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REPRESENTATIONS OF MULTIVARIATE POLYNOMIALS BY SUMS OF UNIVARIATE POLYNOMIALS IN LINEAR FORMS

BY

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In memory of Andrzej Mąkowski

Abstract. The paper is concentrated on two issues: presentation of a multivariate polynomial over a field K, not necessarily algebraically closed, as a sum of univariate polynomials in linear forms defined over K, and presentation of a form, in particular a zero form, as the sum of powers of linear forms projectively distinct defined over an algebraically closed field. An upper bound on the number of summands in presentations of all (not only generic) polynomials and forms of a given number of variables and degree is given. Also some special cases of these problems are studied.

1. Introduction. Let d be a positive integer and K a field of characteristic not dividing d. If $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ it is known (see [12]) that every polynomial $F \in K[x_1, \ldots, x_n]$ of degree d can be written as

$$F = \sum_{\mu=1}^{m} f_\mu(l_\mu), \tag{1}$$

where $m \le \binom{n+d-1}{d}$, $f_\mu \in K[z]$ and $l_\mu \in K[x_1, \ldots, x_n]$ is a linear form $(1 \le \mu \le m)$. For $d \le 3$ we have a better bound $m \le \binom{n+d-2}{d-1}$ (see [13]) and we conjecture that this holds in general. For infinite fields and for finite fields of not too small cardinality this conjecture follows from Theorem 1 below. But before we present the theorem, we recall that an n-ary form is said to *essentially depend on n variables* if it cannot be expressed in fewer than n variables after an invertible linear substitution (sometimes such a form is called nondegenerate, see e.g. [5]). Later we shall also use this terminology for collections of forms: a collection essentially depends on n variables if the forms from the collection cannot be simultaneously expressed in fewer than n variables after the same linear substitution.

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THEOREM 1. *Let either* $\operatorname{char} K = 0$, *or* $\operatorname{char} K > d$ *and* $|K| > 2d - 3$. *Every polynomial* $F \in K[x_1, \ldots, x_n]$ *of degree* d *has a presentation* (1), *where*

$$m \le \binom{n+d-2}{d-1}.$$

Moreover, if K *is infinite and the leading form of* F *essentially depends on* n *variables, then for every finite subset* $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s\}$ *of* $K^n \setminus \{\mathbf{0}\}$, *there exists a presentation* (1) *with* $m \le \binom{n+d-2}{d-1}$ *and* $l_\mu(\boldsymbol{p}_r) \ne 0$ $(1 \le r \le s)$.

This theorem contains as special cases Theorem 4 of [12] $(n = 2)$ and Theorem 1 of [13] $(d = 3)$. It also implies immediately

COROLLARY 1. *Under the same assumption on* K *as in Theorem* 1, *every form* $F \in K[x_1, \ldots, x_n]$ *of degree* d *has a presentation*

$$\sum_{\mu=1}^{m} a_\mu l_\mu^d, \qquad a_\mu \in K, \tag{2}$$

where $m \le \binom{n+d-2}{d-1}$, *and if* F *essentially depends on* n *variables, the linear forms* l_μ *can be chosen in such a way that* $l_\mu(\boldsymbol{p}_r) \ne 0$ $(1 \le r \le s)$ *for any given finite subset* $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s\}$ *of* $K^n \setminus \{\mathbf{0}\}$.

If K is algebraically closed, we may put $a_\mu = 1$, $\mu = 1, \ldots, m$. For $K = \mathbb{C}$ this improves the result of Ellison [4]. For $K = \mathbb{C}$, the first part of the corollary has been ascribed in [3] to B. Reznick, but his proof was never published.

COROLLARY 2. *Under the assumption of Theorem* 1 *every polynomial* $F \in K[x_1, \ldots, x_n]$ *of degree* d *can be written as*

$$\sum_{\mu=1}^{m} a_\mu (l_\mu + b_\mu)^d,$$

where $m \le \binom{n+d-1}{d-1}$, $a_\mu, b_\mu \in K$, *and* l_μ *is a linear form over* K.

Let $\mathfrak{F}_K(n,d)$ denote the space of forms with coefficients in K of degree d in n variables $x_1, \ldots, x_n$. We shall sometimes write $L_K(n)$ in place of $\mathfrak{F}_K(n,1)$. In case K is fixed, we shall write $\mathfrak{F}(n,d)$ and $L(n)$ instead of $\mathfrak{F}_K(n,d)$ and $L_K(n)$, respectively.

For $\operatorname{char} K = 0$, K algebraically closed, $n = 3$, $d = 3, 4$, $F \in \mathfrak{F}_K(n,d)$, Kleppe [8, Chapters 2 and 3], obtained better bounds for m than that given in Corollary 1, namely $m \le 5$ and $m \le 7$, respectively, while our Corollary 1 gives in these cases $m \le 6$ and $m \le 10$ and the easy (Ellison) bound gives $m \le 10$ and $m \le 15$. However, in Corollary 1, we require that the linear forms l_i, whose dth powers appear in the presentation, do not belong to a union of a fixed finite family of hyperplanes in $\mathfrak{F}(n,1)$, and it may be the case that under such restrictions our bound is the best possible. A better but incorrect result for $d = 3$, n arbitrary is claimed in [3, p. 106].

For generic forms over an algebraically closed field of characteristic 0 or $> d$, Corollary 1 can be much improved, namely as deduced by Iarrobino [6] and Iarrobino and Kanev [7, Corollary 1.62] from a result of Alexander and Hirschowitz [1], $\binom{n+d-2}{d-1}$ can be replaced by

$$r_0 = \begin{cases} n & \text{if } d = 2, \\ \left\lceil \frac{1}{n}\binom{n+d-1}{d} \right\rceil + 1 & \text{if } \langle d, n\rangle = \langle 3,5\rangle,\ \langle 4,3\rangle, \langle 4,4\rangle \text{ or } \langle 4,5\rangle, \\ \left\lceil \frac{1}{n}\binom{n+d-1}{d} \right\rceil & \text{otherwise.} \end{cases}$$

Though Theorem 1 can be considered as the main result of the paper, we also present other results. They mainly concern representability of a given form as a linear combination or, in particular, as a sum of powers of linear forms and are connected with the Waring problem and its extensions to the case where one considers representability not only of generic forms but also of specific forms. In the rest of the introduction we shall describe these results.

Problems of representability of a given form as a linear combination of powers of linear forms, in particular problems of uniqueness of such representations, lead to questions concerning linear dependence of powers of linear forms and related problems concerning presentations of the zero form. First, we explain our results concerning linear dependence of powers of linear forms. In a recent paper [2] A. Chlebowicz and M. Wołowiec-Musiał considered the problem of when linear forms $l_1, \ldots, l_m$ over a field of characteristic 0 have the property that $l_1^d, \ldots, l_m^d$ are linearly independent or, in their terminology, $l_1, \ldots, l_m$ are d-independent. In order to describe our results in this direction, let us notice that $\mathfrak{F}(n, d)$ is spanned by dth powers of linear forms. Moreover, forms $l_1, \ldots, l_r$ are d-independent if and only if the matrix of coefficients of their d-powers is of rank r. Next,

if $r \le \binom{n+d-1}{d-1} = \dim \mathfrak{F}(n, d)$, then a generic collection of linear forms $l_1, \ldots, l_r$ in n variables is d-independent. Moreover, every d-independent collection of linear forms can be extended to a collection of linear forms whose d-powers form a base of $\mathfrak{F}(n, d)$.

In particular, in the space $\mathfrak{F}(n, 1)^r$ of all collections $(l_1, \ldots, l_r)$, where $l_i \in \mathfrak{F}(n, 1)$ and $r \le \binom{n+d-1}{d-1}$, the subset composed of d-independent collections $(l_1, \ldots, l_r)$ is non-empty and open.

In general, checking if a given family is d-independent can be a cumbersome task. In the case of forms in one variable the theory is trivial. For two variables, $\dim(\mathfrak{F}(n,d)) = d+1$ and a collection $l_1, \ldots, l_r$, where $r \le d+1$, is d-independent iff the forms $l_1, \ldots, l_r$ are projectively different. This follows immediately from the well known properties of the Vandermonde determinant. As pointed out to us by A. Sładek, it follows easily from Lemma 2.4 of [2] that if $\operatorname{char} K = 0$, $m = d(n-1)+1$, and $l_i \in K[x_1, \ldots, x_n]$ are linear forms $(1 \le i \le m)$ such that any n of them are linearly independent, then the l_i are d-independent. We shall complete this result by showing that it also holds if $\operatorname{char} K > d$ and that, for all fields in question of cardinality at least m, it is best possible.

THEOREM 2. *Let* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *If linear forms*

$$l_i \in K[x_1, \ldots, x_n] \quad (1 \le i \le m),$$

where $m = d(n-1)+1$, *have the property that any* n *of them are linearly independent, then* $l_1^d, \ldots, l_m^d$ *are linearly independent. This is no longer true if* $m = d(n-1)+2 \le |K|+1$.

If $n = 2$, then the theorem gives the above mentioned facts concerning d-independence of linear forms. Hence Theorem 2 can be considered as a multivariable generalization of properties of the Vandermonde determinant. It seems that other similar results concerning relations between d-dependence and d'-dependence for different d and d' would also be of some interest.

The above results concerning linear dependence do not depend on arithmetical properties of the field K. However, in order to go further, we assume that the field K is algebraically closed. Then in (2) we may assume that $a_\mu = 1$ for $\mu = 1, \ldots, m$.

As already mentioned, the problems of presentation of a given form and in particular problems of uniqueness of such presentations, lead to questions concerning presentations of the zero form. We will be interested in presentations of the zero form by powers of forms jointly essentially depending on n variables. To describe our results in this direction we introduce some terminology. We shall call a presentation (2) of F a *presentation of length* m or an *m-presentation*. We shall call a presentation (2) of F a *representation of length* m or an *m-representation* if $f_\mu = z^d$ $(1 \le \mu \le m)$, the forms l_μ are non-zero and projectively different. In [7] such a presentation is called *normalized*.

We shall say that $F \in \mathfrak{F}(n,d)$ *has a lot of representations of length* m if for every finite subset $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s\}$ of $K^n \setminus \{\mathbf{0}\}$, F has a representation (2) such that $a_\mu = 1$ and $l_\mu(\boldsymbol{p}_r) \ne 0$ $(1 \le \mu \le m,\ 1 \le r \le s)$. We shall say that a representation is *simple* if no proper subsum is zero. Let us call two representations $F = l_1^d + \cdots + l_r^d = m_1^d + \cdots + m_s^d$ *disjoint* if $l_i^d \ne m_j^d$ for all $i \le r$ and $j \le s$. We say that a form given as a sum $l_1^d + \cdots + l_r^d$ has a *disjoint representation of length* s if there exist forms $m_1, \ldots, m_s$ such that

$l_1^d + \cdots + l_r^d = m_1^d + \cdots + m_s^d$, where $l_i^d \neq m_j^d$ for all $i \leq r$, $j \leq s$. Disjoint representations of $x_1^d + \cdots + x_n^d$ are of particular importance, because it is clear that they provide representations of zero by powers of forms jointly essentially depending on n variables and vice versa. Finally, we shall say that a representation of a given form F is *unique* if any other representation of the same length differs by a permutation of summands.

Theorem 2 implies

COROLLARY 3. *For all n and d there exists a simple representation of* 0 *of length $d(n-1)+2$ by powers of linear forms jointly essentially depending on n variables.*

We conjecture that, if $\operatorname{char} K = 0$ or $\operatorname{char} K > d$, then $d(n-1)+2$ is the least number with the above property. This is true for $n \leq 4$ by virtue of Theorems 4, 7 and 8 below.

We next have

PROPOSITION 1. *Let $n \geq 2$. Assume that $F \in \mathfrak{F}(n,d)$ admits an $(r-1)$-presentation. If there are no representations of F of length r, then the dimension of the space of r-presentations of F is* 1 *greater than the dimension of the space of $(r-1)$-presentations of F.*

Let $\phi_{n,r,d}$ be the morphism of $L(n)^r$ into $\mathfrak{F}(n,d)$ given by

$$\phi_{n,r,d}\,(l_1, \ldots, l_r) = l_1^d + \cdots + l_r^d.$$

PROPOSITION 2. *Let $n \geq 2$. Assume a generic form F in $\phi_{n,r,d}(L(n)^r)$ admits two different r-representations. Then F admits two disjoint r-representations*

$$F = l_1^d + \cdots + l_r^d = z_1^d + \cdots + z_r^d.$$

From the results of [6], [7] we shall draw the following consequences.

COROLLARY 4. *Let $n \geq 2$ and either* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *Then for $r \geq r_0$ the dimension of the space of r-representations of a generic form is equal to $nr - \dim \mathfrak{F}(n,d)$ and thus is the same as the dimension of the space of r-presentations.*

COROLLARY 5. *Assume that $n \geq 2$,* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$, *and $F \in \mathfrak{F}(n,d)$. Let $r \geq r_0$. Assume that F admits an r-representation and let q be the dimension of the space of r-representations of F. Then $q \geq nr - \dim \mathfrak{F}(n,d)$. Moreover, if there are no $(r+1)$-representations of F, then $q \geq nr - \dim \mathfrak{F}(n,d) + n - 1$.*

PROPOSITION 3. *Let $n \geq 2$ and either* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *For every $t \geq 2r_0 + 1$ there exists a t-representation of the zero form whose summands include at least $r = \lfloor t/2 \rfloor$ dth powers of algebraically independent generic linear forms $l_1, \ldots, l_r$.*

THEOREM 3. *Assume that the field K is algebraically closed and*

$$r > \max\{1, (n-1)(d-1)\}.$$

Then $x_1^d + \dots + x_n^d$ admits a lot of r-representations.

In the rest of the introduction we shall describe our results concerning mainly forms in two, three and four variables. In the case of forms in two variables our results are the most complete and satisfactory. In fact, we have the following

THEOREM 4. *Let K be an algebraically closed field and $r \geq \max\{2, d\}$. Then $x_1^d + x_2^d$ has a lot of r-representations by powers of linear forms in two variables. If either $\operatorname{char} K = 0$, or $\operatorname{char} K = p$, $p^\nu \parallel d+1$ and $p^{\nu+1} > d+1$, then conversely the existence of such an r-representation implies $r = 2$ or $r \geq \max\{2, d\}$.*

Let $\mathfrak{F}(n, d, s)$ be the subset of $\mathfrak{F}(n, d)$ consisting of all forms for which the minimal length of a representation is equal to s. For binary forms over a field of characteristic 0 or $> d$ we have

THEOREM 5. *Let K be an algebraically closed field of characteristic 0 or $> d$. If $F \in \mathfrak{F}(2, d, s)$ and F has an r-representation, then either $r = s \leq (d+1)/2$ or F has infinitely many r-representations.*

Moreover, for every binary form of degree d:

(a) *the set of representations of length $d+1$ has a component of dimension $d+1$,*
(b) *the set of representations of length $r \geq d+1$ has a component of dimension $2r - (d+1)$,*
(c) *there are a lot of representations of any length $r \geq d+1$.*

THEOREM 6. *Let K be an algebraically closed field of characteristic 0 or $> d$. Every binary form over K of degree $d \geq 2$ essentially depending on two variables admits a lot of representations of length d.*

The existence of representations of length at most d has already been proved by Kleppe [8, Chapter 1]. Theorem 6 is best possible as shown by

PROPOSITION 4. *Let $F = x_1^{d_1} x_2^{d_2}$, $c = \max(d_1, d_2) < d_1 + d_2$. Then*

$$F \in \mathfrak{F}(2, d_1 + d_2, c+1)$$

and there are a lot of representations of F of length $c+1$.

The first part of this proposition for $d_1 = 1$ can be found both in [8, p. 11] and in [12, p. 656].

We know from Theorem 5 that every form $F \in \mathfrak{F}(2, d, s)$, where $s > (d+1)/2$, admits infinitely many representations of any length $r \geq s$, if it has at least one such r-representation. However, it is not known if it admits a representation of every length $r \geq s$, and whether it admits a lot of r-representations when it admits infinitely many r-representations. Moreover, Kleppe [8] proved that $\mathfrak{F}(2,d,d)$ consists exactly of the forms equivalent to $x_1 x_2^{d-1}$ by a linear invertible substitution. A similar description of $\mathfrak{F}(2,d,r)$, where $(d+3)/2 \leq r \leq d-1$, is not known, but it is evident that all these sets are not empty.

Every binary form F over an algebraically closed field can be written as a product

$$F = \prod_{i=1}^{j} (a_i x_1 + b_i x_2)^{d_i}, \tag{$\star$}$$

where the factors $a_i x_1 + b_i x_2$ are projectively different.

However, except in the cases where $j = 1, 2$, or 3, the minimal length of representations of F depends not only on j and the sequence of exponents $d_1, \ldots, d_j$, but also on specific arithmetic properties of $a_1, b_1, \ldots, a_j, b_j$. Hence the description of the minimal length of a representation of a specific form F given as in $(\star)$ may be a difficult task. However, it would also be interesting to know the minimal length of a representation of a generic form F, given as in $(\star)$, for fixed $d_1, \ldots, d_j$.

We are only able to partially extend the above results about binary forms to forms in three or four variables. We have the following results in these directions.

THEOREM 7. *Let $d \geq 2$, K be an algebraically closed field, and $r \geq 2d-1$. Then $x_1^d + x_2^d + x_3^d$ has a lot of r-representations. Conversely, if $\operatorname{char} K = 0$ or $\operatorname{char} K > d$, and $x_1^d + x_2^d + x_3^d$ has a disjoint r-representation, then $r \geq 2d-1$.*

COROLLARY 6. *If, under the assumptions of Theorem 7, a form $F \in \mathfrak{F}(3,d)$ over K essentially depends on three variables and admits two disjoint representations of length s and r, respectively, then $s + r \geq 2d + 2$.*

COROLLARY 7. *If, under the assumptions of Theorem 7, a form $F \in \mathfrak{F}(3,d,s)$ admits some but only finitely many representations of length $r \leq 2d + 1 - s$, then $r = s$.*

COROLLARY 8. *If $F \in \mathfrak{F}(3,4,s)$ has some but only finitely many r-representations over an algebraically closed field K with $\operatorname{char} K \neq 2, 3$, then $r = s \leq 5$.*

THEOREM 8. *Let K be an algebraically closed field. Let the number n of variables be even, $n = 2m$. Then $x_1^d + \cdots + x_n^d$ admits a disjoint representation of every length $r \geq md$.*

Moreover, if $\operatorname{char} K = 0$ or $\operatorname{char} K > d$, and if $n = 4$ and $r \leq 3d-3$, then every disjoint r-representation is obtained by adding an s-representation of

$x_i^d + x_j^d$ *and an* $(r-s)$*-representation of* $x_k^d + x_l^d$, *where* $\{i,j\} \cup \{k,l\} = \{1,2,3,4\}$ *and* $r - d \geq s \geq d$. *In particular* $r \geq 2d$.

COROLLARY 9. *Let* K *be an algebraically closed field and* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *If a form* $F \in \mathfrak{F}(4,d)$ *over* K *essentially depends on four variables and admits two disjoint representations of lengths* s *and* r, *then* $s + r \geq 2d + 4$.

COROLLARY 10. *Let* K *be an algebraically closed field and* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *If a form* $F \in \mathfrak{F}(4,d,s)$ *essentially depending on four variables admits some but only finitely many representations of length* $r \leq 2d + 3 - s$, *then* $r = s$.

M. Wołowiec-Musiał [15] asked whether, if a representation of a form F over an algebraically closed field is unique, then the length of the representation is minimal. For binary forms and $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ the affirmative answer follows from Theorem 5. For quartic ternary forms and $\operatorname{char} K \neq 2, 3$ the affirmative answer follows from Corollary 7. For quaternary forms and $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ a partial affirmative answer follows from Corollary 9. Here are some other results pointing towards the affirmative answer.

THEOREM 9. *If* K *is algebraically closed*, $F \in \mathfrak{F}(n,d,s)$ *and* F *has only* $a < \infty$ *representations of length* r, *then*

$$\begin{aligned} r &= s && \text{if } s = 1,\ a > 0, \\ r &\leq s + d - 3 && \text{if } s \geq 2,\ a \geq 0. \end{aligned}$$

COROLLARY 11. *Let* K *be an algebraically closed field. If* $F \in \mathfrak{F}(n,2,s)$, *where* $s \geq 2$, *then* F *has infinitely many representations of length* r *for every* $r \geq s$.

COROLLARY 12. *Let* K *be an algebraically closed field. If* $F \in \mathfrak{F}(n,3,s)$ *and* F *admits some*, *but only finitely many*, *representations of length* r, *then* $r = s$.

At this point, it should be mentioned that, in general, uniqueness of s-representations of a given form is not implied by the fact that the form has only finitely many such representations. For example, a general form of degree d in n variables, where $d \geq n > 2$, has a unique representation of minimal length if and only if $n = 3, d = 5$ (see Theorem 1 in [10]), though for $n = 3$, it has finitely many such representations whenever $d \geq 5$ is not divisible by 3.

However, notice that (by the cited theorem of Iarrobino), for $r < r_0$ a generic form in $\mathfrak{F}(n,d,r)$ admits only finitely many r-representations, and Mella has proved (see Remark 4.6 in [10]) that, for $r < r_0 - 1$, such a representation is unique. This improves a theorem of Iarrobino and Kanev [7, Theorem 2.6(ii), pp. 62–63].

We should like to thank A. Sładek for many remarks which have helped to improve the presentation.

2. Proof of Theorem 1. Denote by $\mathrm{Lin}(l_1,\dots,l_k)$ the linear space spanned over K by the elements $l_1,\dots,l_k$ of $L_K(n)$, and by l^* the coefficient vector of $l \in L_K(n)$.

LEMMA 2.1. *Let* $\mathrm{char}\,K = 0$ *or* $\mathrm{char}\,K > d$. *If* $F \in \mathfrak{F}(n,d)$ *and* $l \in L_K(n) \setminus \{0\}$ *satisfy* $\partial F/\partial l = 0$, *then* F *essentially depends on fewer than* n *variables* $l_1,\dots,l_k$, *which all satisfy* $l_j(l^*) = 0$.

Proof. Let $l_1 = l$, $l_2,\dots,l_n$ be a basis for $L_K(n)$, where $l_j(l^*) = 0$ for $1 < j \le n$ if $l(l^*) \ne 0$, and $l_j(l^*) = 0$ for $1 \le j < n$ otherwise. Notice that in the second case $l_n(l) \ne 0$. We have

$$F = F_0(l_1, l_2, \dots, l_n)$$

for some $F_0 \in \mathfrak{F}(n,d)$. Then, in the first case,

$$0 = \frac{\partial F}{\partial l} = \frac{\partial F_0}{\partial x_1}\, l(l^*),$$

and in the second case,

$$0 = \frac{\partial F}{\partial l} = \frac{\partial F_0}{\partial x_n}\, l_n(l^*).$$

Hence either $\partial F_0/\partial x_1 = 0$ or $\partial F/\partial x_n = 0$, and F depends on fewer than n variables l_j, which all satisfy $l_j(l^*) = 0$.

LEMMA 2.2. *Let* $n > 1$, $d > 1$, $s \ge 0$, $\mathrm{char}\,K = 0$ *or* $\mathrm{char}\,K > d$ *and* $|K| > 2s + 1$. *If a form* $F \in \mathfrak{F}(n,d)$ *essentially depends on* n *variables and* $l_1,\dots,l_s \in L_K(n) \setminus \{0\}$, *then there exists a form* $l \in L_K(n)$ *such that* $l/l_r \notin K$ $(1 \le r \le s)$ *and*

$$\frac{\partial^2 F}{\partial l^2} \ne 0, \qquad \frac{\partial^2 F}{\partial l \partial l_1} \ne 0,\ \dots,\ \frac{\partial^2 F}{\partial l \partial l_s} \ne 0.$$

Proof. For fixed $r = 1,\dots,s$, the set of forms l such that $\partial^2 F/\partial l \partial l_r = 0$ is a proper Zariski closed subset in $L_K(n)$. In fact, if it is not a proper subset, we should have

$$\frac{\partial}{\partial l}\left(\frac{\partial F}{\partial l_r}\right) = 0$$

for all $l \in L_K(n)$ and hence $\partial F/\partial l_r \in K$. Since F is of degree $d > 1$ it follows that $\partial F/\partial l_r = 0$, hence by Lemma 2.1, F depends essentially on fewer than n variables, contrary to the assumption. Hence if K is infinite, the set of

$l \in L_K(n)$ such that

$$\frac{\partial^2 F}{\partial l \partial l_1} \neq 0, \ \ldots, \ \frac{\partial^2 F}{\partial l \partial l_s} \neq 0$$

is not empty and open in $L_K(n)$. If K is finite the set of $l \in L_K(n)$ satisfying the above condition has at least $|K|^n - 1 - s(|K|^{n-1} - 1)$ elements.

It remains to consider the set of forms $l \in L_K(n)$ such that $\partial^2 F/\partial l^2 = 0$. Suppose that $\partial^2 F/\partial l^2 = 0$ for all $l \in L_K(n)$. Then, in particular, $\partial^2 F/\partial x_i^2 = 0$ for $i = 1, \ldots, n$. Hence $d \leq n$, and F is a linear combination of square free monomials

$$(**) \qquad \prod_{j=1}^{d} x_{i_j}$$

for some distinct indices $i_j \leq n$. Now

$$\frac{\partial^2 (\prod_{j=1}^{d} x_{i_j})}{\partial (x_{i_1} + x_{i_2})^2} = 2 \prod_{j=3}^{d} x_{i_j} \neq 0.$$

Moreover, the partial derivatives

$$\frac{\partial^2}{\partial (x_{i_1} + x_{i_2})^2},$$

for fixed x_{i_1}, x_{i_2}, when applied to different square free monomials yield 0 or different monomials of degree $d-2$. Hence

$$\frac{\partial^2 F}{\partial (x_{i_1} + x_{i_2})^2} \neq 0.$$

Thus the set of forms l for which $\partial^2 F/\partial l^2 \neq 0$ is non-empty and for K infinite the lemma is proved. For K finite there are at most $(2|K| - 1)|K|^{n-2} - 1$ non-zero forms l satisfying

$$\frac{\partial^2 F}{\partial l^2} = 0$$

(see [9, Theorems 6.26 and 6.27]).

Therefore, the number of forms l in $L_K(n)$ such that

$$\frac{\partial^2 F}{\partial l^2} \neq 0, \qquad \frac{\partial^2 F}{\partial l \partial l_r} \neq 0 \qquad (1 \leq r \leq s)$$

is at least

$$|K|^n - s(|K|^{n-1} - 1) - (2|K| - 1)|K|^{n-2} \geq |K|^2 - (s+2)|K| + (s+1);$$

on the other hand, the number of forms $l \in L_K(n) \setminus \{0\}$ such that $l/l_r \in K$ for some $r \leq s$ is at most

$$s(|K| - 1).$$

The condition $|K| > 2s+1$ guarantees that the former quantity is greater than the latter.

DEFINITION 1. Let $P \subset K^n \setminus \{\mathbf{0}\}$. Let $S(F,P)$ be the minimal length of a presentation (2) of the form F with the condition that

(A) $l_\mu(\boldsymbol{p}) \neq 0$ for all $\boldsymbol{p} \in P$,
(B) l_μ^{d-1} are linearly independent $(1 \leq \mu \leq m)$ in $\mathfrak{F}(n, d-1)$.

In case such presentations do not exist, $S(F,P)$ is defined to be ∞.

$S(n,d,s)$ is defined as the maximum of all $S(F,P)$ for all forms $F \in \mathfrak{F}(n,d)$ essentially depending on n variables and for all $P \subset K^n \setminus \{\mathbf{0}\}$ of cardinality s.

For $\boldsymbol{p} = (p_1, \dots, p_n) \in K^n$, $\partial F/\partial \boldsymbol{p}\boldsymbol{x}$ denotes the partial derivative of F with respect to $p_1 x_1 + \cdots + p_n x_n$.

LEMMA 2.3. *Let $F = F_1(l_1, \dots, l_k)$, $F_1 \in \mathfrak{F}(k,d)$, $l_j \in L_K(n)$ and F_1 essentially depend on k variables. If $\boldsymbol{p}_r \in K^n$ $(1 \leq r \leq s)$ and for each r,*

$$\frac{\partial F}{\partial \boldsymbol{p}_r \boldsymbol{x}} \neq 0, \tag{3}$$

then

$$F = \sum_{\mu=1}^{m} a_\mu l_{0\mu}^d, \quad \textit{where } m \leq S(k,d,s),\ a_\mu \in K,\ l_{0\mu} \in \mathrm{Lin}(l_1, \dots, l_k),$$

$l_{0\mu}(\boldsymbol{p}_r) \neq 0$ for all $r \leq s$ and $l_{0\mu}^{d-1}$ $(1 \leq \mu \leq m)$ are linearly independent.

Proof. For each $r \leq s$,

$$\langle l_1(\boldsymbol{p}_r), \dots, l_s(\boldsymbol{p}_r)\rangle \neq \mathbf{0} \tag{4}$$

since otherwise we should obtain

$$\frac{\partial F}{\partial \boldsymbol{p}_r \boldsymbol{x}} = \sum_{j=1}^{k} \frac{\partial F_1}{\partial x_j}\bigg|_{(l_1,\dots,l_k)} \cdot \frac{\partial l_j}{\partial \boldsymbol{p}_r \boldsymbol{x}} = \sum_{j=1}^{k} \frac{\partial F}{\partial x_j} \cdot l_j(\boldsymbol{p}_r) = 0,$$

contrary to (3). From (4) and the definition of $S(k,d,s)$ it follows that

$$F_1 = \sum_{\mu=1}^{m} a_\mu l_{1\mu}^d, \qquad m \leq S(k,d,s),\ a_\mu \in K,\ l_{1\mu} \in L_K(n),$$
$$l_{1\mu}(l_1(\boldsymbol{p}_r), \dots, l_k(\boldsymbol{p}_r)) \neq 0 \qquad (1 \leq \mu \leq m,\ 1 \leq r \leq s)$$

and the $l_{1\mu}^{d-1}$ are linearly independent. Now, it suffices to take

$$l_{0\mu} = l_{1\mu}\,(l_1, \dots, l_k)\,.$$

Lemma 2.4. *If* $d > 1$, $s \geq 0$, *and either* $\operatorname{char} K = 0$, *or* $\operatorname{char} K > d$ *and* $|K| > 2d + 2s - 3$, *then*

$$S(n,d,s) \leq \binom{n+d-2}{d-1}.$$

Proof. We proceed by induction on $n+d$. If $n+d=3$ we have $n=1$, $d=2$ and the assertion is trivially true. Assume that it is true for all $n \geq 1$, $d \geq 2$ with $n+d < N$ and consider $F \in \mathfrak{F}(n,d)$ essentially depending on n variables, where $n \geq 1$, $d \geq 2$, $n+d=N$, and a finite subset $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s\}$ of $K^n \setminus \{\mathbf{0}\}$. If $n=1$, the assertion is trivially true. If $n \geq 2$, by Lemma 2.2, there exists $l \in L_K(n)$ such that l^* is projectively different from $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s$ and

$$\frac{\partial^2 F}{\partial l^2} \neq 0, \qquad \frac{\partial^2 F}{\partial l \partial \boldsymbol{p}_r \boldsymbol{x}} \neq 0 \quad (1 \leq r \leq s). \tag{5}$$

Consider first $d=2$. Taking

$$F_0 = \left(\frac{\partial F}{\partial l}\right)^2 \Big/ 2\,\frac{\partial^2 F}{\partial l^2} \tag{6}$$

we obtain

$$\frac{\partial(F-F_0)}{\partial l} = 0,$$

hence, by Lemma 2.1, $F - F_0$ essentially depends on $n_1 < n$ variables,

$$F - F_0 = F_1(l_1, \ldots, l_{n_1}), \tag{7}$$

where $F_1 \in \mathfrak{F}(n_1, d)$, $l_j \in L_K(n)$ and $l_j(l^*) = 0$.

We have $n_1 = n-1$, since otherwise F would depend on fewer than n variables. Since the l_j are linearly independent and $l_j(l^*) = 0$ $(1 \leq j \leq n-1)$, and l^* is projectively different from $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s$, for all $r \leq s$ we have

$$\langle l_1(\boldsymbol{p}_r), \ldots, l_{n-1}(\boldsymbol{p}_r)\rangle \neq \mathbf{0}.$$

Now, by the inductive assumption we have

$$F_1(x_1, \ldots, x_{n-1}) = \sum_{\mu=2}^{n} a_\mu l_{1\mu}^2, \qquad \text{where } a_\mu \in K,\ l_{1\mu} \in L_K(n-1), \tag{8}$$

$$l_{1\mu}(l_1(\boldsymbol{p}_r), \ldots, l_s(\boldsymbol{p}_r)) \neq 0 \quad (1 \leq r \leq s) \tag{9}$$

and the forms $l_{1\mu}$ $(2 \leq \mu \leq n)$ are linearly independent. Now, taking

$$\begin{aligned} a_1 &= \frac{1}{2\partial^2 F/\partial l^2}, \qquad l_{01} = \frac{\partial F}{\partial l}, \\ l_{0\mu} &= l_{1\mu}(l_1, \ldots, l_{n-1}) \quad (2 \leq \mu \leq n), \end{aligned} \tag{10}$$

from (6)–(8) we obtain

$$F = \sum_{\mu=1}^{n} a_\mu l_{0\mu}^2, \tag{11}$$

where, by (5) and (9), $l_{0\mu}(\boldsymbol{p}_r) \neq 0$ $(1 \leq r \leq s)$.

Assume now that

$$\sum_{\mu=1}^{n} A_\mu l_{0\mu} = 0, \quad A_\mu \in K.$$

If $A_1 \neq 0$, then l_{01} depends linearly on $l_{0\mu}$ $(2 \leq \mu \leq n)$, hence by (10) and (11), F depends on l_j $(1 \leq j \leq n-1)$, contrary to the assumption. Thus $A_1 = 0$ and by the linear independence of the $l_{1\mu}$, $A_\mu = 0$ for all $\mu \leq n$.

Assume now that $d \geq 3$. The form $\partial F/\partial l$ of degree $d-1$ essentially depends on $n_0 \leq n$ variables. By (5) and Lemma 2.3 it follows that

$$\frac{\partial F}{\partial l} = \sum_{\mu=1}^{m_0} a_\mu l_{0\mu}^{d-1}, \quad \text{where } m_0 \leq S(n_0, d-1, s+1),\ a_\mu \in K,\ l_{0\mu} \in L_K(n),$$

$$l_{0\mu}(l^*) \neq 0, \quad l_{0\mu}(\boldsymbol{p}_r) \neq 0 \quad (1 \leq r \leq s) \tag{12}$$

and

$$l_{0\mu}^{d-2} \ (1 \leq \mu \leq m_0) \text{ are linearly independent.} \tag{13}$$

Taking

$$F_0 = \sum_{\mu=1}^{m_0} \frac{a_\mu l_{0\mu}^d}{d l_{0\mu}(l^*)}$$

we obtain

$$\frac{\partial(F - F_0)}{\partial l} = 0,$$

hence, by Lemma 2.1, $F - F_0 = F_1^0(l_1^0, \ldots, l_{n-1}^0)$, where $l_j^0 \in L_K(n)$ and $l_j^0(l^*) = 0$. Let T be a minimal subset of $\{1, \ldots, m_0\}$ with the property that for a certain point $\boldsymbol{p}$ projectively different from $\boldsymbol{0}, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_s$ and some linearly independent forms $l_1, \ldots, l_{n_1}$ in $L_K(n)$, and a certain F_1 in $\mathfrak{F}(n_1, d)$,

$$F - \sum_{\mu \in T} \frac{a_\mu l_{0\mu}^d}{d l_{0\mu}(l^*)} = F_1(l_1, \ldots, l_{n_1}), \quad l_j(\boldsymbol{p}) = 0, \tag{14}$$

$$l_{0\mu} \notin \operatorname{Lin}(l_1, \ldots, l_{n_1}) \quad \text{for } \mu \in T. \tag{15}$$

(The set $\{1, \ldots, m_0\}$ has this property with $\boldsymbol{p} = l^*$, $l_j = l_j^0$, $F_1 = F_1^0$, so such sets exist.) Since $l_j(\boldsymbol{p}) = 0$ $(1 \leq j \leq n_1)$ we have $n_1 \leq n-1$. We assert that $n_1 = n-1$. Indeed, supposing the contrary we could find $\mu_1 \in T$ such that $l_{0\mu_1} \notin \operatorname{Lin}(l_1, \ldots, l_{n_1})$ (otherwise F would depend on n_1 variables) and a point $\boldsymbol{p} \neq \mathbf{0}$ such that $l_{0\mu_1}(\boldsymbol{p}) = 0 = l_{1j}(\boldsymbol{p})$ $(1 \leq j \leq n_1)$. Since $l_{0\mu_1}(\boldsymbol{p}_r) \neq 0$, $\boldsymbol{p}$ is projectively different from $\boldsymbol{p}_r$ $(1 \leq r \leq s)$. Taking

$$T_1 = T \setminus \{\mu \leq m_0 : l_{0\mu} \in \mathrm{Lin}(l_{0\mu_1}, l_1, \ldots, l_{n_1})\}$$

we should obtain (14) and (15) with T replaced by $T_1 \subsetneq T$, contrary to the choice of T. Thus $n_1 = n - 1$ and since the l_j are linearly independent $(1 \leq j \leq n-1)$ and $l_j(\boldsymbol{p}) = 0$, for every $r \leq s$ we have

$$\langle l_1(\boldsymbol{p}_r), \ldots, l_{n-1}(\boldsymbol{p}_r)\rangle \neq \mathbf{0}.$$

Hence, by the definition of $S(n-1, d, s)$ we have

$$F_1(x_1, \ldots, x_{n-1}) = \sum_{\mu=m_0+1}^{m_0+m_1} a_\mu l_{1\mu}^d, \qquad a_\mu \in K,\ l_{1\mu} \in L_{n-1}(K), \tag{16}$$

where $m_1 \leq S(n-1, d, s)$,

$$l_{1\mu}(l_1(\boldsymbol{p}_r), \ldots, l_{n-1}(\boldsymbol{p}_r)) \neq 0 \qquad (m_0 < \mu \leq m_0 + m_1,\ 1 \leq r \leq s) \tag{17}$$

and the $l_{1\mu}^{d-1}$ are linearly independent.

It follows from (14) and (16) that

$$F = \sum_{\mu \in T} \frac{a_\mu l_{0\mu}^d}{d l_{0\mu}(l^*)} + \sum_{\mu=m_0+1}^{m_0+m_1} a_\mu l_{1\mu}(l_1, \ldots, l_{n-1})^d, \tag{18}$$

where, by (12) and (17), the relevant linear forms do not vanish at $\boldsymbol{p}_r$ $(1 \leq r \leq s)$. Assume now that

$$\sum_{\mu \in T} A_\mu l_{0\mu}^{d-1} + \sum_{\mu=m_0+1}^{m_0+m_1} A_\mu l_{1\mu}(l_1, \ldots, l_{n-1})^{d-1} = 0, \qquad A_\mu \in K. \tag{19}$$

By (15) there exists $z \in L_K(n) \setminus \{0\}$ such that $l_{0\mu}(z^*) \neq 0$ $(\mu \in T)$ and $l_j(z^*) = 0$ $(1 \leq j \leq n-1)$. Differentiating (19) with respect to z we obtain

$$\sum_{\mu \in T} A_\mu (d-1) l_{0\mu}^{d-2} l_{0\mu}(z^*) = 0,$$

hence, by (13), $A_\mu = 0$ for all $\mu \in T$, and by (19),

$$\sum_{\mu=m_0+1}^{m_0+m_1} A_\mu l_{1\mu}(l_1, \ldots, l_{n-1})^{d-1} = 0.$$

Therefore, by the linear independence of the $l_{1\mu}^{d-1}$ we have $A_\mu = 0$ for all μ. It now follows from (18) and the inductive assumption that

$$\begin{aligned} S(F, s) &\leq |T| + m_1 \leq m_0 + m_1 \leq S(n_0, d-1, s+1) + S(n-1, d, s) \\ &\leq \binom{n+d-3}{d-2} + \binom{n+d-3}{d-1} = \binom{n+d-2}{d-1}. \end{aligned}$$

Since F was an arbitrary element of $\mathfrak{F}(n,d)$ essentially depending on n variables, it follows that

$$S(n,d,s) \leq \binom{n+d-2}{d-1},$$

and the inductive proof is complete.

LEMMA 2.5. *If the forms l_μ^{d-1} $(1 \leq \mu \leq m)$, where $l_\mu \in L_K(n)$, span the space $\mathfrak{F}(n,d-1)$, then for every $i \leq d$ the forms l_μ^{d-i} $(1 \leq \mu \leq m)$ span $\mathfrak{F}(n,d-i)$.*

Proof. Let $F \in \mathfrak{F}(n,d-i)$ be given as

$$F = \sum_{j=0}^{d-i} A_j x_n^{d-i-j}, \qquad A_j \in K[x_1,\ldots,x_{n-1}] \cap \mathfrak{F}(n-1,j).$$

By the assumption

$$F_i := \sum_{j=0}^{d-i} A_j \, \frac{x_n^{d-1-j}}{\prod_{k=1}^{i-1}(d-j-k)} = \sum_{\mu=1}^{m} a_\mu l_\mu^{d-1}, \qquad a_\mu \in K,\ l_\mu \in L_K(n),$$

hence

$$F = \frac{\partial^{i-1} F_1}{\partial x_n^{i-1}} = \sum_{\mu=1}^{m} a_\mu l_\mu^{d-i} \left(\frac{\partial l_\mu}{\partial x_n}\right)^{i-1}.$$

Proof of Theorem 1. For $d=1$ the assertion is obvious. For $d \geq 2$, let $F = \sum_{\delta=0}^{d} F_\delta$, where $F_\delta \in \mathfrak{F}(n,\delta)$. Assume F_d essentially depends on $n_d \leq n$ variables. By Lemma 2.4 with $s=0$ we have $F_d = \sum_{\mu=1}^{m_d} a_{d\mu} l_\mu^d$, where

$$m_d \leq \binom{n_d+d-2}{d-1} \leq \binom{n+d-2}{d-1},$$

$a_{d\mu} \in K$, $l_\mu \in L_K(n)$ and the l_μ^{d-1} are linearly independent $(1 \leq \mu \leq m_d)$. Since the dimension of $\mathfrak{F}(n,d-1)$ is $\binom{n+d-2}{d-1}$ and, by Lemma 2.4, this space is spanned by l^{d-1}, $l \in L_K(n)$, there exist forms l_μ $\left(m_d < \mu \leq \binom{n+d-2}{d-1}\right)$ such that the forms l_μ^{d-1} $\left(1 \leq \mu \leq \binom{n+d-2}{d-1}\right)$ span the space $\mathfrak{F}(n,d-1)$, thus, by Lemma 2.5, for each $\delta < d$ we have

$$F_\delta = \sum_{\mu=1}^{\binom{n+d-2}{d-1}} a_{\delta\mu} l_\mu^\delta$$

and it suffices to take $f_\mu = \sum_{\delta=0}^{d} a_{\delta\mu} z^\delta$, where for $\delta = d$ and $\mu > m_d$ we take $a_{\delta\mu} = 0$.

This proves the first part of the theorem. In order to prove the second part observe that if K is infinite we can apply Lemma 2.4 with an arbitrary s.

Proof of Corollary 2. It suffices to apply Theorem 1 to the polynomial $x_{n+1}^d F(x_1/x_{n+1}, \ldots, x_n/x_{n+1}) \in \mathfrak{F}(n+1, d)$ and then substitute $x_{n+1} = 1$.

3. Proof of Theorem 2 and Corollary 3. We shall prove the first assertion of the theorem by induction on d. For $d = 1$ the assertion is obvious. Assume it is true for the exponent $d-1$ and let l_i be the linear forms in question. By an invertible linear transformation we can achieve that

$$l_i = x_{i-(d-1)(n-1)} \quad \text{for } i > (d-1)(n-1). \tag{20}$$

For $i \le (d-1)(n-1)$ let

$$l_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad a_{ij} \in K.$$

If $a_{i1} = 0$, then the forms l_i and $x_k = l_{k+(d-1)(n-1)}$, for $k = 2, \ldots, n$, are linearly dependent. Hence by the assumption $a_{i1} \neq 0$ for all $i \le (d-1)(n-1)$. Assume now that

$$\sum_{i=1}^{d(n-1)+1} A_i l_i^d = 0, \qquad A_i \in K. \tag{21}$$

Differentiating with respect to x_1 we obtain

$$\sum_{i=1}^{(d-1)(n-1)} dA_i a_{i1} l_i^{d-1} + dA_{(d-1)(n-1)+1} x_1^{d-1} = 0.$$

By the inductive assumption we have

$$dA_i a_{i1} = 0 \quad (1 \le i \le (d-1)(n-1)) \quad \text{and} \quad dA_{(d-1)(n-1)+1} = 0,$$

hence $A_i = 0$ $(1 \le i \le (d-1)(n-1))$. It now follows from (20) and (21) that

$$\sum_{i=1}^{n} A_{(d-1)(n-1)+i} x_i^d = 0,$$

hence $A_i = 0$ for all i.

In order to prove the second assertion of the theorem take $m = d(n-1) + 2$, $l_i = \sum_{j=1}^{n} a_i^{j-1} x_j$ $(1 \le i < m)$, $l_m = x_n$, where the a_i are distinct elements of K (we assume $0^0 = 1$). It is easy to check that any n among the l_i are linearly independent. On the other hand, we have

$$l_i^d = \sum_{k=0}^{d(n-1)} a_i^k f_k \quad (1 \le i < m), \qquad l_m^d = x_n^d = f_{d(n-1)},$$

where

$$f_k = \sum_{\substack{i_2+2i_3+\cdots+(n-1)i_n=k \\ i_1+\cdots+i_n=d}} \binom{d}{i_1 \ldots i_n} \prod_{q=1}^{n} x_q^{i_q}. \tag{22}$$

Hence the dimension of the space generated by the l_i^d is at most $d(n-1)+1$.

Proof of Corollary 3. Since the forms l_i^d constructed in the proof of the second part of Theorem 2 are linearly dependent we have

$$\sum_{i=1}^{d(n-1)+2} A_i l_i^d = 0, \qquad A_i \in K.$$

On the other hand, for every $j \le d(n-1)+2$ the forms l_i^d $(i \ne j)$ are linearly independent, by the first part of Theorem 2, hence all proper subsums of the above sum are different from 0, and in particular $A_i \ne 0$. Moreover, $l_i/l_j \in K$ implies $i = j$, and the l_i jointly essentially depend on n variables. Explicitly, we have

$$A_i = -\prod_{\substack{j=1 \\ j\ne i}}^{d(n-1)+1} (a_j - a_i)^{-1} \quad (1 \le i \le d(n-1)+1), \qquad A_{d(n-1)+2} = 1.$$

4. Proofs of Propositions 1–3 and Corollaries 4–5. Let

$$F \in \phi_{n,r,d}(L_K(n)^r), \qquad F \ne 0.$$

Let W be an irreducible component of $\phi_{n,r,d}^{-1}(F)$. Assume that W does not contain any representation of F. That means that for every $(l_1, \ldots, l_r) \in W$, there exist $i, j = 1, \ldots, r$, $i \ne j$, and $a \in K$ such that $l_i = al_j$. Since W is irreducible, we may find $i, j = 1, \ldots, r$, $i \ne j$, such that for all $(l_1, \ldots, l_r) \in W$, $l_i = al_j$ for some $a \in K$. We may assume that $i = r-1$, $j = r$. Then for some $b \in K$, we have an $(r-1)$-presentation $F = l_1^d + \cdots + (bl_{r-1})^d$. Let W' be an irreducible component of $\phi_{n,r-1,d}^{-1}(F)$ containing all such presentations. Then we will prove the following

LEMMA 4.1. $\dim W = \dim W' - 1$.

We shall use the following classical result:

(∗) *Let $\phi : X \to Y$ be a morphism of irreducible algebraic varieties defined over an algebraically closed field K. Assume that $\overline{\phi(X)} = Y$. Then, for every $y \in \phi(X)$, the dimension of every irreducible component of $\phi^{-1}(y)$ is at least $\dim X - \dim Y$. Moreover, for a generic point $y \in Y$, the dimensions of all irreducible components of $\phi^{-1}(y)$ are equal to $\dim X - \dim Y$.*

This result, under the assumption $\phi(X) = Y$, can be found in [14, Chapter I, Sec. 6.3, first part of Theorem 7]. The proof presented there works under the weaker assumption $\overline{\phi(X)} = Y$.

Proof of Lemma 4.1. We use the notation introduced above. We may represent $\phi_{n,r,d}$ as the composition $\phi^2_{n,r,d} \circ \phi^1_{n,r,d}$, where

$$\phi^1_{n,r,d} : L_K(n)^r \to \phi^1_{n,r,d}(L_K(n)^r) \subset \mathfrak{F}(n,d)^r, \quad \phi^1_{n,r,d}(l_1,\dots,l_r) = (l_1^d,\dots,l_r^d),$$

and

$$\phi^2_{n,r,d} : \Phi^1_{n,r,d}(L_K(n)^r) \to \mathfrak{F}(n,d), \quad \phi^2_{n,r,d}(f_1,\dots,f_r) = f_1 + \dots + f_r.$$

Notice that $\phi^1_{n,r,d}$ is a finite morphism. For a generic r-presentation $F = l_1^d + \dots + l_r^d$ of F contained in W, let $l_r = al_{r-1}$ and $f_1 = l_1^d, \dots, f_{r-1} = (1+a^d)l_{r-1}^d$. Then

$$F = f_1 + \dots + f_{r-1}.$$

Let $\psi : W \to \phi^1_{n,r-1,d}(W')$ be the rational map defined by

$$\psi(l_1,\dots,l_r) = (l_1^d,\dots,l_{r-2}^d,(1+a^d)l_{r-1}^d).$$

Then $\psi(W) = \phi_{n,r-1,d}(W')$ and a generic fiber of ψ is of dimension 1. Thus by $(*)$,

$$\dim W = \dim(\phi_{n,r-1,d}(W')) + 1.$$

Since $\phi^1_{n,r-1,d}$ is a finite morphism,

$$\dim(\phi^1_{n,r-1,d}(W')) = \dim W',$$

and thus our lemma has been proved.

The lemma implies at once Proposition 1.

Proof of Proposition 2. Assume that the proposition is not true. Consider independent generic linear forms $l_1,\dots,l_r$ in n variables and a generic $F = l_1^d + \dots + l_r^d$. Then $(l_1,\dots,l_r) \in \phi^{-1}_{n,r,d}(F)$. Let W be the irreducible component of $\phi^{-1}_{n,r,d}(F)$ containing $(l_1,\dots,l_r)$. For every $(z_1,\dots,z_r) \in W$ we have $z_i^d = l_j^d$ for some $i,j = 1,\dots,r$. Since the algebraic properties of all independent generic collections of linear forms indexed by $(1,\dots,r)$ are the same, we have proved that there exists an integer i such that for every independent generic collection $(l_1,\dots,l_r)$ of linear forms and for $(z_1,\dots,z_r)$ belonging to the irreducible component of $\phi^{-1}_{n,r,d}(l_1^d,\dots,l_r^d)$ containing $(l_1,\dots,l_r)$ we have $z_i = l_i$. On the other hand, since every permutation σ of $(1,\dots,r)$ determines another independent generic collection $(l_{\sigma(1)},\dots,l_{\sigma(r)})$, we find that the above property of the integer i holds for all integers $1,\dots,r$. Thus $z_1 = l_1,\dots,z_r = l_r$ and this means that there is only one point in the irreducible component. This contradicts our assumption.

Proof of Corollaries 4 and 5. By Iarrobino's theorem [6] for every $r \geq r_0$ the mapping $\phi_{n,r,d}$ is dominant, hence the dimensions of generic fibers of $\phi_{n,r+1,d}$ and $\phi_{n,r,d}$ differ by n. For $n \geq 2$ a generic fiber of $\phi_{n,r,d}$ corresponds to an r-representation. Now an application of $(*)$ quoted in the proof of Lemma 4.1 gives Corollaries 4 and 5.

Proof of Proposition 3. In Proposition 2 and its proof we may replace the assumption that "F admits two different representations" by "F admits two different projectively inequivalent representations" and then we conclude that the forms l_i, z_j are also projectively inequivalent. The assumption is valid for $r > r_0$. Since $0 = l_1^d + \cdots + l_r^d - z_1^d - \ldots - z_r^d$, this proves the assertion of Proposition 3 for t even $> 2r_0$. In order to prove it for t odd $> 2r_0$ note that the same type of argument gives the following. Let F be a generic form in $\mathfrak{F}(n,d)$, $r \geq r_0$. Then F admits an r-representation and an $(r+1)$-representation. Moreover, there exist representations $F = l_1^d + \cdots + l_r^d = z_1^d + \cdots + z_{r+1}^d$ such that l_i/z_j is non-constant for $i = 1, \ldots, r$ and $j = 1, \ldots, r+1$.

5. Proofs of Theorems 3 and 4

Lemma 5.1. *Let*

$$B_i(z_1, \ldots, z_m) = (-1)^i \prod_{\substack{1 \leq j < k \leq m \\ j \neq i \neq k}} (z_k - z_j). \tag{23}$$

Then for every $\mu \leq m-2$,

$$P_\mu := \sum_{i=1}^{m} B_i z_i^\mu = 0. \tag{24}$$

Proof. B_i is of degree $\binom{m-1}{2}$, hence P_μ is either 0 or of degree $\binom{m-1}{2} + \mu < \binom{m}{2}$. On the other hand, if $p < q$ and $z_p = z_q$, then $B_i = 0$ for $i \neq p, q$,

$$P_\mu = (B_p + B_q) z_p^\mu \tag{25}$$

and

$$\begin{aligned}
(B_p + B_q) &\prod_{\substack{1 \leq j < k \leq m \\ \{j,k\} \cap \{p,q\} = \emptyset}} (z_k - z_j)^{-1} \\
&= (-1)^p \prod_{\substack{j<q \\ j \neq p}} (z_q - z_j) \prod_{j>q} (z_j - z_q) + (-1)^q \prod_{j<p} (z_p - z_j) \prod_{\substack{j>p \\ j \neq q}} (z_j - z_p) \\
&= (-1)^p \prod_{\substack{j<q \\ j \neq p}} (z_p - z_j) \prod_{j>q} (z_j - z_p) + (-1)^q \prod_{j<p} (z_p - z_j) \prod_{\substack{j>p \\ j \neq q}} (z_j - z_p) \\
&= (-1)^{p+q-2} \prod_{j \neq p,q} (z_j - z_p) + (-1)^{p+q-1} \prod_{j \neq p,q} (z_j - z_p) = 0.
\end{aligned}$$

Thus by (25), P_μ is divisible by $\prod_{p<q}(z_p - z_q)$ and is either 0 or of degree at least $\binom{m}{2}$. Conclusion (24) follows.

LEMMA 5.2. *For $d > 1$, $x_1^d + \cdots + x_n^d$ essentially depends on n variables.*

Proof. Assume that

$$x_1^d + \cdots + x_n^d = F(l_1, \ldots, l_{n-1}),$$

where $F \in \mathfrak{F}(n-1,d) \cap K[y_1, \ldots, y_{n-1}]$, $l_i \in L(n)$. Differentiating with respect to x_i we obtain

$$dx_i^{d-1} = \sum_{j=1}^{n-1} \frac{\partial F}{\partial y_j} \frac{\partial l_j}{\partial x_i}.$$

Since by the assumption made at the beginning of the paper $d \not\equiv 0 \pmod{\operatorname{char} K}$, it follows that $x_1^{d-1}, \ldots, x_n^{d-1}$ are linearly dependent, which is false for $d > 1$.

Proof of Theorem 3. Let

$$I := \{n+1, \ldots, n+r\}.$$

Choose $(a_1, \ldots, a_n) \in K^n$ such that $a_i \neq a_j$ for $i \neq j$. Let $U \subset K^r$ be composed of all $(a_{n+1}, \ldots, a_{n+r})$ such that all a_i are different from $a_1, \ldots, a_n$ and $a_i \neq a_j$ for $i \neq j$ in I. Then the group $\mathcal{S}$ of permutations of I acts on U.

For $i = 1, \ldots, n$, let

$$l_i = x_1 + a_i x_2 + \cdots + a_i^{n-1} x_n$$

and for $\boldsymbol{a} = (a_{n+1}, \ldots, a_{n+r}) \in U$ and $i = n+1, \ldots, n+r$, let

$$l_i(\boldsymbol{a}) = x_1 + a_i x_2 + \cdots + a_i^{n-1} x_n.$$

Then by Lemma 5.1 applied with $m = n + r \geq d(n-1) + 2$,

$$\sum_{i=1}^{n+1} B_i(a_1, \ldots, a_{n+r}) l_i^d = \sum_{\mu=0}^{d(n-1)} f_\mu \sum_{i=1}^{n+r} B_i(a_1, \ldots, a_{n+r}) a_i^\mu = 0,$$

where f_μ is given by (22). It follows that

$$\sum_{i=1}^{n} A_i(\boldsymbol{a}) l_i^d = -\sum_{i=n+1}^{n+r} A_i(\boldsymbol{a}) l_i(\boldsymbol{a})^d, \tag{26}$$

where

$$A_i(\boldsymbol{a}) = \frac{B_i(a_1, \ldots, a_n, \boldsymbol{a})}{D}, \qquad D = \prod_{n<j<k\leq n+r} (a_k - a_j). \tag{27}$$

Now by (22) and (27), $A_1(\boldsymbol{a}), \ldots, A_{n+r}(\boldsymbol{a})$ do not vanish on U, are symmetric, and for $\tau \in \mathcal{S}$ and $i \in I$ we have

$$A_i(\tau(\boldsymbol{a})) = A_{\tau(i)}(\boldsymbol{a}).$$

By a linear substitution (or a linear change of variables) $l_i \mapsto y_i$, where $i = 1, \dots, n$, we obtain from (26) the equality

$$A_1(\boldsymbol{a})y_1^d + A_2(\boldsymbol{a})y_2^d + \cdots + A_n(\boldsymbol{a})y_n^d = -\sum_{i=n+1}^{n+r} A_i(\boldsymbol{a})\bar{l}_i(\boldsymbol{a})^d, \tag{28}$$

where $\bar{l}_i(\boldsymbol{a})$ is the linear form (in the variables $y_1, \dots, y_n$) obtained from $l_i(\boldsymbol{a})$ by the above substitution.

Now, another substitution in (28),

$$\sqrt[d]{A_i(\boldsymbol{a})}\, y_i \mapsto z_i,$$

where $i = 1, \dots, n$, gives for every $\boldsymbol{a} \in U$ the equality

$$z_1^d + \cdots + z_n^d = -\sum_{i=n+1}^{n+r} A_i(\boldsymbol{a})m_i(\boldsymbol{a})^d, \tag{29}$$

where, for $i = n+1, \dots, n+r$, $m_i(\boldsymbol{a})$ is the linear form (in $z_1, \dots, z_n$) obtained from $\bar{l}_i$ by this substitution. In contrast to the previous substitution, the substitution depends on $\boldsymbol{a} \in U$ and, to be precise, in order to consider roots of degree d of the functions A_i we should replace U by its properly determined cover, also denoted by U (e.g. one may take normalization of U in the extension of $K(a_{n+1}, \dots, a_{n+r})$, where $a_{n+1}, \dots, a_{n+r}$ are considered as variables, obtained by adjoining all roots of $A_i(\boldsymbol{a})$ of degree d, for $i = 1, \dots, n$).

Moreover, notice that, since the functions $A_i(\boldsymbol{a})$ for $i = 1, \dots, n$ are symmetric, the group $\mathcal{S}$ acts on the described family of representations of $z_1^d + \cdots + z_n^d$. More exactly, the image of a representation (29) under $\tau \in \mathcal{S}$ is the representation

$$z_1^d + \cdots + z_n^d = -\sum_{i=n+1}^{n+r} A_i(\tau(\boldsymbol{a}))m_i(\tau(\boldsymbol{a})).$$

Since $l_i(\tau(\boldsymbol{a})) = l_{\tau(i)}(\boldsymbol{a})$ and the above substitutions are invariant under the action of $\mathcal{S}$, we still have

$$m_i(\tau(\boldsymbol{a})) = m_{\tau(i)}(\boldsymbol{a}) \quad \text{for } i = n+1, \dots, n+r.$$

Hence the image of (29) under τ can be written as

$$z_1^d + \cdots + z_n^d = -\sum_{i=n+1}^{n+r} A_i(\tau(\boldsymbol{a}))m_{\tau(i)}(\boldsymbol{a}).$$

We are going to prove that there are a lot of representations of $z_1^d + \cdots + z_n^d$ in the above described family parametrized by U. Since U is irreducible, if this is not the case, then there exists $\boldsymbol{p} = (p_1, \dots, p_n) \in K^n \setminus \{\mathbf{0}\}$ such that, for every representation (29) and for every $\boldsymbol{a} \in U$, there exists an index

$i \in \{n+1, \dots, n+r\}$ such that $m_i(\boldsymbol{a})(\boldsymbol{p}) = 0$. Again, since U is irreducible, there exists a fixed index i_0 such that, for all $\boldsymbol{a} \in U$, we have

$$m_{i_0}(\boldsymbol{a})(\boldsymbol{p}) = 0.$$

However, for every $\tau \in \mathcal{S}$,

$$m_{\tau(i_0)}(\boldsymbol{a})(\boldsymbol{p}) = m_{i_0}(\tau(\boldsymbol{a}))(\boldsymbol{p}) = 0.$$

Thus, for all indices $i = n+1, \dots, n+r$ and all $\boldsymbol{a} \in U$, we have

$$m_i(\boldsymbol{a})(\boldsymbol{p}) = 0.$$

This means that the forms $m_{n+1}(\boldsymbol{a}), \dots, m_{d(n-1)+2}(\boldsymbol{a})$ jointly depend on n–1 variables. For $d > 1$ this contradicts the equality (29), since by Lemma 5.2 the left hand side essentially depends on n variables. This completes the proof except for the case $d = 1$, which is trivial.

LEMMA 5.3. *In every r-representation of* 0 *by sums of dth powers of linear forms we have*

$$r > \min\{d, d_1\} + 1,$$

where $d_1 = d + 1$ *if* $\operatorname{char} K = 0$, *and* $d_1 = p^{\nu+1}\left\{\frac{d+1}{p^{\nu+1}}\right\}$ *if* $\operatorname{char} K = p$ *and* $p^\nu \parallel d + 1$.

Proof. We may assume without loss of generality that one of the linear forms in question is x, hence it is enough to show impossibility of the equation

$$x_1^d = \sum_{j=1}^{r} (a_j x_1 + l_j)^d, \tag{30}$$

where $\min\{d, d_1\} \geq r > 1$, $a_j \in K$, $l_j \in L_K(n-1) \cap K[x_2, \dots, x_n]$ and

(31) the forms $a_j x_1 + l_j$ are non-zero and projectively different.

If $\operatorname{char} K = 0$, we have $\binom{d}{i} \neq 0$ for all $i \leq r$. If $\operatorname{char} K = p$, let $d = \sum_{i=0}^{k} c_i p^i$, $0 \leq c_i < p$. By the definition of ν we have $c_i = p - 1$ $(0 \leq i < \nu)$, $c_\nu < p - 1$ and by the Lucas theorem $\binom{d}{i} \not\equiv 0 \bmod p$ for $i < (c_\nu + 1)p^\nu = d_1$.

Therefore, the identity (30) gives

$$\sum_{j=1}^{r} a_j^i l_j^{d-i} = 0 \quad (0 \leq i < r). \tag{32}$$

By (31) we may assume that $l_j \neq 0$ for all $j < r$. If $l^r \neq 0$, then we infer from (32) that

$$D := \det\,(a_j^i l_j^{d-i})_{\substack{0 \leq i < r\\ 1 \leq j \leq r}} = 0.$$

However, by a reduction to a Vandermonde determinant

$$D = \prod_{j=1}^{r} l_j^d \prod_{1 \leq i < j \leq r} \left(\frac{a_j}{l_j} - \frac{a_i}{l_i}\right) \tag{33}$$

and, by (31), $D \neq 0$. This shows that $l_r = 0$ and (32) gives

$$D_1 := \det\,(a_j^i l_j^{d-i})_{\substack{0 \le i < r-1 \\ 1 \le j \le r-1}} = 0.$$

Since

$$D_1 = \prod_{j=1}^{r-1} l_j^d \prod_{1 \le i < j \le r} \left(\frac{a_j}{l_j} - \frac{a_i}{l_i} \right) \neq 0, \tag{34}$$

we have a contradiction.

REMARK 5.4. For $\operatorname{char} K = p > 0$ the number 1 in Lemma 5.3 cannot be replaced by 2, as shown by the example $d = p^2 + p - 1$, $d_1 = p$,

$$0 = -x_1^d + \sum_{j=1}^{p+1} (x_1 + \zeta_{p+1}^j x_2)^d,$$

where ζ_{p+1} is a primitive root of unity of order $p + 1$ in K.

Proof of Theorem 4. The first part of the theorem follows from Theorem 1. The second part follows from Lemma 5.3 and the trivial observation that if $p^{\nu+1} > d + 1$, then

$$p^{\nu+1} \left\{ \frac{d+1}{p^{\nu+1}} \right\} = d + 1.$$

6. Proofs of Theorems 5–7 and Proposition 4

Proof of Theorem 5. Let $F \in \mathfrak{F}(2, d, s)$ have an r-representation. If $r \le (d+1)/2$, then $s \le r \le (d+1)/2$ and unless $r = s$ we have a representation of $0 = F - F$ of length at most $r + s < d + 1$, contrary to Lemma 5.3. If $r > (d + 1)/2$, then $r \ge r_0$ and by Corollary 5 the set of r-representations of F has dimension at least $2r - (d + 1) > 0$, thus it is infinite. This proves the first part of the theorem.

In order to prove the second part notice that by Theorem 2 for a generic point $(l_1, \ldots, l_{d+1}) \in L_K(2)^{d+1}$ the set $\{l_1^d, \ldots, l_{d+1}^d\}$ is a basis of $\mathfrak{F}(2, d)$. Hence for every form F, the set $\phi_{2,d+1,d}^{-1}(F)$ intersected with a non-empty open subset of $L_K(2)^{d+1}$ is of dimension $d + 1$. This proves (a).

For a fixed form F of degree d and for fixed linear forms $l_{d+2}, \ldots, l_r$, $\phi_{2,d+1,d}^{-1}(F + l_{d+2}^d + \cdots + l_r^d)$ has a component of dimension $d + 1$, and this implies that $\phi_{2,r,d}^{-1}(F)$ has a component of dimension $d + 1 + 2(r - d - 1) = 2r - (d + 1)$. This proves (b), and since the condition $l_i(\boldsymbol{p}) \neq 0$ does not influence the dimension, also (c).

LEMMA 6.1. *Let K be an infinite field with* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$, *and let $f \in K[x]$ be of degree d. If $f(x) \neq a(x + b)^d + c$ for all a, b, c in K,* *then for every finite subset S of K there exist $\beta_1,\ldots,\beta_{d-1}$ and $b_1,\ldots,b_d$ in K such that $\beta_i \neq \beta_j$ for $i \neq j$, $\beta_i \notin S$ $(1 \le i < d)$, $b_i \neq 0$ $(1 \le i \le d)$ and*

$$f(x) = \sum_{\mu=1}^{d-1} b_\mu (x+\beta_\mu)^d + b_d. \tag{35}$$

Proof. Let $\tau_i(y_1,\ldots,y_l)$ be the ith elementary symmetric polynomial of $y_1,\ldots,y_l$ and let

$$f(x) = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i}. \tag{36}$$

We follow the proof of Theorem 4 in [12] and put

$$\begin{aligned} G(y_1,\ldots,y_{d-2}) = {} & \prod_{1\le i<j\le d-2} (y_j-y_i) \sum_{i=2}^{d} (-1)^{i-1} a_{d-i}\tau_{i-2}(y_1,\ldots,y_{d-2}) \\ & \times \prod_{j=1}^{d-2} \Big(a_{d-1} + \sum_{i=2}^{d-1} (-1)^{i-1} a_{d-i} (\tau_{i-1}(y_1,\ldots,y_{d-2}) + y_j \tau_{i-2}(y_1,\ldots,y_{d-2})) \\ & \qquad + (-1)^{d-1} a_0 y_j \tau_{d-2}(y_1,\ldots,y_{d-2}) \Big). \end{aligned}$$

Further, for $k \le d-2$ we put

$$\begin{aligned} H_k(y_1,\ldots,y_{d-2}) = {} & \sum_{i,j=0}^{d-3} (-1)^{i+j} (a_{d-2-i}a_{d-2-j} - a_{d-1-i}a_{d-3-j}) \\ & \times \tau_i(y_1,\ldots,y_{k-1},y_{k+1},\ldots,y_{d-2}) \tau_j(y_1,\ldots,y_{k-1},y_{k+1},\ldots,y_{d-2}), \\ H_{d-1}(y_1,\ldots,y_{d-2}) = {} & \sum_{i=2}^{d} (-1)^{i-1} a_{d-i}\tau_{i-2}(y_1,\ldots,y_{d-2}), \end{aligned}$$

$$\begin{aligned} & H_d(y_1,\ldots,y_{d-2}) \\ & = a_d - (-1)^d \sum_{k=1}^{d-2} \frac{H_k(y_1,\ldots,y_{d-2}) y_k^d}{\prod_{\substack{j=1\\ j\neq k}}^{d-2} (y_j-y_k) \Big(\sum_{i=1}^{d-1} (-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) - y_k H_{d-1} \Big)} \\ & \quad + \frac{\Big(\sum_{i=1}^{d-1} (-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) \Big)^d}{\prod_{k=1}^{d-2} \Big(\sum_{i=1}^{d-1} (-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) - y_k H_{d-1} \Big) H_{d-1}}. \end{aligned} \tag{37}$$

Since $a_0 \neq 0$ we have $G \neq 0$, and in particular $H_{d-1} \neq 0$. Since $f(x) \neq a(x+b)^d + c$, we have $a_{d-2-i}^2 - a_{d-1-i}a_{d-3-i} \neq 0$ for at least one $i \le d-3$, hence $H_k \neq 0$ for all $k \leq d-2$. Also for the same reason

$$\sum_{i=1}^{d-1}(-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2})/H_{d-1} \notin K. \tag{38}$$

Finally, the rational function H_d is not identically 0, since it is of order 1 with respect to y_1. Indeed, denoting by $\operatorname{ord} R$ the order of a rational function R with respect to y_1 we have

$$\operatorname{ord} a_d = 0, \quad \operatorname{ord} H_1 = 0, \quad \operatorname{ord} y_1^d = d,$$
$$\operatorname{ord} \prod_{j=2}^{d-2}(y_j - y_1) = d-3,$$
$$\operatorname{ord}\Big(\sum_{i=1}^{d}(-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) - y_1 H_{d-1}\Big) = 2$$

and for $2 \leq k \leq d-2$,

$$\operatorname{ord} H_k = 2, \quad \operatorname{ord} y_k^d = 0, \quad \operatorname{ord} \prod_{\substack{j=1\\ j\neq k}}^{d-2}(y_j - y_k) = 1,$$
$$\operatorname{ord}\Big(\sum_{i=1}^{d}(-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) - y_k H_{d-1}\Big) = 1.$$

Finally,

$$\operatorname{ord}\Big(\sum_{i=1}^{d-1}(-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2})\Big)^d = d,$$
$$\operatorname{ord}\Big(\sum_{i=1}^{d-1}(-1)^i a_{d-i}\tau_{i-1}(y_1,\ldots,y_{d-2}) - y_k H_{d-1}\Big) = \operatorname{ord} H_{d-1} = 1 \quad (k>1),$$

hence all terms in the sum (37) except one have order 0 and the exceptional term has order 1. Thus there exist $\beta_1,\ldots,\beta_{d-2}$ in K such that

$$G(\beta_1,\ldots,\beta_{d-2})\prod_{k=1}^{d} H_k(\beta_1,\ldots,\beta_{d-2}) \neq 0, \tag{39}$$

moreover $\beta_k \notin S$ $(1 \leq k \leq d-2)$ and by (38),

$$\beta_{d-1} := \sum_{i=1}^{d-1}(-1)^i a_{d-i}\tau_{i-1}(\beta_1,\ldots,\beta_{d-2})/H_d(\beta_1,\ldots,\beta_{d-2}) \notin S.$$

Since $G(\beta_1,\ldots,\beta_{d-2})\neq 0$ we have $\beta_i\neq\beta_j$ $(1\le i<j<d)$. Now, for $k\le d-2$ put

$$b_k=\frac{(-1)^d H_k(\beta_1,\ldots,\beta_{d-2})}{\prod_{j=1,\,j\neq k}^{d-1}(\beta_j-\beta_i)H_{d-1}(\beta_1,\ldots,\beta_{d-2})}$$

and

$$b_{d-1}=\frac{-H_{d-1}(\beta_1,\ldots,\beta_{d-2})}{\prod_{k=1}^{d-2}(\beta_{d-1}-\beta_k)}.$$

It follows that

$$b_d:=a_d-\sum_{\mu=1}^{d-1}b_\mu\beta_\mu^d=H_d(\beta_1,\ldots,\beta_{d-2}) \tag{40}$$

and, by (39), $b_\mu\neq 0$ $(1\le\mu\le d)$.

Now, a tedious computation based on Lemma 3 of [12] shows that for $k<d$,

$$b_k=(-1)^{d+k+1}\begin{vmatrix}1 & \ldots & & 1 & a_0\\ \beta_1 & \ldots\ \beta_{k-1} & \beta_{k+1} & \ldots\ \beta_{d-1} & a_1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \beta_1^{d-2} & \ldots\ \beta_{k-1}^{d-2} & \beta_{k+1}^{d-2} & \ldots\ \beta_{d-1}^{d-2} & a_{d-2}\end{vmatrix}\Bigg/\begin{vmatrix}1 & \ldots & 1\\ \beta_1 & \ldots & \beta_{d-1}\\ \cdots & \cdots & \cdots\\ \beta_1^{d-2} & \ldots & \beta_{d-1}^{d-2}\end{vmatrix},$$

and since, by the choice of β_{d-1},

$$\begin{vmatrix}1 & \ldots & 1 & a_0\\ \beta_1 & \ldots & \beta_{d-1} & a_1\\ \cdots & \cdots & \cdots & \cdots\\ \beta_1^{d-2} & \ldots & \beta_{d-1}^{d-2} & a_{d-1}\end{vmatrix}=0$$

we have

$$\sum_{\mu=1}^{d-1}b_\mu\beta_\mu^j=a_j\qquad(0\le j<d).$$

Now, (35) follows from (36) and (40).

Proof of Theorem 6. If F has a representation of length 2 the assertion follows from Theorem 4. If F has no representation of length 2, let $\{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_s\}$ be a subset of $K^2\setminus\{\mathbf{0}\}$. Choose $\xi\in K$ such that $F(1,\xi)\neq 0$ and

$$p_{r2}-\xi p_{r1}\neq 0\ (1\le r\le s),$$

and put

$$S=\left\{\frac{p_{r1}}{\xi p_{r1}-p_{r2}}:1\le r\le s\right\}.$$

By Lemma 6.1 there exist elements $\beta_1, \ldots, \beta_{d-1}$, $b_1, \ldots, b_d$ of K such that $\beta_i \neq \beta_j$ for $i \neq j$, $\beta_i \notin S$, $b_i \neq 0$ and

$$F(x, \xi x + 1) = \sum_{\mu=1}^{d-1} b_\mu (x + \beta_\mu)^d + b_d.$$

It follows that

$$F(x_1, x_2) = \sum_{\mu=1}^{d-1} b_\mu (x_1 + \beta_\mu(x_2 - \xi x_1))^d + b_d (x_2 - \xi x_1)^d.$$

The linear forms $x_1 + \beta_\mu(x_2 - \xi x_1)$ $(1 \leq \mu \leq d)$ and $x_2 - \xi x_1$ are projectively different, since the β_μ are distinct. Moreover, for all $r \leq s$,

$$p_{r1} + \beta_\mu(p_{r2} - \xi p_{r1}) \neq 0, \qquad p_{r2} - \xi p_{r1} \neq 0$$

by the choice of ξ and S.

Proof of Proposition 4. Assume that $c = \max(d_1, d_2) = d_2, d_1 > 0$. First, notice that there is no representation of $x_1^{d_1} x_2^{d_2}$ of length smaller than $c+1$. In fact, if

$$x_1^{d_1} x_2^{d_2} = l_1^d + \cdots + l_r^d,$$

then taking $(\partial/\partial x_1)^{d_1}$ of both sides, we obtain a representation of $d_1! x_2^{d_2}$ of length at most r, thus a representation of the zero form of length at most $r+1$ by (d_2)th powers of binary linear forms. Thus by Theorem 4, either $r + 1 \geq d_2 + 2$ and $r \geq d_2 + 1 = c + 1$, or $r = 1$, $l_1 = a x_2$, which is impossible for $d_1 > 0$.

On the other hand, there exist a lot of representations of $x_1^{d_1} x_2^{d_2}$ of length $c + 1$. In fact, let again $c = d_2$. Then for every $a \in K \setminus \{0\}$, we have

$$(d_2 + 1) \binom{d_1 + d_2}{d_2} x_1^{d_1} x_2^{d_2} = \sum_{j=0}^{d_2} a^{-d_2} \zeta^j (x_1 + a\zeta^j x_2)^{d_1 + d_2},$$

where ζ is a primitive root of 1 of degree $d_2 + 1$. This completes the proof of Proposition 4.

Proof of Theorem 7. The first part of the theorem follows from Theorem 3.

To prove the second part we proceed by induction on d. For $d = 2$ the assertion is obvious. Assume that it is true for the exponent $d - 1$ $(d \geq 3)$ and let

$$x_1^d + x_2^d + x_3^d = l_1^d + \cdots + l_v^d \tag{41}$$

be a disjoint v-representation with the least possible v. We may assume that $l_i \in K[x_1, x_2]$ exactly for $i \leq u$. Then for $i > u$,

$$l_i = a_i x_3 + m_i, \qquad a_i \in K^*,\ m_i \in L_K(2).$$

By the choice of v we have $l_i/x_j \notin K$ $(1 \le i \le u,\ j = 1,2)$ and $m_i \neq 0$ $(u < i \le v)$. Differentiating (41) with respect to x_3 we obtain

$$x_3^{d-1} = \sum_{i=u+1}^{v} a_i l_i^{d-1}. \tag{42}$$

If $\dim \mathrm{Lin}(m_{u+1}, \ldots, m_v) = 2$, then by the inductive assumption

$$v - u + 1 \ge 2(d-1) + 2 = 2d, \quad \text{hence} \quad v \ge 2d - 1.$$

If $\dim \mathrm{Lin}(m_{u+1}, \ldots, m_v) = 1$, then $m_i = b_i m$ $(u < i \le v)$, $b_i \in K^*$, $m \in L_K(2)$ and the equation (42) gives, by Lemma 5.3,

$$v - u + 1 \ge d + 1. \tag{43}$$

Moreover, after substitution $x_3 = 0$, the equation (41) gives

$$x_1^d + x_2^d = \sum_{i=1}^{u} l_i^d + m^d \sum_{i=u+1}^{v} b_i^d,$$

hence, by Theorem 4,

$$u + 1 \ge d. \tag{44}$$

Adding the inequalities (43) and (44) we obtain $v \ge 2d - 1$.

Proof of Corollary 6. Two disjoint representations of F of length r and s, respectively, would give a representation of $0 = F - F$ of length at most $r + s$ by dth powers of linear forms jointly essentially depending on three variables. If $r + s < 2d + 2$ this contradicts Theorem 7.

Proof of Corollary 7. Cancelling the identical terms in the representations of length r and s, respectively, we obtain disjoint representations $F' = l_1^d + \cdots + l_{r'}^d = m_1^d + \cdots + m_{s'}^d$, where $r - r' = s - s'$. If F' essentially depends on at most two variables, then since F' has only finitely many r'-representations we have, by Corollary 5, $s' \le r' \le (d+1)/2$, hence unless $r' = s'$ there is a representation of $0 = F' - F'$ by fewer than $d+2$ dth powers of linear forms, contrary to Lemma 5.3. Thus $r' = s'$, whence $r = s$ or F' essentially depends on three variables. In the latter case, by Theorem 7, $r' + s' \ge 2d + 2$, hence $r + s \ge 2d + 2$, contrary to the assumption.

Proof of Corollary 8. Here $r_0 = 6$, thus if $r \ge 6$ then there exist, by Corollary 5, either 0 or infinitely many r-representations, so $s \le r \le 5$ and either $r = s = 5$ or $r + s \le 9$. In the latter case $r = s$ by Corollary 7.

Proof of Theorem 8. Since, by Theorem 4, $x_i^d + x_{i+1}^d$ admits a lot of representations of every length $\ge d$, $x_1^d + \cdots + x_{2m}^d$ admits a disjoint representation of every length $r \ge md$.

The second part of the theorem is proved by double induction on d and r. For $d \le 2$ or $r \le 3$ there is no representation of $x_1^d + x_2^d + x_3^d + x_4^d$ of length

$r \le 3d-3$, thus the assertion holds trivially. Assume that it holds for the exponent $d-1$ (where $d \ge 3$) and for representations of length $\le r-1$ with exponent d (where $r \ge 4$). Let

$$x_1^d + x_2^d + x_3^d + x_4^d = l_1^d + \cdots + l_r^d \tag{45}$$

be a disjoint r-representation. If $l_w/x_j = c \in K$, then $c^d \ne 1$ and

$$\sum_{\substack{i=1\\ i\ne j}}^{4} x_i^d + (1-c^d)x_j^d = \sum_{\substack{v=1\\ v\ne w}}^{r} l_v^d.$$

Thus by the inductive assumption there exist i,k,l and $S \subset \{1,\dots,r\} \setminus \{w\}$ such that $\{i,j,k,l\} = \{1,2,3,4\}$ and

$$x_i^d + (1-c^d)x_j^d = \sum_{v\in S} l_v^d, \qquad x_k^d + x_l^d = \sum_{v\in\{1,\dots,r\}\setminus(S\cup\{w\})} l_v^d.$$

Then

$$x_i^d + x_j^d = \sum_{v\in S} l_v^d + l_w^d,$$

so the inductive assertion holds.

Therefore assume that

$$l_v/x_j \notin K \quad \text{ for } v \le r,\ j \le 4. \tag{46}$$

We may assume that $l_v \in K[x_1,x_2,x_3]$ exactly for $v \le u$. Thus

$$l_v = a_v x_4 + m_v,$$

where $a_v \in K^*,\ m_v \in L_K(3) \setminus \{\mathbf{0}\}$ and $u < v \le r$.

Differentiating (45) with respect to x_4 we obtain

$$x_4^{d-1} = \sum_{v>u} a_v l_v^{d-1}.$$

If

$$\delta := \dim \operatorname{Lin}(m_v; v>u) = 3$$

we have $\dim \operatorname{Lin}(x_4, l_v;\ v>u) = 4$, and since $r-u+1 \le 3d-2$, by the inductive assumption there exist w and a subset S of $\{u+1,\dots,r\} \setminus \{w\}$ such that $u < w \le r$, $r-u-d-2 \ge |S| \ge d-1$ and

$$x_4^{d-1} - a_w l_w^{d-1} = \sum_{v\in S} a_v l_v^{d-1}. \tag{47}$$

Since $|S| \le r-u-d-2 \le 2d-5$, by Theorem 7, the forms x_4, l_w, l_v for $v \in S$ jointly depend on two variables, hence there exists $m \in L_K(3) \setminus \{\mathbf{0}\}$ such that

$$l_v = a_v x_4 + b_v m$$

for $v = w$ or $v \in S$. Equation (47) implies

$$x_4^d - l_w^d = \sum_{v \in S} l_v^d - cm^d,$$

where $c \in K$, which subtracted from (45) gives

$$x_1^d + x_2^d + x_3^d = \sum_{v \notin S \cup \{w\}} l_v^d + cm^d.$$

Since the number of summands on the right hand side is $r - |S| \leq 2d - 2$, by Theorem 7 the representation is not disjoint and by (46) we have $cm^d = x_k^d$ for some $k \leq 3$. Taking $\{i, j\} = \{1, 2, 3\} \setminus \{k\}$ we obtain

$$x_i^d + x_j^d = \sum_{v \notin S \cup \{w\}} l_v^d,$$

hence by (45),

$$x_k^d + x_4^d = \sum_{v \in S \cup \{w\}} l_v^d.$$

Moreover, by Theorem 2, $s = |S| + 1$ satisfies $r - d \geq s \geq d$, which proves the inductive assertion in the case $\delta = 3$.

If $\delta \leq 2$ we choose $\boldsymbol{p} = (p_1, p_2, p_3) \neq \mathbf{0}$ such that

$$m_v(\boldsymbol{p}) = 0 \quad \text{for } u < v \leq r.$$

Substituting $x_4 = 0$ in (45) and differentiating with respect to $p_1x_1 + p_2x_2 + p_3x_3$ we obtain

$$p_1 x_1^{d-1} + p_2 x_2^{d-1} + p_3 x_3^{d-1} = \sum_{v=1}^{u} l_v(\boldsymbol{p}) l_v^{d-1},$$

hence by (46) and Theorems 4 and 7,

$$u \geq d - 1. \tag{48}$$

On the other hand, differentiating (45) with respect to x_4 we obtain

$$x_4^{d-1} = \sum_{v > u} a_v l_v^{d-1},$$

thus by Theorems 4 and 7,

$$r - u + 1 \geq \begin{cases} 2d & \text{if } \delta = 2, \\ d + 1 & \text{if } \delta = 1. \end{cases} \tag{49}$$

Adding the inequalities (48) and (49) for $\delta = 2$ we obtain

$$r + 1 \geq 3d - 1,$$

contrary to $r \leq 3d - 3$. Thus $\delta = 1$, $m_v = b_v m_0$ for some $m_0 \in L_K(3) \setminus \{\mathbf{0}\}$

and $b_v \in K$ $(u < v \le r)$, and substituting $x_4 = 0$ in (45) we obtain

$$x_1^d + x_2^d + x_3^d = \sum_{v=1}^{u} l_v^d + m_0^d \sum_{v>u} b_v^d. \tag{50}$$

If the representations are disjoint, Theorem 7 yields

$$u + 1 \ge 2d - 1,$$

which together with (49) gives $r + 2 \ge 3d$, contrary to $r \le 3d - 3$. Thus the representations (50) are not disjoint and by (46) there exists $k \le 3$ such that

$$x_k^d = m_0^d \sum_{v>u} b_v^d.$$

Applying the above long argument with x_k in place of x_4 we infer that either the inductive assertion holds or all forms l_v with the coefficient of x_k different from 0 belong to $K[x_k, x_4]$, hence the l_v for $v \le u$ are in $K[x_i, x_j]$ where $\{i, j\} = \{1, 2, 3\} \setminus \{k\}$. It follows from (45) that

$$x_i^d + x_j^d = \sum_{v=1}^{u} l_v^d, \qquad x_k^d + x_4^d = \sum_{v>u} l_v^d,$$

and by Theorem 4,

$$r - d \ge u \ge d.$$

The inductive assertion follows in full generality, the second statement of the theorem is proved, and the last statement follows from the double inequality for r.

Proof of Corollary 9. Two disjoint representations of F of respective lengths r and s would give a representation of $0 = F - F$ of length at most $r + s$ by dth powers of linear forms jointly essentially depending on four variables. If $r + s < 2d + 4$, this contradicts Theorem 8.

Proof of Corollary 10. Cancelling the identical terms in the representations of length r and s, respectively, we obtain disjoint representations

$$F' = l_1^d + \cdots + l_{r'}^d = m_1^d + \cdots + m_{s'}^d, \quad \text{where} \quad r - r' = s - s'.$$

If F' essentially depends on at most two variables, then since F' has only finitely many r'-representations, we have, by Corollary 5,

$$s' \le r' \le (d + 1)/2;$$

hence, unless $r' = s'$, there is a representation of $0 = F' - F'$ by fewer than $d + 2$ dth powers of linear forms, contrary to Lemma 5.3. Thus $r' = s'$, whence either $r = s$ or F' essentially depends on at least three variables. In the latter case, by Corollary 6,

$$r' + s' \ge 2d + 2,$$

hence either $r + s \ge 2d + 4$, contrary to the assumption, or $r = r'$, $s = s'$. In the latter case F essentially depends on four variables and by Corollary 9, $r+s \geq 2d+4$, contrary to the assumption.

7. Proof of Theorem 9. Assume first that $s=1$ and F has a representation of length $r>s$,

$$F=\sum_{i=1}^{r} l_i^d. \tag{51}$$

Since $s=1$ we have

$$F=l_0^d \neq 0. \tag{52}$$

Without loss of generality we may assume that $\{l_0,\dots,l_q\}$ is a linear basis of $\mathrm{Lin}(l_0,\dots,l_r)$ over K, hence $l_0,\dots,l_q$ are linearly independent, and so algebraically independent over K. Since l_1,l_2 are projectively different, we have $q\geq 1$ and

$$l_i=\sum_{j=0}^{q} a_{ij} l_j, \qquad a_{ij}\in K \qquad (1\leq i\leq r). \tag{53}$$

It follows from (51) and (53) that

$$l_0^d=F=\sum_{i=1}^{r}\Big(\sum_{j=0}^{q} a_{ij} l_j\Big)^d,$$

hence by the algebraic independence of $l_0,\dots,l_q$,

$$F=\sum_{i=1}^{r}\Big(a_{i0}l_0+t\sum_{j=1}^{q} a_{ij} l_j\Big)^d \qquad (t\in K).$$

Since $a_{11}=1$ by (53), this gives infinitely many r-representations of F.

Now assume that $s\geq 2$ and we have a representation

$$F=\sum_{i=1}^{s} l_i^d,$$

where $l_1,\dots,l_t$ are linearly independent of l_{s-1},l_s, while $l_i=z_{i-t}(l_{s-1},l_s)$ $(t<i\leq s-2)$ with $z_i\in L_K(2)$ $(1\leq i\leq s-t-2)$. Since $s\geq 2$ we have $d\geq 2$. If $r\geq s+d-2$ we apply Theorem 4 and infer the existence of infinitely many identities

$$F=\sum_{i=1}^{t} l_i^d+\sum_{i=t+1}^{s-2} z_{i-t}(l_{s-1},l_s)^d+\sum_{j=1}^{r-s+2} l_{rj}(l_{s-1},l_s)^d, \tag{54}$$

where the forms l_{rj} are projectively different from each other and from $0, z_{i-t}$. Therefore, $l_{rj}(l_{s-1},l_s)$ are non-zero and projectively different from l_i $(1\leq i\leq s-2)$. Hence (54) gives infinitely many r-representations of F.

Corollaries 11 and 12 follow at once from Theorem 9.

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