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VERY SLOWLY VARYING FUNCTIONS. II

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Abstract. This paper is a sequel to papers by Ash, Erdős and Rubel, on very slowly varying functions, and by Bingham and Ostaszewski, on foundations of regular variation. We show that generalizations of the Ash–Erdős–Rubel approach—imposing growth restrictions on the function h, rather than regularity conditions such as measurability or the Baire property—lead naturally to the main result of regular variation, the Uniform Convergence Theorem.

1. Introduction. We work with the Karamata theory of regular and slow variation; see [BGT] for a monograph account. Here the main result is the Uniform Convergence Theorem (UCT) which asserts that the defining pointwise convergence for slow variation in fact holds uniformly on compact sets if the function h in question is either (Lebesgue) measurable, or has the Baire property, but not in general (see Theorem 3 below). The outstanding foundational question of the theory—raised and left open in [BG1], [BG2], [BGT]—is what common generalization of measurability and the Baire property suffices. This question is answered in [BOst1], where we obtain sets of conditions on h, each necessary and sufficient for UCT (see Theorem UCT below). In [BOst2] this motivates a unified approach to the Karamata theory of the two cases by regarding each as a subfamily of a single family of functions, one that is defined by combinatorial properties shared by both. An alternative unification (see [BOst3]) derives the measure and category forms of their shared infinite combinatorics from a single topological result, the Category Embedding Theorem, by specialization to two topologies—the Euclidean topology and the density topology.

A very few papers in regular variation are able to make progress without imposing regularity conditions. Foremost among these are the Ash–Erdős–Rubel paper [AER], where a growth condition is used instead, and the work of Heiberg [Hei] and Seneta [Sen1], [Sen2], where side-conditions involving the limsup are imposed instead. Informed by the viewpoint of [BOst1], we generalize the results of these papers.

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In Section 2 we apply the Theorem UCT of [BOst1] to derive a new, simple, necessary and sufficient condition on a function h so that it obeys the UCT and obtain a "second Heiberg–Seneta theorem" (see Theorem 1). In Section 3 we show how the simple conditions may be usefully relaxed and give *a* "generalized Heiberg–Seneta theorem" (Theorem 2). We use this in Section 4 to show why the example of [AER] does not satisfy the UCT (see Theorem 3). We close in Section 5 with some complements.

2. Main result. We begin by defining the key notions of the theory of regular variation. Then we recall the definitions and two theorem of [BOst1] which we will need here. The theory is concerned with the consequences of a relationship of the form

$$\text{(RV)} \qquad f(\lambda x)/f(x) \to g(\lambda) \quad (x \to \infty) \quad \forall \lambda > 0,$$

for functions defined on $\mathbb{R}_+$. The limit function g must satisfy the Cauchy functional equation

$$\text{(CFE)} \qquad g(\lambda\mu) = g(\lambda)g(\mu) \quad \forall \lambda, \mu > 0.$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$(\varrho) \qquad g(\lambda) = \lambda^{\varrho} \quad \forall \varrho > 0.$$

Then f is said to be *regularly varying* with *index* ϱ, written $f \in R_{\varrho}$.

The case $\varrho = 0$ is basic. A function f is called *slowly varying* if $f \in R_0$, i.e. if

$$\text{(SV)} \qquad f(\lambda x)/f(x) \to 1 \quad (x \to \infty) \quad \forall \lambda > 0.$$

Slowly varying functions are often written ℓ (for *lente*, or *langsam*). The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that under appropriate assumptions, if (SV) holds, then the convergence is *uniform* on compact sets of λ values in $(0, \infty)$. Necessary and sufficient assumptions for UCT have only recently been given (in [BOst1]) and are quoted below for convenience. While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. If we write $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), the relation above becomes

$$(\mathrm{SV}_+) \qquad h(x+u) - h(x) \to 0 \quad (x \to \infty) \quad \forall u \in \mathbb{R}.$$

Here the functions are defined on $\mathbb{R}$, whereas in the multiplicative notation functions are defined on $\mathbb{R}_+$. We find it helpful to use the notation $h_x(u) = h(x+u) - h(x)$.

DEFINITIONS

(i) The *ε-level set* (of h_x) is defined to be the set

$$H^{\varepsilon}(x) = \{t : |h(t+x) - h(x)| < \varepsilon\}.$$

(ii) For $\mathbf{x} = \{x_n : n \in \omega\}$ an arbitrary sequence tending to infinity, the $\mathbf{x}$-*stabilized* ε-*level set* (of h) is defined to be the set

$$T_k^{\varepsilon}(\mathbf{x}) = \bigcap_{n=k}^{\infty} H^{\varepsilon}(x_n) \quad \text{for } k \in \omega.$$

Here ω denotes the set of natural numbers $0, 1, 2, \ldots$. Note that

(1) $$T_0^{\varepsilon}(\mathbf{x}) \subseteq T_1^{\varepsilon}(\mathbf{x}) \subseteq T_2^{\varepsilon}(\mathbf{x}) \subseteq \cdots \text{ and } T_k^{\varepsilon}(\mathbf{x}) \subseteq T_k^{\eta}(\mathbf{x}) \text{ whenever } \varepsilon < \eta.$$

If h is slowly varying, then $\mathbb{R} = \bigcup_{k \in \omega} T_k^{\varepsilon}(\mathbf{x})$.

(iii) The basic *No Trumps* combinatorial principle (there are several), denoted $\mathbf{NT}(\{T_k : k \in \omega\})$, refers to a family of subsets of reals $\{T_k : k \in \omega\}$ and means the following. For every bounded sequence $\{u_m : m \in \omega\}$ of reals there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite set $\mathbb{M} \subseteq \omega$ such that

$$t + u_m \in T_k \quad \text{for all } m \text{ in } \mathbb{M}.$$

In words: the translate of some *subsequence* of $\{u_m\}$ is contained in some T_k. We will also say that $\{T_k : k \in \omega\}$ *traps sequences by translation.*

We now quote from [BOst1].

Theorem (UCT). *For h slowly varying, the following are equivalent.*

(i) *The UCT holds for h.*

(ii) *The principle* $\mathbf{1\text{-}NT}_h$ *holds: for every $\varepsilon > 0$ and every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^{\varepsilon}(\mathbf{x}) : k \in \omega\}$ of h trap bounded sequences by translation. In loose notation:*

$$(\forall \varepsilon > 0)(\forall \mathbf{x})\ \mathbf{NT}(\{T_k^{\varepsilon}(\mathbf{x}) : k \in \omega\}).$$

(iii) *For every $\varepsilon > 0$ and for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^{\varepsilon}(\mathbf{x}) : k \in \omega\}$ of h contain all the bounded sequences.*

The property in (iii) is called the full-inclusion or F-analogue of $\mathbf{1\text{-}NT}_h$. For the proof see [BOst1], where it is also shown that either of the conditions (ii) or (iii) holds for measurable h, and also for h with the Baire property. We will also need the following result from [BOst1].

Theorem (Equivalence Theorem). *For h a slowly varying function the following are equivalent.*

(i) *The family $\{T_n^{\varepsilon}(\mathbf{x}) : n \in \omega\}$ traps bounded sequences for any sequence $\mathbf{x}$ tending to infinity, and any positive ε.*

(ii) *Whenever $\{u_n\}$ is a bounded sequence, and $\{x_n\}$ tends to infinity,*

(2) $$\lim_{n\to\infty} (h(u_n + x_n) - h(x_n)) = 0.$$

(iii) *For any sequence* $\mathbf{x}$ *tending to infinity, and any positive* ε, *the family* $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ *ultimately contains almost all of any bounded sequence* $\{u_n\}$. *That is, for any bounded sequence* $\{u_n\}$ *there is* k *such that*

$$\{u_m : m > k\} \subseteq T_n^\varepsilon(\mathbf{x}) \quad \textit{for all } n > k. \tag{3}$$

(iv) *The UCT holds for* h.

Definition. We say that h satisfies the *Heiberg–Lipschitz condition* if there are two positive functions φ, g defined on $\mathbb{R}_+$ such that:

(i) $g(x)$ is decreasing to 0 as $x \to \infty$;
(ii) $\varphi(t) \to \infty$ as $t \to \infty$;
(iii) for all $x, t > 0$, there is $x(t)$ between x and $x + t$ such that

$$|h(t + x) - h(x)| = \varphi(t)g(x(t)). \tag{4}$$

The final condition is modelled after the mean-value theorem. Note that the assumptions imply that for all $x, t > 0$,

$$\varphi(t)g(x + t) \le |h(t + x) - h(x)| \le \varphi(t)g(x).$$

This is the information which makes the proof of our main theorem (Theorem 1 below) transparent; we show later how to relax these assumptions to obtain a more useful formulation of the basic paradigm. When studying slowly varying functions h in the context of the Uniform Convergence Theorem (UCT) it helps to paraphrase the concepts by reference to the notation introduced earlier:

$$h_x(u) = h(u + x) - h(x).$$

Regarding x as a parameter and $h_x(u)$ as an "approximately-additive" function of u, slow variation is just pointwise convergence to zero of the family $\{h_x\}$ as $x \to \infty$ (at all single points u). Thus UCT is the qualified assertion that pointwise convergence of the family $\{h_x\}$ implies uniform convergence over compact sets of u. In this language, the simple Heiberg–Lipschitz condition (4) "factorizes out of h_x its dependence on x" locally. The original (i.e. First—see Section 5 below) Heiberg–Seneta Theorem factorizes out "dependence on x at infinity", studying in essence an appropriate application of L'Hospital's Rule. Our Generalized Heiberg–Seneta Theorem 2 of Section 3 is then the "direct comparison" analogue.

Remark. The preceding definition subsumes the case of any increasing, differentiable concave function $h(x)$ satisfying the celebrated "Inada conditions" of Economic Theory, introduced in [Inada]. This class includes $\log x$ and the power functions x^ϱ with $0 < \varrho < 1$. Indeed, for h satisfying the Heiberg–Lipschitz condition, we have, for $t > 0$, for some x^* with

$x < x^* < x + t$,

$$0 < h(x+t) - h(x) = h'(x^*)t.$$

Thus with $g(x) = h'(x)$ and $x(t) = x^*$ the conditions are met since $g(x)$ is decreasing to 0 as $x \to \infty$.

OBSERVATION. *If h satisfies the Heiberg–Lipschitz condition, then h is slowly varying.*

For,

$$\lim_{x\to\infty} |h(t+x) - h(x)| \le \lim_{x\to\infty} \varphi(t)g(x) = 0.$$

Our main result follows (for details of the *First Heiberg–Seneta Theorem* see the closing discussion in Section 5). This new theorem thus complements [Hei], [Sen1], [Sen2] (cf. [BGT, Theorem 1.4.3, pp. 18–19]).

THEOREM 1 (Second Heiberg–Seneta Theorem). *For h satisfying the Heiberg–Lipschitz condition the following are equivalent.*

(i) *UCT holds for h.*
(ii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ traps sequences by translation.*
(iii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains almost all terms of every bounded sequence.*
(iv) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains every bounded sequence.*

Proof. We will show first (a) that (ii) implies (i), and then (b) that (i) implies (ii).

Clearly (iv)⇒(iii) and (iii)⇒(ii). The proof will thus be complete when in (c) we explain how to adapt the notation used in the proof of (b) so that it reads as a proof of (i)⇒(iv).

(a) *Proof that* (ii)⇒(i). Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity, let $\mathbf{u} = \{u_m\}$ be any bounded sequence and suppose that the condition of the Equivalence Theorem, namely $\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0$, fails. Thus we suppose that for some $\varepsilon > 0$ and for $n = 1, 2, \ldots$ we have

$$|h(x_n + u_n) - h(x_n)| > 2\varepsilon. \tag{5}$$

Working by analogy with ε-level sets, define the reduced level sets by

$$H_n^- = \{t : \varphi(t) < \varepsilon/g(x_n)\}.$$

Thus

$$H_n^- \subseteq \{y : |h(x_n + y) - h(x_n)| < \varepsilon\} = H^\varepsilon(x_n).$$

Observe next that

$$\begin{aligned} H_n^- - u_m &= \{y : (\exists t)[y = t - u_m \,\&\, t \in H_n^-]\} \\ &= \{y : (\exists t)[t = u_m + y \,\&\, \varphi(t) < \varepsilon/g(x_n)]\} \\ &= \{y : \varphi(u_m + y) < \varepsilon/g(x_n)\}. \end{aligned}$$

Since $\{\varphi^{-1}((0,n)) : n \in \omega\}$ is sequence trapping, there are N, y and infinite $\mathbb{M}$ such that

$$\{u_m + y : m \in \mathbb{M}\} \subseteq \varphi^{-1}((0,N)), \quad \text{i.e.} \quad \{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N).$$

But, for some k large enough, we have $\varepsilon/g(x_k) > N$. Hence, for this y, we have, for $n \geq k$,

$$\{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N) \subseteq (0, \varepsilon/g(x_n)).$$

Thus by definition of H_n, for all $m \in \mathbb{M}$,

$$y \in \bigcap_{n=k}^{\infty} H_n^- - u_m.$$

We now claim that, for any $n \geq k$ with $n \in \mathbb{M}$,

$$|h(x_n + u_n) - h(x_n + u_n + y)| \geq \varepsilon. \tag{6}$$

Indeed, otherwise, for any such n,

$$|h(x_n + u_n) - h(x_n + u_n + y)| < \varepsilon. \tag{7}$$

But referring to $x = x_n$ and $t = u_n + y$ in clause (iii) of the Heiberg–Lipschitz condition we have, since $y \in H_n^- - u_n$,

$$|h(u_n + y + x_n) - h(x_n)| \leq \varphi(u_n + y) g(x_n) < \varepsilon,$$

and this combined with (7) yields

$$|h(x_n + u_n) - h(x_n)| < 2\varepsilon,$$

a contradiction to our standing assumption (5).

Define $v_n = x_n + u_n$ (which tends to infinity). Then (6) yields

$$|h(v_n + y) - h(v_n)| \geq \varepsilon$$

for infinitely many n, which contradicts the fact that h is slowly varying.

(b) *Proof that* (i)⇒(ii). Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity and let $\mathbf{u} = \{u_m\}$ be any positive bounded sequence (otherwise pass to a subsequence). Assume for some $b > 0$ that for all $m \in \omega$ we have

$$0 \leq u_m \leq b.$$

Again working by analogy with ε-level sets, define the expanded level sets by

$$H_n^+ = \{t : \varphi(t) < \varepsilon/g(x_n + t)\}.$$

Thus

$$H^\varepsilon(x_n) = \{y : |h(x_n + y) - h(x_n)| < \varepsilon\} \subseteq H_n^+,$$

since, for $t \in H^\varepsilon(x_n)$,

$$\varphi(t) g(x_n + t) \leq |h(t + x_n) - h(x_n)| < \varepsilon.$$

Now if UCT holds, then by the Theorem UCT of [BOst1], $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ traps sequences, so for some y, infinite $\mathbb{M}$ and $k \in \omega$, we have

$$\{y + u_m : m \in \mathbb{M}\} \subseteq T_k^\varepsilon(\mathbf{x}) = \bigcap_{n=k}^{\infty} H^\varepsilon(x_n) \subseteq H^\varepsilon(x_k)$$
$$\subseteq H_k^+ = \{t : \varphi(t) < \varepsilon/g(x_k + t)\},$$

i.e.

$$\{y + u_m : m \in \mathbb{M}\} \subset \{t : \varphi(t) < \varepsilon/g(x_k + t)\}.$$

Thus, for $m \in \mathbb{M}$,

$$\varphi(y + u_m) < \varepsilon/g(x_k + y + u_m) \le \varepsilon/g(x_k + y + b).$$

Choose an integer N such that $\varphi(t) > \varepsilon/g(x_k + y + b)$ for all $t > N$. Then

$$\{y + u_m : m \in \mathbb{M}\} \subset \varphi^{-1}(0, N),$$

as required.

(c) *Modification to* (b). Given the Theorem UCT of [BOst1] we may clearly adapt the proof just given in (b) to show that (i) implies (iv) by putting $y = 0$ and $\mathbb{M} = \omega$. ∎

3. A generalization. In this section we show one possible way to move away from the context dictated by the mean-value theorem and still have a corresponding Second Heiberg–Seneta Theorem. Some further alternative formulations are discussed in Section 4.

Definition. We say that h satisfies the *generalized Heiberg–Lipschitz condition* if

(a) there is a function φ defined on $\mathbb{R}_+$ such that $\varphi(t) \to \infty$ as $t \to \infty$;
(b) there are functions g_+, g_- defined on $\mathbb{R}_+^2$ such that, for $x, t > 0$,

$$g_-(x, \varphi(t)) \le |h(x + t) - h(x)| \le g_+(x, \varphi(t)), \tag{8}$$

(c) for all $\varepsilon > 0$ small enough, the solution sets of $g_\pm(x, y) < \varepsilon$ are bounded in y and take the form $\{y : y < \psi_\pm(x, \varepsilon)\}$ for some functions $\psi_\pm(x, \varepsilon)$;
(d) $\lim_{x\to\infty} \psi_+(x, \varepsilon) = \infty$ for all $\varepsilon > 0$ small enough.

Observation. *If h satisfies the generalized Heiberg–Lipschitz condition, then h is slowly varying.*

Indeed, given $t, \varepsilon > 0$ there exists $X > 0$, by condition (d), such that $\psi_+(x, \varepsilon) > \phi(t)$ for $x > X$, or equivalently, from (c), such that $g_+(x, \varphi(t)) < \varepsilon$ for $x \ge X$. In this case we conclude, for $x > X$, that

$$|h(x + t) - h(x)| < \varepsilon.$$

Note that the observation relies only on the right-hand inequality in (8).

THEOREM 2 (Generalized Heiberg–Seneta Theorem). *For h satisfying the generalized Heiberg–Lipschitz condition the following are equivalent.*

(i) *UCT holds for h.*
(ii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ traps sequences by translation.*
(iii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains almost all terms of every bounded sequence.*
(iv) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains every bounded sequence.*

Proof. We follow the proof structure of the Second Heiberg–Seneta Theorem.

(a) *Proof of UCT from* (ii). As before, suppose for some $\varepsilon > 0$ and for $n = 1, 2, \ldots$ that

$$|h(x_n + u_n) - h(x_n)| \geq 2\varepsilon. \tag{9}$$

As expected, put $H_n^- = \{t : g_+(x_n, \varphi(t)) < \varepsilon\}$. Thus

$$H_n^- = \{y : |h(y + x_n) - h(x_n)| < \varepsilon\} \subseteq H^\varepsilon(x_n),$$

since

$$|h(y + x_n) - h(x_n)| \leq g_+(x_n, \varphi(y)) < \varepsilon.$$

As before,

$$H_n^- - u_m = \{y : g_+(x_n, \varphi(u_m + y)) < \varepsilon\} \subseteq \{y : \varphi(y + u_m) < \psi_+(x_n, \varepsilon)\}.$$

Since $\{\varphi^{-1}((0,n)) : n \in \omega\}$ is sequence trapping, there are N, y and infinite $\mathbb{M}$ such that

$$\{u_m + y : m \in \mathbb{M}\} \subseteq \varphi^{-1}((0,N)), \quad \text{i.e.} \quad \{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N).$$

Since $\lim_{x\to\infty} \psi_+(x,\varepsilon) = \infty$, for some k large enough we have $\psi_+(x_n, \varepsilon) > N$ for all $n \geq k$. Hence, for this y and $n \geq k$,

$$\{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N) \subseteq (0, \psi_+(x_n, \varepsilon)).$$

Thus by definition of H_n^-, for all $m \in \mathbb{M}$,

$$y \in \bigcap_{n=k}^{\infty} H_n^- - u_m.$$

We now claim that, for any $n \geq k$ with $n \in \mathbb{M}$,

$$|h(x_n + u_n) - h(x_n + u_n + y)| \geq \varepsilon. \tag{10}$$

Indeed, otherwise for any such n,

$$|h(x_n + u_n) - h(x_n + u_n + y)| < \varepsilon. \tag{11}$$

But referring to $x = x_n$ and $t = u_n + y$ in clause (b) of the generalized Heiberg–Lipschitz condition we have, since $y \in H_n^- - u_n$,

$$|h(u_n + y + x_n) - h(x_n)| \leq g_+(x_n, \varphi(y + u_n)) < \varepsilon,$$

and this combined with (11) yields

$$|h(x_n + u_n) - h(x_n)| < 2\varepsilon,$$

a contradiction to our standing assumption (9).

Define $v_n = x_n + u_n$ (which tends to infinity); then (10) yields

$$|h(v_n + y) - h(v_n)| \geq \varepsilon$$

for infinitely many n, which contradicts the assumption that h is slowly varying.

(b) *Proof that UCT implies* (ii). As expected put $H_n^+ = \{t : g_-(x_n, \varphi(t)) < \varepsilon\}$. Thus

$$H^\varepsilon(x_n) = \{y : |h(y + x_n) - h(x_n)| < \varepsilon\} \subseteq H_n^+,$$

since, for $y \in H^\varepsilon(x_n)$,

$$g_-(x_n, \varphi(y)) \leq |h(y + x_n) - h(x_n)| < \varepsilon.$$

Now if UCT holds then, by the No Trumps Theorem of [BOst1, Section 3], $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ traps sequences, so for some y, infinite $\mathbb{M}$, and k we have, as before,

$$\{y + u_m : m \in \mathbb{M}\} \subseteq T_k^\varepsilon(\mathbf{x}) \subseteq H^\varepsilon(x_k) \subseteq H_k^+ = \{t : g_-(x_k, \varphi(t)) < \varepsilon\},$$

i.e.

$$\{y + u_m : m \in \mathbb{M}\} \subset \{t : \varphi(t) < \psi_-(x_k, \varepsilon)\}.$$

Thus, for $m \in \mathbb{M}$,

$$\varphi(y + u_m) < \psi_-(x_k, \varepsilon).$$

Choose an integer N such that $\varphi(t) > \psi_-(x_k, \varepsilon)$ for all $t > N$. Then $\{y + u_m : m \in \mathbb{M}\} \subseteq \varphi^{-1}(0, N)$, as required.

(c) *Modifications to* (b). Now if UCT holds then, by part (iii) of the Theorem UCT of [BOst1] (see Section 1 above), $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ contains all sequences, so the proof in (b) may be re-read with $y = 0$ and $\mathbb{M} = \omega$. ■

4. An application. We now take the view that $\mathbb{R}$ is a vector space over the field $\mathbb{Q}$. For the purposes of the next result, we need to assume the existence of a (Hamel) basis in this vector space. Its existence is ensured by the Axiom of Choice (AC); as is well-known, (AC) implies that every vector space has a basis. We note in passing that the converse is also true; see [Bl].

Fix a Hamel basis H which includes 1. Let $n(t)$ be the cardinality of the smallest subset of H which spans t (over $\mathbb{Q}$). We now use the Generalized Heiberg–Seneta Theorem to explain why the following slowly varying function, introduced in [AER], does not obey UCT. Whilst our proof is slightly longer than that in [BGT, pp. 10–11], we feel that it casts rather more light on what is happening.

THEOREM 3. *The slowly varying function $h(x) = \log(x+n(x))$ does not satisfy UCT.*

Proof. We begin by establishing the right inequality of (8) for all $x \in \mathbb{R}_+$ and the left inequality for all *rational* $x \in \mathbb{R}_+$. (The former implies that h is slowly varying.)

Note that $n(x+t) \le n(x) + n(t)$ and $n(x) = n(-x)$. Therefore $n(t) = n(t+x-x) \le n(x+t) + n(x)$, hence for $x \in \mathbb{Q}_+$, as $n(x) = 1$,

$$n(t) - 1 \le n(x+t) \le n(t) + 1. \tag{12}$$

The right inequality of (8) follows from the mean-value theorem applied to the logarithm function: for $h(x) := \log(x+n(x))$ and $x, t > 0$,

$$|h(t+x) - h(x)| \le \frac{|t+n(x+t)-n(x)|}{x} \le \frac{t+|n(x+t)-n(x)|}{x} \le \frac{t+n(t)}{x},$$

as both points $x+n(x)$ and $t+x+n(x+t)$ are to the right of x. (Note that $n(x+t) < n(x)$ is possible.)

Note that $n(x+t) \ge 1$ for $t, x > 0$ (as $x + t \ne 0$), so if $x \in \mathbb{Q}_+$, then $n(x) = 1 \le t + n(x+t)$, implying

$$x + n(x) < x + t + n(x+t).$$

For $t > 0$ and $x \in \mathbb{Q}_+$ apply the mean-value theorem again to the logarithm function, to obtain

$$\frac{[t+n(x+t)]-1}{x+[t+n(x+t)]} = \frac{[t+n(x+t)]-n(x)}{x+t+n(x+t)} \le h(t+x) - h(x).$$

But for $x, z > 0$, the expression $(z-1)/(z+x)$ is increasing in z, so taking $z = \varphi(t) := t + n(t) - 1 > 0$, that being the lowest value of $t + n(x+t)$ by (12), we obtain, for all $t > 0$ and $x \in \mathbb{Q}_+$,

$$\frac{\varphi(t)-1}{x+\varphi(t)} \le |h(t+x) - h(x)|.$$

Putting

$$g_-(x,\varphi(t)) := \frac{\varphi(t)-1}{x+\varphi(t)}, \qquad g_+(x,\varphi(t)) := \frac{\varphi(t)+1}{x},$$

and noting that $\varphi(t)$ tends to infinity, we finally have

$$g_-(x,\varphi(t)) \le |h(t+x) - h(x)| \qquad (x \in \mathbb{Q}_+),$$
$$|h(t+x) - h(x)| \le g_+(x,\varphi(t)) \qquad (x \in \mathbb{R}_+).$$

Let $0 < \varepsilon < 1$. The solution set of $g_-(x,y) \le \varepsilon$ is bounded for each x by the line

$$y = \psi_-(x,\varepsilon) = \frac{\varepsilon x + 1}{1-\varepsilon}.$$

With (8) established for $x \in \mathbb{Q}_+$, we may now apply the general theorem to show that UCT fails. This we may do by restricting attention to any sequence of rationals $\{x_n\}$ that tends to infinity. By the Theorem UCT in [BOst1] all we need do is check that the family of sets $T_k = \{t : \varphi(t) \le k\}$ is not sequence trapping. Indeed, choose t_m in $[0, 1]$ so that $n(t_m) = m$. By passing to a subsequence we may, without loss of generality, assume that t_m converges. But for any y and any infinite $\mathbb{M}$ the subsequence $\varphi(t_m + y)$ for m in $\mathbb{M}$ is unbounded, since $y + t_m + n(t_m) - 1 \le \varphi(t_m + y) + n(y)$. Hence $\{t_m + y : m \in \mathbb{M}\}$ is not trapped by T_k for any k. ∎

5. Complements. *No Trumps.* The term No Trumps in the definition of Section 2, a combinatorial principle, is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond ♢ [Je] and Ostaszewski's Club ♣ [Ost] and its weakening in another direction: "Stick" in [FSS]. The argument in the proof of the No Trumps Theorem is implicit in [CsEr] and explicit in [BG1, p. 482], and [BGT, p. 9]. The intuition behind our formulation may be gleaned from forcing arguments in [Mil1]–[Mil3].

De Haan theory. The study of functional relations of the form (RV), or (RV$_+$), is *Karamata theory*, in the terminology of [BGT, Chs. 1, 2]. Related is the study of *de Haan theory*—that of relations of the form

$$\text{(deH)} \qquad \frac{f(\lambda x) - f(x)}{g(x)} \to h(\lambda) \quad (x \to \infty) \quad \forall \lambda > 0$$

[BGT, Ch. 3]. See [BGT, §3.0] for the inter-relationships between the two (de Haan theory both contains Karamata theory, and refines it by filling in "gaps"). Our approach here to Karamata theory extends to de Haan theory along similar lines.

In de Haan theory, the relevant limit function in (deH) is

$$h(\lambda) = \begin{cases} (\lambda^\varrho - 1)/\varrho, & \varrho \neq 0, \\ \log \lambda, & \varrho = 0. \end{cases}$$

The Ash–Erdős–Rubel results [AER] and Heiberg–Lipschitz condition have something of a de Haan rather than a Karamata character. See e.g. [BGT, Th. 3.1.10a,c] for illustrations of this.

Weakening quantifiers. It is both interesting and useful to see to what extent the quantifier ∀ in (RV), (deH) may be weakened to "for some", plus some side-condition. The prototypical result here is the following [BGT, Th. 1.4.3 in the Karamata case, Th. 3.2.5 in the de Haan case].

Theorem (First Heiberg–Seneta Theorem). *Write*

$$g^*(\lambda) := \limsup_{x \to \infty} f(\lambda x)/f(x),$$

and assume that

$$\limsup_{\lambda \downarrow 1} g^*(\lambda) \leq 1.$$

Then for a positive function f, the following are equivalent:

(i) (RV) *and* (ϱ) *hold for some* ϱ.
(ii) *The limit* $g(\lambda)$ *in* (RV) *exists for all* λ *in a set of positive measure, or a non-meagre Baire set.*
(iii) $g(\lambda)$ *exists and is finite for all* λ *in a dense subset of* $(0, \infty)$.
(iv) $g(\lambda)$ *exists and is finite for* $\lambda = \lambda_1, \lambda_2$ *with* $(\log \lambda_1)/(\log \lambda_2)$ *finite and irrational.*

This question of weakening of quantifiers is treated in detail in [BG1] (where the above is Th. 5.7). The original motivation was the study of Frullani integrals; see [BG2, §6], [BGT, §1.6.4], [Ber, pp. 466–467].

Further generalizations. We note that the lower bound may be taken in the form $g_-(x + \varphi(t))\varphi(t)$, provided that for all $\varepsilon > 0$ small enough, the solution set of $g_-(x + y)y < \varepsilon$ is bounded for each x and takes the form $\{y : y \leq \psi_-(x, \varepsilon)\}$. Rewriting the solution set as

$$S(x, \varepsilon) = \{y : 0 \leq y < G(y) = \varepsilon/g_-(x + y)\},$$

we see that $0 \in S$. Thus $\psi_-(x, \varepsilon)$ is well-defined iff $\sup S(x, \varepsilon) < \infty$. Geometrically, the assumption requires the graph of $G(y)$ to cross the ray of slope 1 from the origin once so as to be ultimately below it. The condition is satisfied in the quoted example of [AER]. Putting $\varphi(t) = t + n(t) - 1$ as before, we see that, for $x \in \mathbb{Q}$,

$$\frac{\varphi(t)}{x + \varphi(t) + 2} = \frac{t + n(t) - 1}{x + t + n(t) + 1} \leq |h(t + x) - h(x)|.$$

Let $0 < \varepsilon < 1/2$. The required solution set is thus bounded by the line

$$\psi_-(x, \varepsilon) = \frac{\varepsilon(x + 2)}{1 - \varepsilon},$$

with slope less than unity.

One can introduce other conditions relaxing the location of the term $x(t)$ of the simple Heiberg–Lipschitz condition (4), say by bounding $|h(t + x) - h(x)|$ above and below "functionally", i.e. in terms of functions of x and functions of t, so long as one can recover corresponding finite functions $\psi_\pm(x, \varepsilon)$ with $\lim_{x\to\infty} \psi_+(x, \varepsilon) = \infty$.

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