*

(iii) $B' \cong S^-_{j_s} \dots S^-_{j_1} B$ *for a reflection sequence of sources* $j_1, \dots, j_s$ *of* Q_B.

For an algebra B, we denote by $\underline{\operatorname{mod}}\,\hat{B}$ the stable category of $\operatorname{mod}\hat{B}$. Recall that the objects of $\underline{\operatorname{mod}}\,\hat{B}$ are the modules in $\operatorname{mod}\hat{B}$ without nonzero projective direct summands, and, for any two objects M and N in $\underline{\operatorname{mod}}\,\hat{B}$, the space $\underline{\operatorname{Hom}}_{\hat{B}}(M,N)$ of morphisms from M to N is the quotient $\operatorname{Hom}_{\hat{B}}(M,N)/P_{\hat{B}}(M,N)$, where $P_{\hat{B}}(M,N)$ is the subspace of $\operatorname{Hom}_{\hat{B}}(M,N)$ consisting of all morphisms which factorize through a projective $\hat{B}$-module. For a morphism $f \in \operatorname{Hom}_{\hat{B}}(M,N)$, the induced morphism $f + P_{\hat{B}}(M,N)$ in $\underline{\operatorname{Hom}}_{\hat{B}}(M,N)$ is denoted by $\underline{f}$. We note that the syzygy operators $\Omega_{\hat{B}}$ and $\Omega^-_{\hat{B}}$ induce two mutually inverse functors $\Omega_{\hat{B}}, \Omega^-_{\hat{B}} : \underline{\operatorname{mod}}\,\hat{B} \to \underline{\operatorname{mod}}\,\hat{B}$.

The following known fact (see [34, p. 56]) will be applied in Section 5.

Lemma 4.4. *Let M and N be two objects of $\underline{\operatorname{mod}}\,\hat{B}$, and $f : M \to N$ a nonzero morphism in* $\operatorname{mod}\hat{B}$. *Assume that f is a monomorphism or an epimorphism. Then $\underline{f}$ is a nonzero morphism in $\underline{\operatorname{mod}}\,\hat{B}$.*

5. Selfinjective algebras of strictly canonical type. In this section we describe the structure and properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type, applying results presented in Sections 3 and 4. The following theorem is crucial.

Theorem 5.1. *Let B be a branch extension* (*respectively*, *branch coextension*) *of a canonical algebra C. Then there exist algebras C_q, B^-_q, B^+_q, B^*_q and $\overline{B}_q$, $q \in \mathbb{Z}$, and a decomposition*

$$\Gamma_{\hat{B}} = \bigvee_{q\in\mathbb{Z}} (\mathcal{X}_q \vee \mathcal{C}_q)$$

of the Auslander–Reiten quiver $\Gamma_{\hat{B}}$ of $\hat{B}$ such that the following statements hold:

(i) *For each $q \in \mathbb{Z}$, $\mathcal{X}_q$ is a family of components of $\Gamma_{\hat{B}}$ containing exactly one simple $\hat{B}$-module S_q.*
(ii) *For each $q \in \mathbb{Z}$, $\mathcal{C}_q$ is a family $(\mathcal{C}_q(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes of $\Gamma_{\hat{B}}$ with $s(\mathcal{C}_q(\lambda)) + p(\mathcal{C}_q(\lambda)) = r(\mathcal{C}_q(\lambda)) - 1$ for any $\lambda \in \mathbb{P}_1(K)$.*
(iii) *For each pair $p, q \in \mathbb{Z}$ with $p < q$, we have* $\operatorname{Hom}_{\hat{B}}(\mathcal{X}_q, \mathcal{X}_p \vee \mathcal{C}_p) = 0$ *and* $\operatorname{Hom}_{\hat{B}}(\mathcal{C}_q, \mathcal{X}_p \vee \mathcal{C}_p \vee \mathcal{X}_{p+1}) = 0$.
(iv) *For each $q \in \mathbb{Z}$, C_q is a canonical algebra, B^-_q is a branch coextension of C_q, B^+_q is a branch extension of C_q, and B^*_q is a quasi-tube enlargement of C_q.*

(v) *For each* $q \in \mathbb{Z}$, C_q, B_q^-, B_q^+, B_q^* *and* $\overline{B}_q$ *are full convex subcategories of* $\hat{B}$ *with* $\hat{B}_q^- = \hat{B} = \hat{B}_q^+$, $\nu_{\hat{B}}(C_q) = C_{q+2}$, $\nu_{\hat{B}}(B_q^-) = B_{q+2}^-$, $\nu_{\hat{B}}(B_q^+) = B_{q+2}^+$, $\nu_{\hat{B}}(B_q^*) = B_{q+2}^*$, $\nu_{\hat{B}}(\overline{B}_q) = \overline{B}_{q+2}$.

(vi) *There exists a reflection sequence of sinks* $i_0, i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_{n-1} i_n$ *of* $Q_{B_0^-}$, *where* n *is the rank of* $K_0(B_0^-) = K_0(B)$, *such that* $B_0^+ = S_{i_{r-1}}^+ \ldots S_{i_0}^+ B_0^-$, $B_1^- = S_{i_r}^+ B_0^+$, $B_1^+ = S_{i_{n-1}}^+ \ldots S_{i_{r+1}}^+ B_1^-$, $B_2^- = S_{i_n}^+ B_1^+$, $B_0^* = T_{i_1,\ldots i_{r-1}}^+ B_0^-$, $\overline{B}_1 = T_{i_r}^+ B_0^+$, $B_1^* = T_{i_{r+1},\ldots i_{n-1}}^+ B_1^-$ *and* $\overline{B}_2 = T_{i_n}^+ B_1^+$.

(vii) *For each* $q \in \mathbb{Z}$, $\mathcal{C}_q$ *is the canonical* $\mathbb{P}_1(K)$*-family of quasi-tubes of* $\Gamma_{B_q^*}$, *obtained from the canonical* $\mathbb{P}_1(K)$*-family* $\mathcal{T}_q^-$ *of coray tubes of* $\Gamma_{B_q^-}$ *by infinite rectangle insertions, and from the canonical* $\mathbb{P}_1(K)$*-family* $\mathcal{T}_q^+$ *of ray tubes of* $\Gamma_{B_q^+}$ *by infinite rectangle insertions.*

(viii) *For each* $q \in \mathbb{Z}$, $\mathcal{X}_q$ *consists of indecomposable* $\overline{B}_q$*-modules.*

(ix) *For each* $q \in \mathbb{Z}$, *we have* $\nu_{\hat{B}}(\mathcal{X}_q) = \mathcal{X}_{q+2}$ *and* $\nu_{\hat{B}}(\mathcal{C}_q) = \mathcal{C}_{q+2}$.

(x) $\hat{B}$ *is locally support-finite.*

(xi) *For each* $q \in \mathbb{Z}$, $\mathrm{Hom}_{\hat{B}}(S_q, \mathcal{C}_q(\lambda)) \neq 0$ *for all* $\lambda \in \mathbb{P}_1(K)$, *and* $\mathrm{Hom}_{\hat{B}}(S_p, \mathcal{C}_q) = 0$ *for* $p \neq q$ *in* $\mathbb{Z}$.

(xii) *For each* $q \in \mathbb{Z}$, $\mathrm{Hom}_{\hat{B}}(\mathcal{C}_q(\lambda), S_{q+1}) \neq 0$ *for all* $\lambda \in \mathbb{P}_1(K)$, *and* $\mathrm{Hom}_{\hat{B}}(\mathcal{C}_q, S_p) = 0$ *for* $p \neq q+1$ *in* $\mathbb{Z}$.

(xiii) *For each* $q \in \mathbb{Z}$, *we have* $\Omega_{\hat{B}}(\mathcal{C}_{q+1}^s) = \mathcal{C}_q^s$ *and* $\Omega_{\hat{B}}(\mathcal{X}_{q+1}^s) = \mathcal{X}_q^s$.

Proof. It follows from Theorem 3.1 and Section 4 that the classes of repetitive algebras (categories) of branch extensions and branch coextensions of a fixed canonical algebra C coincide. Therefore, we may assume (without loss of generality) that B is a branch coextension of a canonical algebra C. Let $B_0^- = B$ and $C_0 = C$. Moreover, if $B = C$, we set $B_0^+ = C$, $B_0^* = C$, $\mathcal{C}_0 = \mathcal{T}^C$, $\mathcal{C}_0(\lambda) = \mathcal{T}_\lambda^C$ for any $\lambda \in \mathbb{P}_1(K)$. Assume $B \neq C$. Applying Theorem 3.1(i), we conclude that there is a reflection sequence of sinks $i_0, i_1, \ldots, i_{r-1}$ of Q_B, for some $r \geq 1$, such that the iterated reflection $B_0^+ = S_{i_{r-1}}^+ \ldots S_{i_0}^+ B_0^-$ of $B_0^- = B$ is a branch extension of $C_0 = C$ and the Auslander–Reiten quiver $\Gamma_{B_0^*}$ of the iterated extension $B_0^* = T_{i_0,\ldots,i_{r-1}}^+ B_0^-$ of $B_0^- = B$ has a decomposition

$$\Gamma_{B_0^*} = \mathcal{P}^{B_0^*} \vee \mathcal{C}^{B_0^*} \vee \mathcal{Q}^{B_0^*},$$

where $\mathcal{P}^{B_0^*} = \mathcal{P}^{B_0^-}$ is a family of components consisting of B_0^--modules and containing all indecomposable projective B_0^--modules, $\mathcal{Q}^{B_0^*} = \mathcal{Q}^{B_0^+}$ is a family of components consisting of B_0^+-modules and containing all indecomposable injective B_0^+-modules, and $\mathcal{C}^{B_0^*}$ is a $\mathbb{P}_1(K)$-family $(\mathcal{C}_\lambda^{B_0^*})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes, separating $\mathcal{P}^{B_0^*}$ from $\mathcal{Q}^{B_0^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_0^-} = (\mathcal{T}_\lambda^{B_0^-})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes of $\Gamma_{B_0^-}$ by iterated infinite rectangle insertions. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, we have $s(\mathcal{C}_\lambda^{B_0^*}) + p(\mathcal{C}_\lambda^{B_0^*}) = r(\mathcal{C}_\lambda^{B_0^*}) - 1$. Further, B_0^- is the iterated reflection $B_0^- = S^-_{\nu(i_0)} \dots S^-_{\nu(i_{r-1})} B_0^+$, and applying Theorem 3.1(ii), we infer that the $\mathbb{P}_1(K)$-family $\mathcal{C}_\lambda^{B_0^*}$ of quasi-tubes of $\Gamma_{B_0^*}$ is obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_0^+} = (\mathcal{T}_\lambda^{B_0^+})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes of $\Gamma_{B_0^+}$ by suitable iterated infinite rectangle coinsertions. We set $\mathcal{C}_0 = \mathcal{C}^{B_0^*}$ and $\mathcal{C}_0(\lambda) = \mathcal{C}_\lambda^{B_0^*}$ for $\lambda \in \mathbb{P}_1(K)$.

Since B_0^+ is a branch extension of $C_0 = C$ (trivial if $B_0^+ = C$), the unique sink of Q_C, say $i_r = 0$, is a sink of $Q_{B_0^+}$. Then we may consider the one-point extension $\overline{B}_1 = T^+_{i_r} B_0^+ = B_0^+[I(i_r)]$ of B_0^+ by the indecomposable injective B_0^+-module $I_{B_0^+}(0)$ at the vertex i_r, and the reflection $B_1^- = S^+_{i_r} B_0^+$ of B_0^+ at i_r. In this process, we create a new canonical algebra C_1 such that the extension vertex $\nu(i_r)$ of $T^+_{i_r} B_0^+$ is the unique source of Q_{C_1}, while the unique source ω of Q_C is the unique sink of Q_{C_1}. Moreover, B_1^- is a branch coextension of C_1, with respect to the canonical family $\mathcal{T}^{C_1} = (\mathcal{T}_\lambda^{C_1})_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes of Γ_{C_1}. Observe also that $\overline{B}_1$ is also the one-point coextension $[P_{B_1^-}(\nu(i_r))]B_1^-$ of B_1^- by the indecomposable projective B_1^--module $P_{B_1^-}(\nu(i_r))$ at the vertex $\nu(i_r)$. Hence, the Auslander–Reiten quiver $\Gamma_{\overline{B}_1}$ of $\overline{B}_1$ has a decomposition

$$\Gamma_{\overline{B}_1} = \mathcal{P}^{B_0^+} \vee \mathcal{T}^{B_0^+} \vee \mathcal{X}_1 \vee \mathcal{T}^{B_1^-} \vee \mathcal{Q}^{B_1^-}$$

given by canonical decompositions

$$\Gamma_{B_0^+} = \mathcal{P}^{B_0^+} \vee \mathcal{T}^{B_0^+} \vee \mathcal{Q}^{B_0^+} \quad \text{and} \quad \Gamma_{B_1^-} = \mathcal{P}^{B_1^-} \vee \mathcal{T}^{B_1^-} \vee \mathcal{Q}^{B_1^-}$$

of the Auslander–Reiten quivers of B_0^+ and B_1^-, where $\mathcal{P}^{B_0^+} = \mathcal{P}^{C_0}$, $\mathcal{Q}^{B_1^-} = \mathcal{Q}^{C_1}$, and $\mathcal{X}_1$ is a family of components containing the simple $\overline{B}_1$-module $S_1 = S_{\overline{B}_1}(\omega)$ at the vertex ω of $Q_{B_1^+}$. We note that ω is the unique common vertex of the quivers Q_{C_0} and Q_{C_1}. Observe that we may have $B_1^- = C_1$. In such a case, we set $B_1^+ = C_1$. Assume $B_1^+ \neq C_1$. Then, applying Theorem 3.1(i) to the branch coextension B_1^- of C_1, we conclude that there exists a reflection sequence of sinks $i_{r+1}, \dots, i_t$ of $Q_{B_1^-}$, for some $t \geq r+1$, such that the iterated reflection $B_1^+ = S^+_{i_t}, \dots, S^+_{i_{r+1}} B_1^-$ of B_1^- is a branch extension of C_1 and the Auslander–Reiten quiver $\Gamma_{B_1^*}$ of the iterated extension $B_1^* = T_{i_{r+1},\dots,i_t} B_1^-$ of B_1^- has a decomposition

$$\Gamma_{B_1^*} = \mathcal{P}^{B_1^*} \vee \mathcal{C}^{B_1^*} \vee \mathcal{Q}^{B_1^*},$$

where $\mathcal{P}^{B_1^*} = \mathcal{P}^{B_1^-}$ is a family of components consisting of B_1^--modules and containing all indecomposable projective B_1^--modules, $\mathcal{Q}^{B_1^*} = \mathcal{P}^{B_1^+}$ is a family of components consisting of B_1^+-modules and containing all indecomposable injective B_1^+-modules, and $\mathcal{C}^{B_1^*}$ is a $\mathbb{P}_1(K)$-family $(\mathcal{C}_\lambda^{B_1^*})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes, separating $\mathcal{P}^{B_1^*}$ from $\mathcal{Q}^{B_1^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_1^-} = (\mathcal{T}_\lambda^{B_1^-})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes of $\Gamma_{B_1^-}$ by iterated infinite rectangle insertions. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, we have $s(\mathcal{C}_\lambda^{B_1^*}) + p(\mathcal{C}_\lambda^{B_1^*}) = r(\mathcal{C}_\lambda^{B_1^*}) - 1$. Further, B_1^- is the iterated reflection $B_1^- = S^-_{\nu(i_r+1)} \dots S^-_{\nu(i_t)} B_1^+$, and applying Theorem 3.1(ii) we infer that the canonical family $\mathcal{C}^{B_1^*}$ of quasi-tubes of $\Gamma_{B_1^*}$ is obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_1^+} = (\mathcal{T}_\lambda^{B_1^+})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes of $\Gamma_{B_1^+}$ by suitable iterated infinite rectangle coinsertions. We set $\mathcal{C}_1 = \mathcal{C}^{B_1^*}$ and $\mathcal{C}_1(\lambda) = \mathcal{C}_\lambda^{B_1^*}$ for $\lambda \in \mathbb{P}_1(K)$.

We now note that $i_0, i_1, \dots, i_{r-1}, i_r, i_{r+1}, \dots, i_t$ is a reflection sequence of sinks of $Q_B = Q_{B_0^-}$ exhausting all vertices of Q_B except the unique source ω of Q_C, and hence $t = n-1$. Moreover, $i_n = \omega$ is a unique sink of Q_{C_1}, and a sink of $Q_{B_1^+}$, because B_1^+ is a branch extension of C_1. Consider the one-point extension $\overline{B}_2 = T^+_{i_n} B_1^+ = B_1^+[I_{B_1^+}(i_n)]$ and the reflection $B_2^- = S^+_{i_n} B_1^+$. Then B_2^- is a branch coextension of a new canonical algebra C_2, and $\overline{B}_2$ is a one-point coextension $[P_{B_2^-}(\nu(i_n))]B_2^-$ of B_2^- by the indecomposable projective B_2^--module $P_{B_2^-}(\nu(i_n))$ at the vertex $\nu(i_n) = \nu(\omega)$. Hence, the Auslander–Reiten quiver $\Gamma_{\overline{B}_2}$ of $\overline{B}_2$ has a decomposition

$$\Gamma_{\overline{B}_2} = \mathcal{P}^{B_1^+} \vee \mathcal{T}^{B_1^+} \vee \mathcal{X}_2 \vee \mathcal{T}^{B_2^-} \vee \mathcal{Q}^{B_2^-}$$

given by canonical decompositions

$$\Gamma_{B_1^+} = \mathcal{P}^{B_1^+} \vee \mathcal{T}^{B_1^+} \vee \mathcal{Q}^{B_1^+} \quad \text{and} \quad \Gamma_{B_2^-} = \mathcal{P}^{B_2^-} \vee \mathcal{T}^{B_2^-} \vee \mathcal{Q}^{B_2^-}$$

of the Auslander–Reiten quivers of B_1^+ and B_2^-, where $\mathcal{P}^{B_1^+} = \mathcal{P}^{C_1}$, $\mathcal{Q}^{B_2^-} = \mathcal{Q}^{C_2}$, and $\mathcal{X}_2$ is a family of components containing the simple $\overline{B}_2$-module $S_2 = S_{\overline{B}_2}(\nu(i_r))$ at the vertex $\nu(i_r) = \nu(0)$ of $Q_{B_2^+}$. Observe that $\nu(0)$ is the unique common vertex of C_1 and C_2.

Identify now $B = B_0^-$ with the full convex subcategory of $\hat{B}$ given by the objects $e_{0,k}$, $k \in \{1, \dots, n\}$. Then

- B_0^+ is the full convex subcategory of $\hat{B}$ given by the objects $e_{0,k}$ with $k \in \{1, \dots, n\} \setminus \{i_0, \dots, i_{r-1}\}$ and $e_{1,i_0} = \nu_{\hat{B}}(e_{0,i_0}), \dots, e_{1,i_{r-1}} = \nu_{\hat{B}}(e_{0,i_{r-1}})$;

- B_1^- is the full convex subcategory of $\hat{B}$ given by the objects $e_{k,0}$ with $k \in \{1, \ldots, n\} \setminus \{i_0, \ldots, i_{r-1}, i_r\}$ and $e_{1,i_0} = \nu_{\hat{B}}(e_{0,i_0}), \ldots, e_{1,i_{r-1}} = \nu_{\hat{B}}(e_{0,i_{r-1}})$, $e_{1,i_r} = \nu_{\hat{B}}(e_{0,i_r})$;
- B_1^+ is the full convex subcategory of $\hat{B}$ given by the objects e_{0,i_n}, $e_{1,i_0} = \nu_{\hat{B}}(e_{0,i_0}), \ldots, e_{1,i_r} = \nu_{\hat{B}}(e_{0,i_r})$, $e_{1,i_{r+1}} = \nu_{\hat{B}}(e_{0,i_{r+1}}), \ldots, e_{1,i_{n-1}} = \nu_{\hat{B}}(e_{0,i_{n-1}})$;
- B_2^- is the full convex subcategory of $\hat{B}$ given by the objects $e_{1,k} = \nu_{\hat{B}}(e_{0,k})$, $k \in \{1, \ldots, n\}$.

In particular, we conclude that the Nakayama automorphism $\nu_{\hat{B}}$ of $\hat{B}$ induces isomorphisms of K-categories (algebras) $B_0^- \cong B_2^-$ and $C = C_0 \cong C_2$.

We define full convex subcategories C_q, B_q^-, B_q^+, B_q^* and $\overline{B}_q$, $q \in \mathbb{Z}$, of $\hat{B}$ as follows:

- For $q = 2p$ even, $C_q = \nu_{\hat{B}}^p(C_0)$, $B_q^- = \nu_{\hat{B}}^p(B_0^-)$, $B_q^+ = \nu_{\hat{B}}^p(B_0^+)$, $B_q^* = \nu_{\hat{B}}^p(B_0^*)$, $\overline{B}_q = \nu_{\hat{B}}^{p-1}(\overline{B}_2)$.
- For $q = 2p + 1$ odd, $C_q = \nu_{\hat{B}}^p(C_1)$, $B_q^- = \nu_{\hat{B}}^p(B_1^-)$, $B_q^+ = \nu_{\hat{B}}^p(B_1^+)$, $B_q^* = \nu_{\hat{B}}^p(B_1^*)$, $\overline{B}_q = \nu_{\hat{B}}^p(\overline{B}_1)$.

Then, for each $q \in \mathbb{Z}$, C_q is a canonical algebra, B_q^- is a branch coextension of C_q, and B_q^+ is a branch extension of C_q. We denote by 0_q the unique sink and by ω_q the unique source of the quiver Q_{C_q} of C_q.

For each $q \in \mathbb{Z}$, the Auslander–Reiten quiver $\Gamma_{B_q^*}$ of B_q^* has a decomposition

$$\Gamma_{B_q^*} = \mathcal{P}^{B_q^*} \vee \mathcal{C}^{B_q^*} \vee \mathcal{Q}^{B_q^*},$$

where $\mathcal{P}^{B_q^*} = \mathcal{P}^{B_q^-}$ is a family of components consisting of B_q^--modules and containing all indecomposable projective B_q^--modules, $\mathcal{Q}^{B_q^*} = \mathcal{Q}^{B_q^+}$ is a family of components consisting of B_q^+-modules and containing all indecomposable injective B_q^+-modules, and $\mathcal{C}^{B_q^*}$ is a $\mathbb{P}_1(K)$-family $(\mathcal{C}_\lambda^{B_q^*})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes, separating $\mathcal{P}^{B_q^*}$ from $\mathcal{Q}^{B_q^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_q^-} = (\mathcal{T}_\lambda^{B_q^-})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes of $\Gamma_{B_q^-}$ by iterated infinite rectangle insertions, and from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_q^+} = (\mathcal{T}_\lambda^{B_q^+})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes of $\Gamma_{B_q^+}$ by iterated infinite rectangle coinsertions. Moreover, for each $\lambda \in \mathbb{P}_1(K)$, we have $s(\mathcal{C}_\lambda^{B_q^*}) + p(\mathcal{C}_\lambda^{B_q^*}) = r(\mathcal{C}_\lambda^{B_q^*}) - 1$. We set $\mathcal{C}_q = \mathcal{C}^{B_q^*}$ and $\mathcal{C}_q(\lambda) = \mathcal{C}_\lambda^{B_q^*}$ for $\lambda \in \mathbb{P}_1(K)$. Since $\mathrm{Hom}_{B_q^*}(\mathcal{C}^{B_q^*}, \mathcal{P}^{B_q^*}) = 0$ and $\mathrm{Hom}_{B_q^*}(\mathcal{Q}^{B_q^*}, \mathcal{C}^{B_q^*}) = 0$, B_q^* is a full convex subcategory of $\hat{B}$, and $\hat{B}$ can be obtained from B_q^* by iterated one-point coextensions by projective modules whose restrictions to B_q^* are modules from the additive category $\mathrm{add}(\mathcal{P}^{B_q^*})$

and iterated one-point extensions by injective modules whose restrictions to B_q^* are modules from the additive category $\mathrm{add}(\mathcal{Q}^{B_q^*})$ of $\mathcal{Q}^{B_q^*}$, applying [36, Corollary 1.7] and its dual, we conclude that $\mathcal{C}_q = (\mathcal{C}_q(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ remains a $\mathbb{P}_1(K)$-family of pairwise orthogonal standard quasi-tubes of $\Gamma_{\hat{B}}$.

Similarly, for each $q \in \mathbb{Z}$, $\overline{B}_q$ is a one-point extension $B_{q-1}^+[I_{B_{q-1}^+}(0_q)]$ of B_{q-1}^+ by the indecomposable injective B_{q-1}^+-module $I_{B_{q-1}^+}(0_{q-1})$ at the unique sink 0_{q-1} of $Q_{C_{q-1}}$, and a one-point coextension $[P_{B_q^-}(\nu(0_{q-1}))]B_q^-$ of B_q^- by the indecomposable projective B_q^--module $P_{B_q^-}(\nu(0_{q-1}))$ at the unique source $\nu(0_{q-1}) = \omega_q$ of Q_{C_q}. Moreover, the Auslander–Reiten quiver $\Gamma_{\overline{B}_q}$ of $\overline{B}_q$ has a decomposition

$$\Gamma_{\overline{B}_q} = \mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+} \vee \mathcal{X}_q \vee \mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-}$$

given by canonical decompositions

$$\Gamma_{B_{q-1}^+} = \mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+} \vee \mathcal{Q}^{B_{q-1}^+} \quad \text{and} \quad \Gamma_{B_q^-} = \mathcal{P}^{B_q^-} \vee \mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-}$$

of the Auslander–Reiten quivers of B_{q-1}^+ and B_q^-, where $\mathcal{P}^{B_{q-1}^+} = \mathcal{P}^{C_{q-1}}$, $\mathcal{Q}^{B_q^-} = \mathcal{Q}^{C_q}$, and $\mathcal{X}_q$ is a family of components containing the simple $\overline{B}_q$-module $S_q = S_{B_{q-1}^+}(\omega_{q-1}) = S_{B_q^-}(0_q)$ at the vertex $\omega_{q-1} = 0_q$, separating $\mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+}$ from $\mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-}$. In particular, we have $\mathrm{Hom}_{\overline{B}_q}(\mathcal{X}_q, \mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+}) = 0$ and $\mathrm{Hom}_{\overline{B}_q}(\mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-}, \mathcal{X}_q) = 0$. Since $\hat{B}$ can be obtained from $\overline{B}_q$ by iterated one-point extensions by indecomposable projective modules whose restrictions to $\overline{B}_q$ are modules from the additive category $\mathrm{add}(\mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+})$ of $\mathcal{P}^{B_{q-1}^+} \vee \mathcal{T}^{B_{q-1}^+}$ and iterated one-point coextensions by indecomposable injective modules whose restrictions to $\overline{B}_q$ are modules from the additive category $\mathrm{add}(\mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-})$ of $\mathcal{T}^{B_q^-} \vee \mathcal{Q}^{B_q^-}$, applying [36, Corollary 1.7] and its dual again, we conclude that $\mathcal{X}_q$ remains a family of components of $\Gamma_{\hat{B}}$.

For each pair of integers $p \le q$, let $B_{p,q}$ be the full subcategory of $\hat{B}$ given by the objects $e_{m,k}$ with $p \le m \le q$ and $k \in \{1,\dots,n\}$. Observe that the module category $\mathrm{mod}\, B_{p,q}$ is the full subcategory of $\mathrm{mod}\, \hat{B}$ consisting of modules with supports contained in $B_{p,q}$. Moreover, every module from $\mathrm{mod}\, \hat{B}$ belongs to a full subcategory $\mathrm{mod}\, B_{p,q}$.

Observe now that $B_{0,1}$ is the iterated extension $B_{0,1} = T^+_{i_0,i_1,\dots,i_n} B$ of $B = B_0^-$. Then it follows from the above discussion that the Auslander–Reiten quiver $\Gamma_{B_{0,1}}$ of $B_{0,1}$ has a decomposition

$$\Gamma_{B_{0,1}} = \mathcal{P}^{B_0^-} \vee \mathcal{C}_0 \vee \mathcal{X}_1 \vee \mathcal{C}_1 \vee \mathcal{X}_2 \vee \mathcal{T}^{B_2^-} \vee \mathcal{Q}^{B_2^-}$$

where $B_2^- = \nu_{\hat{B}}(B_0^-)$, $\mathcal{T}^{B_2^-} = \nu_{\hat{B}}(\mathcal{T}^{B_0^-})$ and $\mathcal{Q}^{B_2^-} = \nu_{\hat{B}}(\mathcal{Q}^{B_0^-})$. Similarly, the Auslander–Reiten quiver $\Gamma_{B_{-1,0}}$ of $B_{-1,0}$ has a decomposition

$$\Gamma_{B_{-1,0}} = \mathcal{P}^{B_{-1}^-} \vee \mathcal{C}_{-1} \vee \mathcal{X}_0 \vee \mathcal{C}_0 \vee \mathcal{X}_1 \vee \mathcal{T}^{B_1^-} \vee \mathcal{Q}^{B_1^-}$$

where $B_{-1}^- = \nu_{\hat{B}}^-(B_1^-)$, $\mathcal{C}_{-1} = \nu_{\hat{B}}^-(\mathcal{C}_1)$, and $\mathcal{X}_0 = \nu_{\hat{B}}^-(\mathcal{X}_2)$. Combining, we conclude that the Auslander–Reiten quiver $\Gamma_{B_{-1,1}}$ of $B_{-1,1}$ has a decomposition

$$\Gamma_{B_{-1,1}} = \mathcal{P}^{B_{-1}^-} \vee \mathcal{C}_{-1} \vee \mathcal{X}_0 \vee \mathcal{C}_0 \vee \mathcal{X}_1 \vee \mathcal{C}_1 \vee \mathcal{X}_2 \vee \mathcal{T}^{B_2^-} \vee \mathcal{Q}^{B_2^-}.$$

Repeating these considerations, we deduce that, for any positive integer p, the Auslander–Reiten quiver $\Gamma_{B_{-p,p}}$ of $B_{-p,p}$ has a decomposition

$$\Gamma_{B_{-p,p}} = \mathcal{P}^{B_{-p}^-} \vee \mathcal{C}_{-p} \vee \Big(\bigvee_{-p<q\le p} (\mathcal{X}_q \vee \mathcal{C}_q)\Big) \vee \mathcal{X}_{p+1} \vee \mathcal{T}^{B_{p+1}^-} \vee \mathcal{Q}^{B_{p+1}^-}.$$

Since $\operatorname{mod} \hat{B}$ is the union of the full subcategories $\operatorname{mod} B_{-p,p}$, $p \ge 1$, we conclude that the Auslander–Reiten quiver $\Gamma_{\hat{B}}$ of $\hat{B}$ has a required decomposition

$$\Gamma_{\hat{B}} = \bigvee_{q\in\mathbb{Z}} (\mathcal{X}_q \vee \mathcal{C}_q)$$

and the statements (i)–(ix) hold. Observe also that, for a fixed object $x = e_{q,k}$ of $\hat{B}$, the full subcategory of $\hat{B}$ given by the supports $\operatorname{supp} M$ of all indecomposable modules from $\operatorname{mod} \hat{B}$ with $M(x) \ne 0$, is contained in the full subcategory $B_{q-1,q+1}$. Therefore, $\hat{B}$ is a locally support-finite category, and so (x) also holds.

We now prove the statements (xi) and (xii). Fix $q \in \mathbb{Z}$. For each $\lambda \in \mathbb{P}_1(K)$, the quasi-tube $\mathcal{C}_q(\lambda)$ contains the unique nonsimple indecomposable C_q-module $E_q^{(\lambda)}$ lying on the mouth of the stable tube $\mathcal{T}_\lambda^{C_q}$ of Γ_{C_q}, having the simple socle isomorphic to $S_q = S_{C_q}(0_q)$ and the simple top isomorphic to $S_{q+1} = S_{C_q}(\omega_q)$. Therefore, we have $\operatorname{Hom}_{\hat{B}}(S_q, E_q^{(\lambda)}) = \operatorname{Hom}_{C_q}(S_q, E_q^{(\lambda)}) \ne 0$ and $\operatorname{Hom}_{\hat{B}}(E_q^{(\lambda)}, S_{q+1}) = \operatorname{Hom}_{C_q}(E_q^{(\lambda)}, S_{q+1}) \ne 0$. Hence, $\operatorname{Hom}_{\hat{B}}(S_q, \mathcal{C}_q(\lambda))$ $\ne 0$ and $\operatorname{Hom}_{\hat{B}}(\mathcal{C}_q(\lambda), S_{q+1}) \ne 0$ for any $\lambda \in \mathbb{P}_1(K)$. Moreover, since $\Gamma_{B_q^*} = \mathcal{P}^{B_q^-} \vee \mathcal{C}_q \vee \mathcal{Q}^{B_q^+}$, $\mathcal{C}_q$ separates $\mathcal{P}^{B_q^-}$ from $\mathcal{Q}^{B_q^+}$, S_q lies in $\mathcal{P}^{B_q^-}$, S_{q+1} lies in $\mathcal{Q}^{B_q^+}$, we conclude that $\operatorname{Hom}_{\hat{B}}(S_{q+1}, \mathcal{C}_q) = 0$ and $\operatorname{Hom}_{\hat{B}}(\mathcal{C}_q, S_q) = 0$. Finally, the support of any indecomposable $\hat{B}$-module from the family $\mathcal{C}_q$ is contained in the full convex subcategory B_q^*. Hence, we obtain $\operatorname{Hom}_{\hat{B}}(S_p, \mathcal{C}_q) = 0$ and $\operatorname{Hom}_{\hat{B}}(\mathcal{C}_q, S_p) = 0$ for any $p \in \mathbb{Z}$ different from q and $q+1$, respectively. Thus the statements (xi) and (xii) hold.

It remains to prove (xiii). The syzygy operators $\Omega_{\hat{B}}$ and $\Omega_{\hat{B}}^-$ are mutually inverse equivalences of the stable category $\underline{\operatorname{mod}}\, \hat{B}$ of $\hat{B}$. Applying (iii), for each $q \in \mathbb{Z}$, we have $\underline{\operatorname{Hom}}_{\hat{B}}(\mathcal{X}_q^s, \mathcal{X}_p^s \vee \mathcal{C}_p^s) = 0$ and $\underline{\operatorname{Hom}}_{\hat{B}}(\mathcal{C}_q^s, \mathcal{C}_p^s \vee \mathcal{X}_{p+1}^s) = 0$

for any $p \in \mathbb{Z}$ with $p < q$. We first show that $\Omega(\mathcal{C}^s_{q+1}) = \mathcal{C}^s_q$ for any $q \in \mathbb{Z}$. Fix $\lambda \in \mathbb{P}_1(K)$. We have three cases to consider, depending on the structure of the quasi-tube $\mathcal{C}_{q+1}(\lambda)$.

Assume first that $\mathcal{C}_{q+1}(\lambda)$ is a stable tube of rank 1. Then, in the above notation, $\mathcal{C}_{q+1}(\lambda)$ is a stable tube $\mathcal{T}^{C_{q+1}}_\lambda$ of rank 1 of the Auslander–Reiten quiver $\Gamma_{C_{q+1}}$ of the canonical algebra C_{q+1}. Then the unique module $E^{(\lambda)}_{C_{q+1}}$ lying on the mouth of $\mathcal{T}^{C_{q+1}}_\lambda = \mathcal{C}_{q+1}(\lambda)$ is an indecomposable C_{q+1}-module having a one-dimensional space at each vertex of $Q_{C_{q+1}}$ (see Section 1), one-dimensional socle $S_{C_{q+1}}(0_{q+1})$ given by the unique sink 0_{q+1} of $Q_{C_{q+1}}$ and one-dimensional top $S_{C_{q+1}}(\omega_{q+1})$ given by the unique source ω_{q+1} of $Q_{C_{q+1}}$. Further, the quiver Q_{C_q} of the canonical algebra C_q has a unique source at the vertex $\omega_q = 0_{q+1}$ and a unique sink at the vertex 0_q such that $\nu_{\hat{B}}(0_q) = \omega_{q+1}$, the indecomposable projective-injective $\hat{B}$-module $P_{\hat{B}}(0_{q+1})$ at 0_{q+1} has a 2-dimensional vector space at the common vertex $\omega_q = 0_{q+1}$ of Q_{C_q} and $Q_{C_{q+1}}$, a one-dimensional vector space at the remaining vertices of Q_{C_q} and $Q_{C_{q+1}}$, and the zero space at the vertices of $Q_{\hat{B}}$ which are not vertices of Q_{C_q} and $Q_{C_{q+1}}$. Then the syzygy module $\Omega_{\hat{B}}(E^{(\lambda)}_{C_{q+1}})$ is an indecomposable C_q-module having a one-dimensional vector space at each vertex of Q_{C_q}, one-dimensional socle $S_{C_q}(0_q)$ at the unique sink 0_q of Q_{C_q} and one-dimensional top $S_{C_q}(\omega_q)$ at the unique source ω_q of Q_{C_q}. Moreover, since $\mathcal{C}_{q+1}(\lambda) = \mathcal{T}^{C_{q+1}}_\lambda$ is a stable tube of rank 1 (hence without simple and projective modules), we conclude that $\Omega_{\hat{B}}(\mathcal{C}_{q+1}(\lambda))$ is a stable tube of rank 1 in $\Gamma_{\hat{B}}$, and consequently $\Omega_{\hat{B}}(\mathcal{C}_{q+1}(\lambda))$ is a stable tube $\mathcal{T}^{C_q}_\varrho$ of rank 1 in Γ_{C_q} for some $\varrho \in \mathbb{P}_1(K)$. Clearly, in that case $\mathcal{T}^{C_q}_\varrho = \mathcal{C}_q(\varrho)$, and $\Omega_{\hat{B}}(\mathcal{C}_{q+1}(\lambda)) = \mathcal{C}_q(\varrho)$.

Assume now that $\mathcal{C}_{q+1}(\lambda)$ is a quasi-tube enlargement of a stable tube $\mathcal{T}^{C_{q+1}}_\lambda$ of rank 1 in $\Gamma_{C_{q+1}}$, with $r(\mathcal{C}_{q+1}(\lambda)) \ge 2$ (equivalently, $\mathcal{C}_{q+1}(\lambda) \ne \mathcal{T}^{C_{q+1}}_\lambda$). Then the branch coextension B^-_{q+1} of C_{q+1} inside $\hat{B}$ contains the one-point coextension $[E^{(\lambda)}_{C_{q+1}}]C_{q+1}$ of C_{q+1} by the unique module $E^{(\lambda)}_{C_{q+1}}$ lying on the mouth of $\mathcal{T}^{C_{q+1}}_\lambda$. According to Theorem 3.1 (and its proof), the quasi-tube $\mathcal{C}_{q+1}(\lambda)$ contains the indecomposable projective-injective $\hat{B}$-module $I_{\hat{B}}(x) = P_{\hat{B}}(\nu_{\hat{B}}(x))$, where x is the coextension vertex of $[E^{(\lambda)}_{C_{q+1}}]C_{q+1}$. Moreover, x is the sink of an arrow with source $0_{q+1} = \omega_q$ on the path of Q_{C_q} from the source ω_q to the sink 0_q corresponding to the parameter λ. Hence, the simple $\hat{B}$-module $S_{\hat{B}}(x) = S_{C_q}(x)$ lies in the stable tube $\mathcal{T}^{C_q}_\lambda$ of Γ_{C_q}, and consequently $S_{\hat{B}}(x)$ lies in the quasi-tube $\mathcal{C}_q(x) = \mathcal{C}^{B^*_q}_\lambda$. Finally, observe that $P_{\hat{B}}(\nu_{\hat{B}}(x))/S_{\hat{B}}(x)$ lies in $\mathcal{C}_{q+1}(\lambda)$, and $\Omega_{\hat{B}}(P_{\hat{B}}(\nu_{\hat{B}}(x))/S_{\hat{B}}(x)) = S_{\hat{B}}(x)$. This shows that $\Omega_{\hat{B}}(\mathcal{C}^s_{q+1}(\lambda)) = \mathcal{C}^s_q(\lambda)$.

Assume that $\mathcal{C}_{q+1}(\lambda)$ is a quasi-tube enlargement of a stable tube $\mathcal{T}_\lambda^{C_{q+1}}$ of $\Gamma_{C_{q+1}}$ of rank at least 2. Then the tube $\mathcal{T}_\lambda^{C_{q+1}}$, and hence $\mathcal{C}_{q+1}(\lambda)$, contains a simple module $S_{\hat{B}}(y) = S_{C_{q+1}}(y)$ at a vertex y which is the source of an arrow with sink 0_{q+1} on the path from ω_{q+1} to 0_{q+1} in $Q_{C_{q+1}}$ corresponding to the parameter λ. Then y is the extension vertex of the one-point extension $C_q[E_{C_q}^{(\lambda)}]$ of C_q by the unique nonsimple module lying on the mouth of the stable tube $\mathcal{T}_\lambda^{C_q}$ of Γ_{C_q}, and $C_q[E_{C_q}^{(\lambda)}]$ is a full convex subcategory of the quasi-tube enlargement B_q^* of C_q inside $\hat{B}$. Applying Theorem 3.1 (and its proof) again, we conclude that the quasi-tube $\mathcal{C}_q(\lambda) = \mathcal{C}_\lambda^{B_q^*}$ contains the indecomposable projective module $P_{\hat{B}}(y) = P_{B_q^*}(y)$, and hence also its radical $\operatorname{rad} P_{\hat{B}}(y)$. Since $\operatorname{rad} P_{\hat{B}}(y) = \Omega_{\hat{B}}(S_{\hat{B}}(y))$, we conclude that $\Omega_{\hat{B}}(\mathcal{C}_{q+1}^s(\lambda)) = \mathcal{C}_q^s(\lambda)$.

Summing up, we proved that $\Omega_{\hat{B}}(\mathcal{C}_{q+1}^s) = \mathcal{C}_q^s$ for any $q \in \mathbb{Z}$. In order to prove that $\Omega_{\hat{B}}(\mathcal{X}_{q+1}^s) = \mathcal{X}_q^s$ for $q \in \mathbb{Z}$, we need a characterization of indecomposable nonprojective modules from a family $\mathcal{X}_p$ in the stable category $\underline{\operatorname{mod}}\,\hat{B}$. Fix $p \in \mathbb{Z}$. Recall that $\mathcal{X}_p$ consists of indecomposable $\overline{B}_p$-modules, where $\overline{B}_p$ is simultaneously the one-point extension $\overline{B}_p = B_{p-1}^+[I_{B_{p-1}^+}(0_{p-1})]$ of the branch extension B_{p-1}^+ of the canonical algebra C_{p-1} by the indecomposable injective B_{p-1}^+-module $I_{B_{p-1}^+}(0_{p-1})$ at the unique sink 0_{p-1} of $Q_{C_{p-1}}$, and the one-point coextension $\overline{B}_p = [P_{B_p^-}(\omega_p)]B_p^-$ of the branch coextension B_p^- of the canonical algebra C_p by the indecomposable projective B_p^--module $P_{B_p^-}(\omega_p)$ at the unique source $\omega_p = \nu_{\hat{B}}(0_{p-1})$ of Q_{C_p}. Further, the Auslander–Reiten quiver $\Gamma_{\overline{B}_p}$ has a decomposition

$$\Gamma_{\overline{B}_p} = \mathcal{P}^{B_{p-1}^+} \vee \mathcal{T}^{B_{p-1}^+} \vee \mathcal{X}_p \vee \mathcal{T}^{B_p^-} \vee \mathcal{Q}^{B_p^-}$$

given by decompositions

$$\Gamma_{B_{p-1}^+} = \mathcal{P}^{B_{p-1}^+} \vee \mathcal{T}^{B_{p-1}^+} \vee \mathcal{Q}^{B_{p-1}^+} \quad \text{and} \quad \Gamma_{B_p^-} = \mathcal{P}^{B_p^-} \vee \mathcal{T}^{B_p^-} \vee \mathcal{Q}^{B_p^-}$$

of the Auslander–Reiten quivers of B_{p-1}^+ and B_p^-. The $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_{p-1}^+}$ of ray tubes of $\Gamma_{B_{p-1}^+}$ separates $\mathcal{P}^{B_{p-1}^+}$ from $\mathcal{Q}^{B_{p-1}^+}$, the indecomposable projective B_{p-1}^+-modules lie in $\mathcal{P}^{B_{p-1}^+} \vee \mathcal{T}^{B_{p-1}^+}$, and hence, for each indecomposable module X in $\mathcal{Q}^{B_{p-1}^+}$, there exists an epimorphism $U \to X$ with U from the additive category $\operatorname{add}(\mathcal{T}^{B_{p-1}^*})$ of $\mathcal{T}^{B_{p-1}^*}$, because a projective cover epimorphism $P_{B_{p-1}^+}(X) \to X$ of X in $\operatorname{mod} B_{p-1}^+$ factors through a module U from $\operatorname{add}(\mathcal{T}^{B_{p-1}^*})$. Dually, the $\mathbb{P}_1(K)$-family $\mathcal{T}^{B_p^-}$ of ray tubes of $\Gamma_{B_p^-}$ separates $\mathcal{P}^{B_p^-}$ from $\mathcal{Q}^{B_p^-}$, the indecomposable injective B_p^--modules lie in $\mathcal{T}^{B_p^-} \vee \mathcal{Q}^{B_p^-}$, and hence, for each indecomposable module Y in $\mathcal{P}^{B_p^-}$, there exists a monomorphism $Y \to V$ with V a module from the additive category $\operatorname{add}(\mathcal{T}^{B_p^-})$ of $\mathcal{T}^{B_p^-}$, because an injective envelope monomorphism $Y \to I_{B_p^-}(Y)$ of Y in $\operatorname{mod} B_p^-$ factors through a module V from $\operatorname{add}(\mathcal{T}^{B_p^-})$.

Observe also that $\mathcal{X}_p$ contains exactly one projective and exactly one injective B_p^--module, namely the indecomposable projective-injective $\hat{B}$-module $P_{\hat{B}}(\omega_p) = P_{\overline{B}_p}(\omega_p) = I_{\overline{B}_p}(0_{p-1}) = I_{\hat{B}}(0_{p-1})$, where $\omega_p = \nu_{\hat{B}}(0_{p-1})$. Moreover, the simple $\hat{B}$-module $S_{p-1} = S_{\hat{B}}(0_{p-1}) = S_{C_{p-1}}(0_{p-1})$ lies in $\mathcal{X}_{p-1}$, and the simple $\hat{B}$-module $S_{p+1} = S_{\hat{B}}(\omega_p) = S_{C_p}(\omega_p)$ lies in $\mathcal{X}_{p+1}$. Since $\overline{B}_p = B_{p-1}^+[I_{B_{p-1}^+}(0_{p-1})]$ and $I_{p-1}^+(0_{p-1})$ lies in $\mathcal{Q}^{B_{p-1}^+}$, the restriction of every module M in $\mathcal{X}_p$ to B_{p-1}^+ belongs to the additive category $\operatorname{add}(\mathcal{Q}^{B_{p-1}^+})$ of $\mathcal{Q}^{B_{p-1}^+}$. In particular, every module M from $\mathcal{X}_p$ contains an indecomposable submodule X from $\mathcal{Q}^{B_{p-1}^+}$. Dually, since $\overline{B}_p = [P_{B_p^-}(\omega_p)]B_p^-$ and $P_{B_p^-}(\omega_p)$ lies in $\mathcal{P}^{B_p^-}$, the restriction of every module N in $\mathcal{X}_p$ to B_p^- belongs to the additive category $\operatorname{add}(\mathcal{P}^{B_p^-})$ of $\mathcal{P}^{B_p^-}$. As a consequence, every module N from $\mathcal{X}_p$ has an indecomposable quotient module Y from $\mathcal{P}^{B_p^-}$. Therefore, we conclude that an indecomposable module Z from $\operatorname{mod} \hat{B}$ belongs to $\mathcal{X}_p$ if and only if there exists a sequence of homomorphisms in $\operatorname{mod} \hat{B}$ of the form

$$U \xrightarrow{e} X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} V$$

where e and g are epimorphisms, f and h are monomorphisms, U is a module from $\operatorname{add}(\mathcal{T}^{B_{p-1}^+})$, X a module from $\mathcal{Q}^{B_{p-1}^+}$, Y a module from $\mathcal{P}^{B_p^-}$, and V a module from $\operatorname{add}(\mathcal{T}^{B_p^-})$. We also note that all modules of $\mathcal{T}^{B_{p-1}^+}$ are indecomposable nonprojective $\hat{B}$-modules contained in $\mathcal{C}_{p-1} = \mathcal{C}^{B_{p-1}^*}$, and all modules of $\mathcal{T}^{B_p^-}$ are indecomposable nonprojective $\hat{B}$-modules contained in $\mathcal{C}_p = \mathcal{C}^{B_p^*}$.

Hence, applying (iii), we infer that the modules X and Y, occurring in the above sequence, belong to $\mathcal{X}_p$. Further, applying Lemma 4.4, we conclude that the homomorphisms e, f, g, h induce nonzero morphisms $\underline{e}$, $\underline{f}$, $\underline{g}$, $\underline{h}$ in the stable category $\underline{\operatorname{mod}}\, \hat{B}$. Moreover, $\underline{\operatorname{Hom}}_{\hat{B}}(U, X) \neq 0$ implies that $\underline{\operatorname{Hom}}_{\hat{B}}(L, X) \neq 0$ for some indecomposable direct summand L of U, and $\underline{\operatorname{Hom}}_{\hat{B}}(Y, V) \neq 0$ implies that $\underline{\operatorname{Hom}}_{\hat{B}}(Y, W) \neq 0$ for some indecomposable direct summand W of V. Therefore, we established the following characterization of modules from $\mathcal{X}_p^s$: an indecomposable nonprojective module Z from $\operatorname{mod} \hat{B}$ belongs to $\mathcal{X}_p^s$ if and only if there exists a sequence of nonzero morphisms in $\underline{\operatorname{mod}}\, \hat{B}$ of the form

$$L \to X \to Z \to Y \to W$$

where L is in $\mathcal{C}^s_{p-1}$, X is indecomposable not in $\mathcal{C}^s_{p-1}$, Y is indecomposable not in $\mathcal{C}^s_p$, and W is in $\mathcal{C}^s_p$. Clearly, X and Y then also belong to $\mathcal{X}^s_p$.

Fix now $q \in \mathbb{Z}$, and take an indecomposable module M in $\mathcal{X}^s_{q+1}$. Then there exists a sequence of nonzero morphisms in $\underline{\mathrm{mod}}\,\hat{B}$ of the form

$$N \to M' \to M \to M'' \to R$$

such that N is in $\mathcal{C}^s_q$, M' is indecomposable not in $\mathcal{C}^s_q$, M'' is indecomposable not in $\mathcal{C}^s_{q+1}$, and R is in $\mathcal{C}^s_{q+1}$. Applying the selfequivalence functor $\Omega_{\hat{B}} : \underline{\mathrm{mod}}\,\hat{B} \to \underline{\mathrm{mod}}\,\hat{B}$ to the above sequence, we obtain a sequence of nonzero morphisms in $\underline{\mathrm{mod}}\,\hat{B}$ of the form

$$\Omega_{\hat{B}}(N) \to \Omega_{\hat{B}}(M') \to \Omega_{\hat{B}}(M) \to \Omega_{\hat{B}}(M'') \to \Omega_{\hat{B}}(R).$$

Since $\Omega_{\hat{B}}(\mathcal{C}^s_q) = \mathcal{C}^s_{q-1}$ and $\Omega_{\hat{B}}(\mathcal{C}^s_{q+1}) = \mathcal{C}^s_q$, we conclude that $\Omega_{\hat{B}}(N)$ lies in $\mathcal{C}^s_{q-1}$, $\Omega_{\hat{B}}(M')$ is indecomposable not in $\mathcal{C}^s_{q-1}$, $\Omega_{\hat{B}}(M'')$ is indecomposable not in $\mathcal{C}^s_q$, and $\Omega_{\hat{B}}(R)$ lies in $\mathcal{C}^s_q$. This implies that $\Omega_{\hat{B}}(M)$ lies in $\mathcal{X}^s_q$. Therefore, $\Omega_{\hat{B}}(\mathcal{X}^s_{q+1}) = \mathcal{X}^s_q$. ∎

Proposition 5.2. *Let B be a branch extension* (*respectively*, *branch coextension*) *of a canonical algebra C. Then there exists a strictly positive automorphism $\varphi_{\hat{B}}$ of $\hat{B}$ such that following statements hold*:

(i) $\varphi_{\hat{B}} = \nu_{\hat{B}}$ *or* $\varphi^2_{\hat{B}} = \nu_{\hat{B}}$.

(ii) *Every torsion-free admissible group G of automorphisms of $\hat{B}$ is an infinite cyclic group generated by a strictly positive automorphism $f\varphi^s_{\hat{B}}$ for some $s \ge 1$ and some rigid automorphism f of $\hat{B}$.*

Proof. We may assume (without loss of generality) that B is a branch coextension of C. We identify B and C with the corresponding full convex subcategories $B_0 = B^-_0$ and C_0 of $\hat{B}$. In the notation of Theorem 5.1, there exists a reflection sequence of sinks $i_0, i_1, \dots, i_{r-1}, i_r$ of Q_B such that the iterated reflection $B^-_1 = S^+_{i_r} \dots S^+_{i_1} S^+_{i_0} B$ is again a branch coextension of a canonical algebra C_1. Further, the iterated Nakayama shifts $C_{2p} = \nu^p_{\hat{B}}(C_0)$ and $C_{2p+1} = \nu^p_{\hat{B}}(C_1)$, $p \ge 0$, form a complete family of full convex canonical subcategories of $\hat{B}$. Clearly, the iterated Nakayama shifts $B^-_{2p} = \nu^p_{\hat{B}}(B^-_0)$ and $B^-_{2p+1} = \nu^p_{\hat{B}}(B^-_1)$, $p \ge 0$, then form a complete family of full convex subcategories of $\hat{B}$ which are branch coextensions of canonical algebras inside $\hat{B}$. We also have $C_{q+2} = \nu_{\hat{B}}(C_q)$ and $B^-_{q+2} = \nu_{\hat{B}}(B^-_q)$ for any $q \in \mathbb{Z}$. Moreover, $\hat{B}^-_q = \hat{B}^-_0 = \hat{B}$ for any $q \in \mathbb{Z}$. We have two possible cases: $B^-_0 \not\cong B^-_1$ or $B^-_0 \cong B^-_1$. If $B^-_0 \not\cong B^-_1$, we take $\varphi_{\hat{B}} = \nu_{\hat{B}}$. In the case $B^-_0 \cong B^-_1$, we denote by $\varphi_{\hat{B}}$ the canonical automorphism of $\hat{B}$ such that $\varphi_{\hat{B}}(B^-_0) = B^-_1$ and $\varphi^2_{\hat{B}} = \nu_{\hat{B}}$.

Let G be a torsion-free admissible group of automorphisms of $\hat{B}$. Then every element $g \in G$ acts on the family C_q, $q \in \mathbb{Z}$, of full convex canonical subcategories of $\hat{B}$. For $g \in G$, let m_g be the integer such that $g(C_0) = C_{m_g}$. Observe that $m_h = -m_g$ for $h = g^{-1}$. Suppose $m_g = 0$ for some $g \in G$. Then g acts on the finite set of objects of C_0, and hence a power g^r of g fixes an object of C_0. Since G is torsion-free and acts freely on the objects of $\hat{B}$, we get $g = 1$. Choose now an element $g \in G$ such that m_g is positive and minimal. Let $h \in G$ and $m_h = tm_g + l$ with $t \in \mathbb{Z}$ and $0 \leq l < m_g$. Then $a = hg^{-t} \in G$, $m_a = l$, and hence $l = 0$, $a = 1$. Therefore, G is an infinite cyclic group generated by g. The automorphism g also acts on the family B_q^-, $q \in \mathbb{Z}$, and $g(C_q) = C_{q+mq}$ forces $g(B_q^-) = B_{q+mg}^-$. If $B_0^- \not\cong B_1^-$, then m_g is even, say $m_g = 2s$ for some $s \geq 1$, and we define $f = g\nu_{\hat{B}}^{-s} = g\varphi_{\hat{B}}^{-s}$. If $B_0^- \cong B_1^-$, we take $s = m_g$ and $f = g\varphi_{\hat{B}}^{-s}$. Observe that $f(C_q) = C_q$ and $f(B_q^-) = B_q^-$ for any $q \in \mathbb{Z}$, and hence f is a rigid automorphism of $\hat{B}$. Consequently, G is an infinite cyclic group generated by $g = f\varphi_{\hat{B}}^{s}$ for some $s \geq 1$ and some rigid automorphism f of $\hat{B}$. ∎

We are now in a position to prove the theorem describing the structure and homological properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type.

THEOREM 5.3. *Let A be a selfinjective algebra of strictly canonical type. The Auslander–Reiten quiver Γ_A of A has a decomposition*

$$\Gamma_A = \bigvee_{q \in \mathbb{Z}/n\mathbb{Z}} (\mathcal{X}_q^A \vee \mathcal{C}_q^A)$$

for some positive integer n, and the following statements hold:

(i) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathcal{C}_q^A = (\mathcal{C}_q^A(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes with $s(\mathcal{C}_q^A(\lambda)) + p(\mathcal{C}_q^A(\lambda)) = r(\mathcal{C}_q^A(\lambda)) - 1$ for each $\lambda \in \mathbb{P}_1(K)$.*

(ii) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathcal{X}_q^A$ is a family of components containing exactly one simple module S_q.*

(iii) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathrm{Hom}_A(S_q, \mathcal{C}_q^A(\lambda)) \neq 0$ for all $\lambda \in \mathbb{P}_1(K)$, and $\mathrm{Hom}_A(S_p, \mathcal{C}_q^A) = 0$ for $p \neq q$ in $\mathbb{Z}/n\mathbb{Z}$.*

(iv) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathrm{Hom}_A(\mathcal{C}_q^A(\lambda), S_{q+1}) \neq 0$ for all $\lambda \in \mathbb{P}_1(K)$, and $\mathrm{Hom}_A(\mathcal{C}_q^A, S_p) = 0$ for $p \neq q+1$ in $\mathbb{Z}/n\mathbb{Z}$.*

(v) *For each $q \in \mathbb{Z}/n\mathbb{Z}$, $\Omega_A((\mathcal{C}_{q+1}^A)^s) = (\mathcal{C}_q^A)^s$ and $\Omega_A((\mathcal{X}_{q+1}^A)^s) = (\mathcal{X}_q^A)^s$.*

Proof. We may assume that $A = \hat{B}/G$, where B is a branch coextension of a canonical algebra C, with respect to the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C$ of stable tubes of Γ_A, and G is an infinite cyclic group generated by a strictly positive automorphism $g = f\varphi_{\hat{B}}^s$ for some positive integer s and some rigid automorphism f of $\hat{B}$. We use the notation introduced in the proof of Theorem 5.1. Let $n = 2s$ if $\varphi_{\hat{B}} = \nu_{\hat{B}}$, and $n = s$ if $\varphi_{\hat{B}}^2 = \nu_{\hat{B}}$. Then for the full convex subcategories C_q, B_q^-, B_q^+, B_q^* and $\overline{B}_q$, $q \in \mathbb{Z}$, from Theorem 5.1, we have $g(C_q) = C_{q+n}$, $g(B_q^-) = B_{q+n}^-$, $g(B_q^+) = B_{q+n}^+$, $g(B_q^*) = B_{q+n}^*$, and $g(\overline{B}_q) = \overline{B}_{q+n}$ for all $q \in \mathbb{Z}$. Consider now the induced actions of G on $\operatorname{mod} \hat{B}$ and $\Gamma_{\hat{B}}^s$. For the decomposition

$$\Gamma_{\hat{B}} = \bigvee_{q\in\mathbb{Z}} (\mathcal{X}_q \vee \mathcal{C}_q)$$

of $\Gamma_{\hat{B}}$ established in Theorem 5.1, we then have $g(\mathcal{X}_q) = \mathcal{X}_{q+n}$ and $g(\mathcal{C}_q) = \mathcal{C}_{q+n}$ for all $q \in \mathbb{Z}$. The push-down functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} \hat{B}/G = \operatorname{mod} A$ associated to the Galois covering $F : \hat{B} \to \hat{B}/G = A$ is exact and preserves Auslander–Reiten sequences, simple modules, and projective modules. Moreover, by Theorem 5.1(x), $\hat{B}$ is a locally support-finite category. Applying Theorem 4.2, we conclude that F_λ induces an isomorphism of the orbit translation quiver $\Gamma_{\hat{B}}/G$ of $\Gamma_{\hat{B}}$, with respect to the action of G, and the Auslander–Reiten quiver Γ_A of $A = \hat{B}/G$. Therefore, Γ_A has a decomposition

$$\Gamma_A = \bigvee_{q\in\mathbb{Z}/n\mathbb{Z}} (\mathcal{X}_q^A \vee \mathcal{C}_q^A)$$

with $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ and $\mathcal{X}_q^A = F_\lambda(\mathcal{X}_q)$, $\mathcal{C}_q^A = F_\lambda(\mathcal{C}_q)$ for $q \in \mathbb{Z}/n\mathbb{Z}$. Further, $\mathcal{C}_q^A = (\mathcal{C}_q^A(\lambda))_{\lambda\in\mathbb{P}_1(K)}$, where $\mathcal{C}_q^A(\lambda) = F_\lambda(\mathcal{C}_q(\lambda))$, $\lambda \in \mathbb{P}_1(K)$, are quasi-tubes such that $s(\mathcal{C}_q^A(\lambda)) + p(\mathcal{C}_q^A(\lambda)) = r(\mathcal{C}_q^A(\lambda)) - 1$, because F_λ preserves the simple and projective modules and ranks of the stable tubes of $\Gamma_{\hat{B}}^s$. Similarly, $\mathcal{X}_q^A = F_\lambda(\mathcal{X}_q)$ is a family of components of Γ_A containing a unique simple A-module $S_q = F_\lambda(S_q)$. This shows the statements (i) and (ii).

Since the push-down functor F_λ is dense, we also have a Galois covering $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} A$ of module categories. In particular, for any indecomposable modules M and N in $\operatorname{mod} \hat{B}$, the functor induces isomorphisms of K-vector spaces

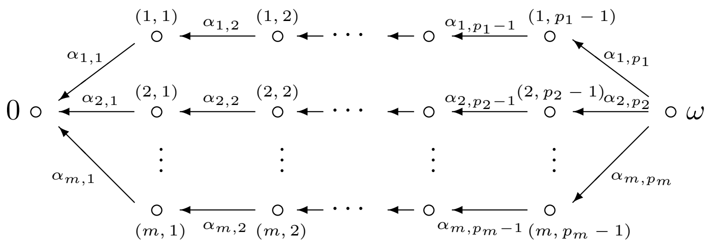
$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\hat{B}}({}^{g^r}M, N) \xrightarrow{\sim} \operatorname{Hom}(F_\lambda(M), F_\lambda(N)),$$

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\hat{B}}(M, {}^{g^r}N) \xrightarrow{\sim} \operatorname{Hom}(F_\lambda(M), F_\lambda(N)).$$

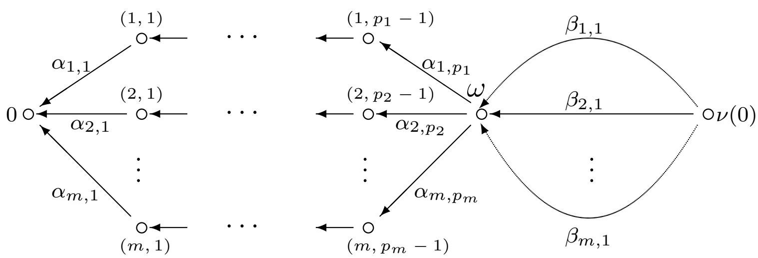
Hence, the statements (iii) and (iv) follow from the statements (xi) and (xii) of Theorem 5.1. Finally, since F_λ is exact and preserves the indecomposable modules and projective covers (see [17]), for any nonprojective indecomposable module M in $\operatorname{mod}\hat{B}$, we have $F_\lambda(\Omega_{\hat{B}}(M)) \cong \Omega_A F_\lambda(M)$. Hence, the statement (v) follows from the statement (xiii) of Theorem 5.1. ∎

We end this section with two examples illustrating possible situations.

EXAMPLE 5.4. Let B be a canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ with a weight sequence $\boldsymbol{p} = (p_1, \dots, p_m)$ and a parameter sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, $m \geq 2$, $\lambda_1 = \infty$, $\lambda_2 = 0$. Then C is the bound quiver algebra $K\Delta(\boldsymbol{p})/I(\boldsymbol{p}, \boldsymbol{\lambda})$, where $\Delta(\boldsymbol{p})$ is the quiver



and $I(\boldsymbol{p}, \boldsymbol{\lambda}) = 0$ for $m = 2$, while $I(\boldsymbol{p}, \boldsymbol{\lambda})$ is the ideal of $K\Delta(\boldsymbol{p}, \boldsymbol{\lambda})$ generated by the elements $\alpha_{j,p_j} \dots \alpha_{j,1} + \alpha_{1,p_1} \dots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \dots \alpha_{2,1}$, $j \in \{3, \dots, m\}$, for $m \geq 3$. Moreover, the Auslander–Reiten quiver Γ_C has a decomposition $\Gamma_A = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$, where $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes, described in Section 1. We use the notation introduced in Theorem 5.1. Hence $B = B_0^- = B_0^* = B_0^+ = C = C_0$ is a trivial quasi-tube enlargement of C. Further, the algebra $\overline{B}_1 = T_0^+ B_0^- = C[I_C(0)]$ is the bound quiver algebra $KQ_{\overline{B}_1}/I_{\overline{B}_1}$, where $Q_{\overline{B}_1}$ is the quiver



and $I_{\overline{B}_1}$ is the ideal of the path algebra $KQ_{\overline{B}_1}$ of $Q_{\overline{B}_1}$ generated by the elements

$$\beta_{1,1}\alpha_{1,p_1},\ \beta_{2,1}\alpha_{2,p_2},\ \beta_{1,1}\alpha_{2,p_2} \dots \alpha_{2,1} - \beta_{2,1}\alpha_{1,p_1} \dots \alpha_{1,1}$$

if $m = 2$, and the elements

$$\begin{aligned}
&\alpha_{j,p_j} \dots \alpha_{j,1} + \alpha_{1,p_1} \dots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \dots \alpha_{2,1}, \quad j \in \{3, \dots, m\},\\
&\beta_{j,1} + \beta_{1,1} + \lambda_j \beta_{2,1}, \quad j \in \{3, \dots, m\},\\
&\beta_{j,1}\alpha_{j,p_j}, \quad j \in \{1, \dots, m\}, \quad \beta_{2,1}\alpha_{1,p_1} \dots \alpha_{1,1} - \beta_{1,1}\alpha_{2,p_2} \dots \alpha_{2,1},
\end{aligned}$$

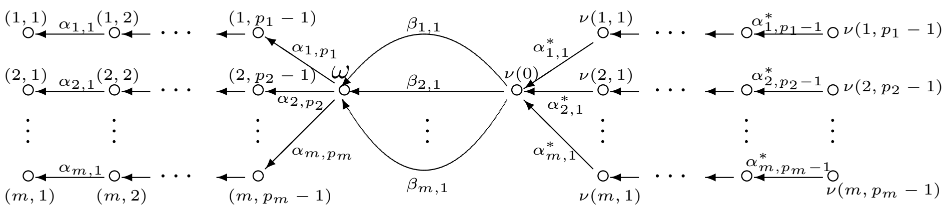
for $m \geq 3$. The reflection $B_1^- = S_0^+ B_0^+ = S_0^+ C$ is the bound quiver algebra $KQ_{B_1^-}/I_{B_1^-}$, where $Q_{B_1^-}$ is the quiver obtained from $Q_{\overline{B}_1}$ by removing the vertex 0 and the arrows $\alpha_{1,1}, \alpha_{2,1}, \dots, \alpha_{m,1}$, and $I_{B_1^-}$ is the ideal of $KQ_{B_1^-}$ generated by $\beta_{1,1}\alpha_{1,p_1}$ (if $p_1 \geq 2$) and $\beta_{2,1}\alpha_{2,p_2}$ (if $p_2 \geq 2$) for $m = 2$, and by the elements

$$\begin{aligned} &\beta_{j,1} + \beta_{1,1} + \lambda_j \beta_{2,1}, \quad j \in \{3, \dots, m\}, \\ &\beta_{j,1}\alpha_{j,p_j} \qquad \text{with } p_j \geq 2, \quad j \in \{1, \dots, m\}, \end{aligned}$$

for $m \geq 3$. Moreover, B_1^- is a branch coextension of the canonical algebra $C_1 = KQ_{C_1}/I_{C_1}$, where Q_{C_1} is the subquiver of $Q_{\overline{B}_1}$ given by the vertices ω, $\nu(0)$, and the arrows $\beta_{1,1}, \beta_{2,1}, \dots, \beta_{m,1}$, $I_{C_1} = 0$ for $m = 2$, and I_{C_1} is generated by $\beta_{j,1} + \beta_{1,1} + \lambda_j \beta_{2,1}$, $j \in \{3, \dots, m\}$, for $m \geq 3$. Observe that C_1 is isomorphic to the path algebra of the Kronecker quiver given by the arrows $\beta_{1,1}$ and $\beta_{2,1}$. Moreover, the vertices

$$(1,1), \dots, (1, p_1 - 1), (2,1), \dots, (2, p_2 - 1), \dots, (m, 1), \dots, (m, p_m - 1)$$

form a reflection sequence of sinks of $Q_{B_1^-}$. Then the quasi-tube enlargement B_1^* of C_1 associated to this reflection sequence of sinks is the bound quiver algebra $KQ_{B_1^*}/I_{B_1^*}$, where $Q_{B_1^*}$ is the quiver



and $I_{B_1^*}$ is generated by I_{C_1}, and the paths

$$\begin{aligned} &\beta_{j,1}\alpha_{j,p_j}, \alpha^*_{j,1}\beta_{j,1}, \qquad j \in \{1, \dots, m\}, \\ &\alpha^*_{1,r} \dots \alpha^*_{1,1}\beta_{2,1}\alpha_{1,p_1} \dots \alpha_{1,r}, \qquad r \in \{2, \dots, p_1 - 1\}, \\ &\alpha^*_{j,r} \dots \alpha^*_{j,1}\beta_{1,1}\alpha_{j,p_j} \dots \alpha_{j,r}, \qquad r \in \{2, \dots, p_j - 1\}, \ j \in \{2, \dots, m\}. \end{aligned}$$

Moreover, the associated iterated reflection algebra B_1^+ is a branch extension of the canonical algebra C_1 and the bound quiver algebra $KQ_{B_1^+}/I_{B_1^+}$, where $Q_{B_1^+}$ is the full convex subquiver of $Q_{B_1^*}$ given by the vertices

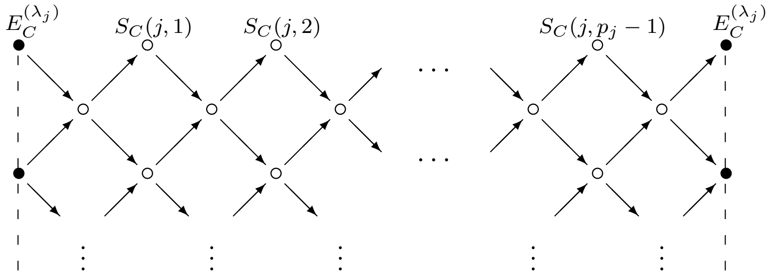
$$\begin{aligned} &\omega, \nu(0), \nu(1,1), \dots, \nu(1, p_1 - 1), \\ &\qquad\qquad \nu(2,1), \dots, \nu(2, p_2 - 1), \dots, \nu(m, 1), \dots, \nu(m, p_m - 1), \end{aligned}$$

and $I_{B_1^+}$ is generated by $\alpha^*_{1,1}\beta_{1,1}$ and $\alpha^*_{1,2}\beta_{2,1}$, for $m = 2$, and by the elements

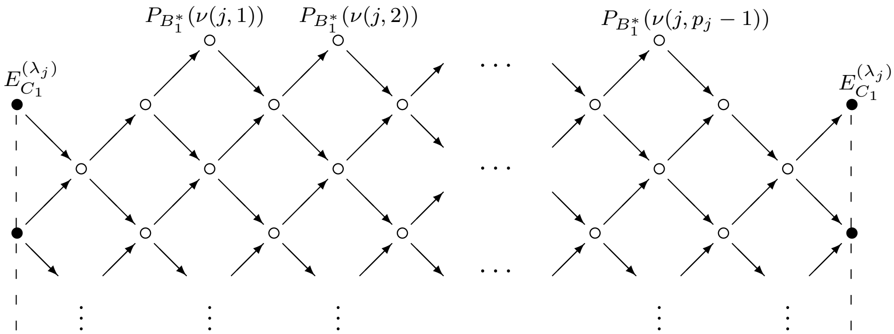
$$\beta_{j,1} + \beta_{1,1} + \lambda_j \beta_{2,1}, \quad j \in \{3, \dots, m\}, \qquad \alpha^*_{j,1}\beta_{j,1}, \quad j \in \{1, \dots, m\},$$

for $m \geq 3$. Observe also that the extension $\overline{B}_2 = T^+_\omega B^+_1$ is the algebra $\overline{B}_1^{\mathrm{op}}$ opposite to $\overline{B}_1$, while the reflection $B^-_2 = S^+_\omega B^+_1 = \nu_{\hat{B}}(B^-_0) = \nu_{\hat{B}}(C_0)$ is the canonical algebra isomorphic to C.

Assume $p_j \geq 2$ for some $j \in \{1, \dots, m\}$. Then $\mathcal{C}_0(\lambda_j) = \mathcal{T}^C_{\lambda_j}$ is a stable tube of rank p_j of the form



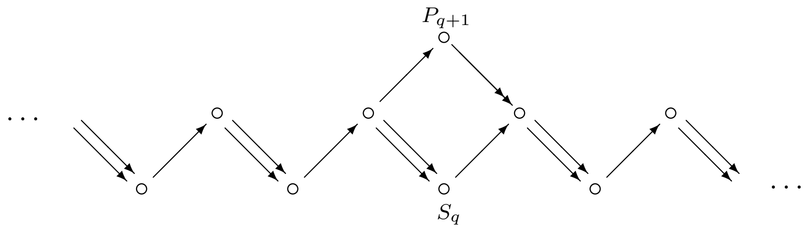
with $p_j - 1 = r(\mathcal{C}_0(\lambda_j)) - 1$ simple modules lying on its mouth. Applying Theorem 3.1 to the branch coextension B^-_1 of C_1, we conclude that the quasi-tube $\mathcal{C}_1(\lambda_j) = \mathcal{C}^{B^*_1}_\lambda$ contains $p_j - 1$ projective modules but no simple modules, and is of the form



Therefore, if the weight sequence $\boldsymbol{p} = (p_1, \dots, p_m)$ is different from $(1, \dots, 1)$, then $B^-_0 \ncong B^-_1$, $\varphi_{\hat{B}} = \nu_{\hat{B}}$, and

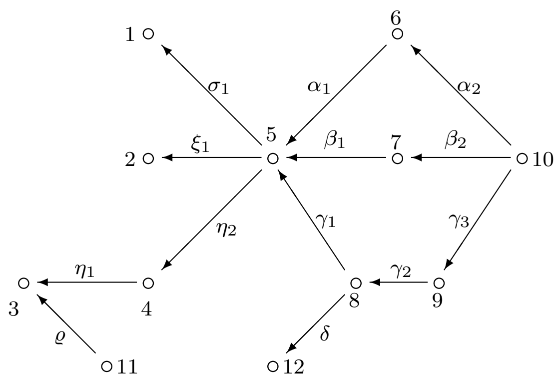
- for even $q \in \mathbb{Z}$, $\mathcal{C}_q = (\mathcal{C}_q(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of stable tubes of $\Gamma_{\hat{B}}$ containing simple $\hat{B}$-modules but no projective $\hat{B}$-modules,
- for odd $q \in \mathbb{Z}$, $\mathcal{C}_q = (\mathcal{C}_q(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes of $\Gamma_{\hat{B}}$ containing projective $\hat{B}$-modules but no simple $\hat{B}$-modules.

For the weight sequence $\boldsymbol{p} = (p_1, \dots, p_m) = (1, \dots, 1)$, each $\mathcal{C}_q = (\mathcal{C}_q(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of stable tubes of $\Gamma_{\hat{B}}$ of rank 1, and hence all simple $\hat{B}$-modules and indecomposable projective $\hat{B}$-modules are located in the families $\mathcal{X}_q$, $q \in \mathbb{Z}$. In fact, each $\mathcal{X}_q$ then consists of one component, which is of the form

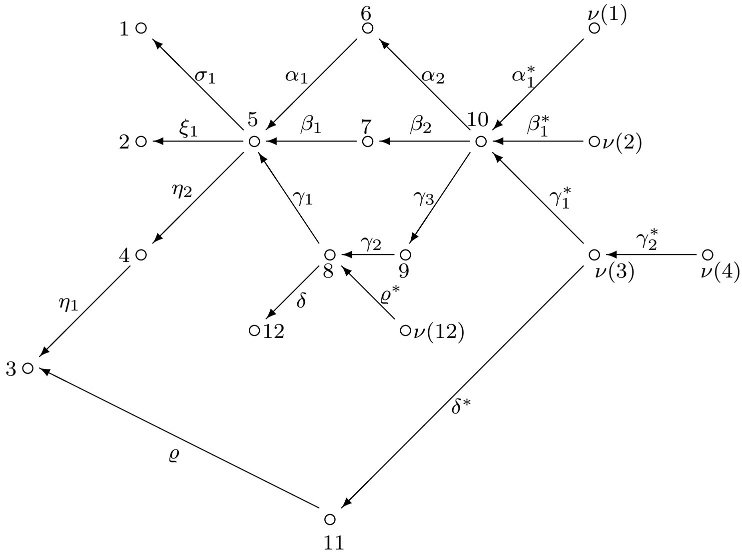


and $P_{q+1} = P_{\hat{B}}(S_{q+1})$ is the projective cover of the simple $\hat{B}$-module $S_{q+1} \in \mathcal{X}_{q+1}$. We note that in this degenerate case, that is, for the Kronecker algebra $B = C$, we have $\varphi_{\hat{B}} \neq \nu_{\hat{B}}$ and $\varphi_{\hat{B}}^2 = \nu_{\hat{B}}$.

EXAMPLE 5.5. Let $B = KQ_B/I_B$, where Q_B is the quiver

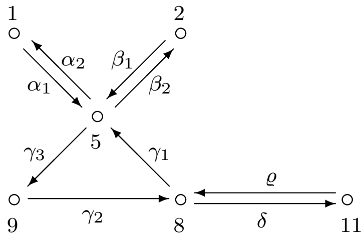


and I_B is the ideal of the path algebra KQ_B of Q_B generated by the elements $\alpha_1\sigma_1$, $\beta_1\xi_1$, $\gamma_1\eta_2$, $\gamma_2\delta$, $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1$. Denote by C the bound quiver algebra $C = KQ_C/I_C$, where Q_C is the full subquiver of Q given by the vertices 5, 6, 7, 8, 9, 10, and I_C is the ideal in the path algebra KQ_C of Q_C generated by $\gamma_3\gamma_2\gamma_1+\alpha_2\alpha_1+\beta_2\beta_1$. Then C is the canonical algebra $C(\boldsymbol{p}, \boldsymbol{\lambda})$ with the weight sequence $\boldsymbol{p} = (2, 2, 3)$ and the parameter sequence $\boldsymbol{\lambda} = (\infty, 0, 1)$. Moreover, B is a branch coextension $B = [E_1, \mathcal{L}_1, E_2, \mathcal{L}_2, E_3, \mathcal{L}_3, E_4, \mathcal{L}_4]C$ with $E_1 = E^{(\infty)} \in \mathcal{T}_\infty^C$, $E_2 = E^{(0)} \in \mathcal{T}_0^C$, $E_3 = E^{(1)} \in \mathcal{T}_1^C$, $E_4 = S(8) \in \mathcal{T}_1^C$, $\mathcal{L}_1$ the branch given by the vertex 1, $\mathcal{L}_2$ the branch given by the vertex 2, $\mathcal{L}_3$ the branch given by the vertices $3, 4, 11$ and arrows η_1, ϱ, and $\mathcal{L}_4$ the branch given by the vertex 12. Then 1, 2, 3, 4, 12 is a reflection sequence of sinks of Q_B, and the iterated extension $B_0^* = T^+_{1,2,3,4,12}B$ is the bound quiver algebra $KQ_{B_0^*}/I_{B_0^*}$, where $Q_{B_0^*}$ is the quiver



and $I_{B_0^*}$ is the ideal of $KQ_{B_0^*}$ generated by the elements $\alpha_1\sigma_1$, $\beta_1\xi_1$, $\gamma_1\eta_2$, $\gamma_2\delta$, $\gamma_3\gamma_2\gamma_1+\alpha_2\alpha_1+\beta_2\beta_1$, $\alpha_1^*\alpha_2$, $\beta_1^*\beta_2$, $\gamma_1^*\gamma_3$, $\gamma_2^*\delta^*$, $\delta^*\varrho-\gamma_1^*\alpha_2\alpha_1\eta_2\eta_1$, $\varrho^*\gamma_1$. Moreover, the iterated reflection $B_0^+ = S_{12}^+S_4^+S_3^+S_2^+S_1^+B_0^-$ of $B_0^- = B$ is the bound quiver algebra $KQ_{B_0^+}/I_{B_0^+}$, where $Q_{B_0^+}$ is the full subquiver of $Q_{B_0^*}$ given by the vertices 5, 6, 7, 8, 9, 10, $\nu(1)$, $\nu(2)$, $\nu(3)$, $\nu(4)$, $\nu(12)$, and $I_{B_0^+}$ is the ideal of $KQ_{B_0^+}$ generated by the elements $\gamma_3\gamma_2\gamma_1+\alpha_2\alpha_1+\beta_2\beta_1$, $\alpha_1^*\alpha_2$, $\beta_1^*\beta_2$, $\gamma_1^*\gamma_3$, $\gamma_2^*\delta^*$, $\varrho^*\gamma_1$, which is the branch extension $C[E_1,\mathcal{L}_1^*,E_2,\mathcal{L}_2^*,E_3,\mathcal{L}_3^*,E_4,\mathcal{L}_4^*]$, with $\mathcal{L}_1^*$ the branch given by the vertex $\nu(1)$, $\mathcal{L}_2^*$ the branch given by the vertex $\nu(2)$, $\mathcal{L}_3^*$ the branch given by the vertices $\nu(3)$, $\nu(4)$, 11 and arrows γ_2^*, δ^*, and $\mathcal{L}_4^*$ the branch given by the vertex $\nu(12)$. Observe also that the reflection $B_1^- = S_5^+B_0^+$ is isomorphic to $B_0^- = B$. Therefore, we have a canonical strictly positive automorphism $\varphi_{\hat{B}}$ of $\hat{B}$, with $\varphi_{\hat{B}}^2 = \nu_{\hat{B}}$ and $\varphi_{\hat{B}} \neq \nu_{\hat{B}}$, such that $\varphi_{\hat{B}}(e_{0,1}) = e_{0,6}$, $\varphi_{\hat{B}}(e_{0,2}) = e_{0,7}$, $\varphi_{\hat{B}}(e_{0,3}) = e_{0,8}$, $\varphi_{\hat{B}}(e_{0,4}) = e_{0,9}$, $\varphi_{\hat{B}}(e_{0,5}) = e_{0,10}$, $\varphi_{\hat{B}}(e_{0,6}) = e_{1,1} = \nu_{\hat{B}}(e_{0,1})$, $\varphi_{\hat{B}}(e_{0,7}) = e_{1,2} = \nu_{\hat{B}}(e_{0,2})$, $\varphi_{\hat{B}}(e_{0,8}) = e_{1,3} = \nu_{\hat{B}}(e_{0,3})$, $\varphi_{\hat{B}}(e_{0,9}) = e_{1,4} = \nu_{\hat{B}}(e_{0,4})$, $\varphi_{\hat{B}}(e_{0,10}) = e_{1,5} = \nu_{\hat{B}}(e_{0,5})$, $\varphi_{\hat{B}}(e_{0,12}) = e_{0,11}$. Moreover, we also have a rigid automorphism f of $\hat{B}$ induced by the automorphism h of the quiver Q_B of order 2 such that $h(1)=2$, $h(2)=1$, $h(3)=3$, $h(4)=4$, $h(5)=5$, $h(6)=7$, $h(7)=6$, $h(8)=8$, $h(9)=9$, $h(10)=10$, $h(11)=11$, $h(12)=12$, and $h(\sigma_1)=\xi_1$, $h(\xi_1)=\sigma_1$, $h(\alpha_1)=\beta_1$, $h(\beta_1)=\alpha_1$, $h(\alpha_2)=\beta_2$, $h(\beta_2)=\alpha_2$, $h(\gamma_1)=\gamma_1$, $h(\gamma_2)=\gamma_2$, $h(\gamma_3)=\gamma_3$, $h(\eta_1)=\eta_1$, $h(\eta_2)=\eta_2$, $h(\varrho)=\varrho$, $h(\delta)=\delta$.

Consider the orbit algebras $A = \hat{B}/(\varphi_{\hat{B}})$ and $A' = \hat{B}/(f\varphi_{\hat{B}})$. Then A and A' are the bound quiver algebras $A = KQ/I$ and $A' = KQ/I'$, where Q is the quiver



and I is the ideal of KQ generated by the elements

$$\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1,\ \alpha_1\alpha_2,\ \beta_1\beta_2,\ \gamma_1\gamma_3,\ \varrho\gamma_1,\ \gamma_2\delta,$$
$$\alpha_2\alpha_1\beta_2\beta_1 - \beta_2\beta_1\alpha_2\alpha_1,\ \gamma_1\gamma_3\gamma_2 - \delta\varrho,$$

and I' is the ideal of KQ generated by the elements

$$\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1,\ \alpha_1\beta_2,\ \beta_1\alpha_2,\ \gamma_1\gamma_3,\ \varrho\gamma_1,\ \gamma_2\delta,$$
$$\alpha_2\alpha_1\alpha_2\alpha_1 - \beta_2\beta_1\beta_2\beta_1,\ \gamma_1\gamma_3\gamma_2 - \delta\varrho.$$

We note that A is a symmetric algebra and A' is not symmetric. According to Theorem 5.3, the Auslander–Reiten quivers Γ_A and $\Gamma_{A'}$ have decompositions

$$\Gamma_A = \mathcal{X}^A \vee \mathcal{C}^A \quad \text{and} \quad \Gamma_{A'} = \mathcal{X}^{A'} \vee \mathcal{C}^{A'}$$

where $\mathcal{C}^A = (\mathcal{C}^A(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ (respectively, $\mathcal{C}^{A'} = (\mathcal{C}^{A'}(\lambda))_{\lambda\in\mathbb{P}_1(K)}$) is a $\mathbb{P}_1(K)$-family of quasi-tubes of Γ_A (respectively, $\Gamma_{A'}$) containing all simple modules and indecomposable projective modules, except the simple module $S(5)$ and the projective module $P(5)$ at the vertex 5, which are located in $\mathcal{X}^A$ (respectively, $\mathcal{X}^{A'}$).

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