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EXISTENCE OF DISCRETE ERGODIC SINGULAR TRANSFORMS FOR ADMISSIBLE PROCESSES

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Abstract. This article is concerned with the study of the discrete version of generalized ergodic Calderón–Zygmund singular operators. It is shown that such discrete ergodic singular operators for a class of superadditive processes, namely, bounded symmetric admissible processes relative to measure preserving transformations, are weak $(1,1)$. From this maximal inequality, a.e. existence of the discrete ergodic singular transform is obtained for such superadditive processes. This generalizes the well-known result on the existence of the ergodic Hilbert transform.

1. Introduction. Let (X, Σ, μ) be a probability space, and $T : X \to X$ be an invertible measure preserving transformation. The operator $S^{\varphi} f = \sum_{i\in\mathbb{Z}} \varphi(i)T^i f$ is called the *discrete ergodic singular transform* (with kernel φ), where $\varphi = \{\varphi(n)\}_{n\in\mathbb{Z}}$ is a discrete singular kernel [AM]. If $\varphi(n) = 1/n$ then $S^{\varphi} f$ is the usual ergodic Hilbert transform. Another example of a singular kernel that generates a discrete ergodic singular transform is $\varphi(n) = 1/(n \log |n|)$, $n \neq 0, \pm 1$. The study of Hilbert transform is strongly connected to the study of singular integrals and approach regions for the existence of such integrals [CaJRW, NS]. The a.e. existence of the ergodic Hilbert transform was first proved by M. Cotlar [C]. Another proof was given by K. Petersen [P]. A. M. Alphonse and S. Madan introduced discrete ergodic singular transforms (which are discrete analogues of singular integrals) and proved that such transforms relative to invertible measure preserving transformations exist a.e. [AM].

All the results above involve additive processes in their respective settings. In [Ç$_1$] the author proved the a.e. existence of the ergodic Hilbert transform for bounded symmetric admissible processes satisfying an additional condition (on the purely subadditive part of the process). In this article we will obtain the a.e. existence of discrete ergodic singular transforms for bounded symmetric admissible processes without any additional conditions (Theorem 2.3 below). In the manuscript [Ç$_2$], which is the Hilbert transform

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version of the current article, the a.e. existence of the ergodic Hilbert transform for such admissible processes was proved without requiring the condition used in [Ç$_1$]. To this end, the maximal inequality of K. Petersen [P] was extended to the setting of admissible processes [Ç$_2$]. The maximal inequality utilized in the current article (Theorem 2.1 below) is a generalization of the maximal inequality in [AM], which, in turn, generalizes the maximal inequality in [P] to the setting of discrete singular transforms. Since the Hilbert transform is a special case of discrete ergodic singular transform, and since admissible processes include additive processes, our result generalizes the a.e. existence theorems in [AM, C, Ç$_1$, Ç$_2$, P]. We also observe that if one considers discrete ergodic singular transforms along sequences satisfying the cone condition, then a.e. existence may fail.

A *T-superadditive process on $\mathbb{Z}$* is a family of functions $F = \{f_i\}_{i\in\mathbb{Z}} \subset L_p$, $1 \le p < \infty$, whose sequence of partial sums $\{F_n\}_{n\in\mathbb{Z}}$ satisfies

$$F_{n+m} \ge F_n + T^n F_m \quad \text{and} \quad F_{-(n+m)} \ge F_{-n} + T^{-n} F_{-m} \quad \text{for all } n, m \ge 0,$$

where $F_n = \sum_{i=0}^{n-1} f_i$ if $n \ge 1$, and $F_n = \sum_{i=n+1}^{0} f_i$ if $n \le -1$. If equalities hold, then F is called a *T-additive process on $\mathbb{Z}$*, which is necessarily of the form $\{T^i f_0\}$. A family $F = \{f_i\}_{i\in\mathbb{Z}}$ is called *symmetric* if $T^{2i} f_{-i} = f_i$ for all $i \in \mathbb{Z}$, and *strongly bounded* if $\sup_{i\in\mathbb{Z}} \|f_i\|_p < \infty$. We call $F = \{f_i\}_{i\in\mathbb{Z}}$ a *T-admissible process on $\mathbb{Z}$* if

$$Tf_i \le f_{i+1} \quad \text{for } i \ge 0, \quad \text{and} \quad T^{-1} f_i \le f_{i-1} \quad \text{for } i \le 0.$$

Obviously, any T-admissible process is T-superadditive.

A *singular kernel* is a sequence $\varphi = \{\varphi(n)\}_{n\in\mathbb{Z}}$ of real numbers with the following properties: there exist constants $d, D > 0$ such that

(i) $\lim_n \sum_{i=-n}^{n} \varphi(i) = 0$,
(ii) $\varphi(0) = 0$ and $|\varphi(n)| \le d/n$ for $n \ne 0$,
(iii) $|\varphi(n) - \varphi(n+1)| \le D/n^2$ for $n \ne 0$.

In what follows, unless stated otherwise, all singular kernels φ on $\mathbb{Z}$ will be odd functions, i.e., $\varphi(-n) = -\varphi(n)$ for all $n \in \mathbb{Z}^+$, and satisfy $\varphi(n) \ge 0$ for $n \ge 0$. Given a T-superadditive family F on $\mathbb{Z}$, the discrete ergodic singular transform of F is defined as $S^\varphi F = \lim_n S_n^\varphi F$, where $S_n^\varphi F(x) = \sum_{i=-n}^{n} \varphi(i) f_i(x)$. Accordingly, $S^{\varphi *} F(x) = \sup_n |S_n^\varphi F(x)|$ is the associated maximal operator and $S_N^{\varphi *} F(x) = \sup_{0<n\le N} |\sum_{i=-n}^{n} \varphi(i) f_i(x)|$ is its truncated version. For a T-admissible process $F = \{f_i\}$ on $\mathbb{Z}$, we always have $f_i - T^i f_0 \ge 0$; hence, if a statement holds for $S^\varphi f_0$, then the same is also valid for $S^\varphi F$ if and only if it is valid for $S^\varphi G$, where $G = \{g_i\}$ with $g_i = f_i - T^i f_0$. Therefore, in such a case we can assume that $f_i \ge 0$ for all $i \in \mathbb{Z}$.

2. Discrete ergodic singular transforms for admissible processes. We begin by stating the following theorem on the maximal estimates for discrete ergodic singular transforms of additive processes. This result will be the basis for the proof of the weak $(1,1)$ maximal inequality in the admissible setting.

THEOREM A ([AM, Theorem 2.4]). *Let φ be a discrete singular kernel and $\mathbf{a} \in l_p$, $1 \le p < \infty$. Then there exist constants C_p such that*

(a) $|\{k \in \mathbb{Z} : S^{\varphi *}\mathbf{a}(k) > \lambda\}| \le \dfrac{C_1}{\lambda}\|\mathbf{a}\|_1$ *for every* $\lambda > 0$,

(b) $\|S^{\varphi *}\mathbf{a}\|_p \le C_p\|\mathbf{a}\|_p$, $1 < p < \infty$,

where $S^{\varphi *}\mathbf{a}(k) = \sup_n |\sum_{i=-n}^{n} \varphi(i) a_{k+i}|$.

Let $F = \{F_n\}_{n\ge 1} \subset L_1$ be a strongly bounded symmetric T-admissible process. It is shown in [Ç$_1$, Proposition 2.1] that if $F = \{f_n\} \subset L_1$ is a positive symmetric strongly bounded T-admissible process, then there exists an increasing sequence $\{u_k\} \subset L_1^+$ with $u_k \uparrow \delta \in L_1$ such that $f_n = T^n u_{|n|}$ for all $n \in \mathbb{Z}$ and $f_n \le T^n\delta$ for all $n \in \mathbb{Z}$. Clearly, $\|\delta\|_1 = \sup_{n\in\mathbb{Z}} \|f_n\|_1$; furthermore, δ is an exact dominant for F (i.e., $\|\delta\|_1 = \sup_{n\in\mathbb{Z}} |n|^{-1}\|F_n\|_1$).

THEOREM 2.1. *Let $F \subset L_1$ be a strongly bounded symmetric T-admissible process and φ be an odd singular kernel. Then, for any $\lambda > 0$, there is a constant C such that*

$$\mu(\{x : S^{\varphi *}F(x) > \lambda\}) \le \frac{C}{\lambda}\|\delta\|_1,$$

where δ is an exact dominant for F.

Proof. Since the assertion holds for additive processes [AM], we can assume that $f_i \ge 0$ for all $i \in \mathbb{Z}$. Let $S_N^{\varphi *}F(x) = \sup_{1\le n\le N} |\sum_{i=-n}^{n} \varphi(i) f_i(x)|$, and $E_N = \{x \in X : S_N^{\varphi *}F(x) > \lambda\}$. Then $E_N = E_N^1 \cup E_N^2$, where

$$E_N^1 = \Big\{x \in X : \sup_{1\le n\le N} \sum_{i=-n}^{n} \varphi(i) f_i(x) > \lambda\Big\},$$

$$E_N^2 = \Big\{x \in X : \inf_{1\le n\le N} \sum_{i=-n}^{n} \varphi(i) f_i(x) < -\lambda\Big\}.$$

Now, consider E_N^1. Since $f_k = T^k v_{|k|}$ for all $k \in \mathbb{Z}$, where $v_k \uparrow \delta$, $v_k \le v_{k+1}$ for all $k \ge 1$, it follows that

$$T^k v_0 \le f_k \le T^k \delta \quad \text{for all } k \in \mathbb{Z}. \tag{1}$$

By the measure preserving property of T, for any $m \in \mathbb{Z}$,

$$\mu(E_N^1) = \mu(T^{-m}E_N^1) = \mu\Big\{x : \sup_n \sum_{i=-n}^{n} \varphi(i) T^m f_i(x) > \lambda\Big\}$$

$$= \mu\Big\{x : \sup_n \sum_{i=-n}^{n} \varphi(i) T^{m+i} v_{|i|}(x) > \lambda\Big\}.$$

By (1), for any $M \geq |k|$,

$$\lambda < \sum_{i=-n}^{n} \varphi(i) T^{m+i} v_{|i|} = \sum_{i=-n}^{-1} \varphi(i) T^{m+i} v_{|i|} + \sum_{i=1}^{n} \varphi(i) T^{m+i} v_{|i|}$$
$$\leq \sum_{i=-n}^{-1} \varphi(i) T^{m+i} v_0 + \sum_{i=1}^{n} \varphi(i) T^{m+i} \delta.$$

Hence,

$$\mu(E_N^1) \leq \Big\{ x : \sup_n \Big[\sum_{i=-n}^{-1} \varphi(i) T^{m+i} v_0(x) + \sum_{i=1}^{n} \varphi(i) T^{m+i} \delta(x) \Big] > \lambda \Big\}.$$

For $x \in X$, not belonging a set of measure zero, and $|m| < M$, define a sequence $\mathbf{a} = \{a_k\}$ by

$$a_k = \begin{cases} T^k \delta(x) & \text{if } m \leq |k| \leq N+m, \\ T^k v_0(x) & \text{if } m-N \leq |k| \leq m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for each $m \in [-M, M]$, we have $\{a_k\} \in l_1$. Thus,

$$\lambda < \sum_{i=-n}^{n} \varphi(i) T^{m+i} v_{|k|}(x) \leq \sum_{i=-n}^{-1} \varphi(i) a_{i+m} + \sum_{i=1}^{n} \varphi(i) a_{i+m},$$

which implies that $E_N^1 \subset \{x : \sup_n \sum_{i=-n}^{n} \varphi(i) a_{i+m} > \lambda\}$.

For E_N^2, similarly, by (1), we have

$$\sum_{i=-n}^{n} \varphi(i) T^{m+i} v_{|i|} \geq \sum_{i=-n}^{-1} \varphi(i) T^{m+i} \delta + \sum_{i=1}^{n} \varphi(i) T^{m+i} v_0.$$

Hence, for $x \in X$ off a set of measure zero, and $|m| < M$, if we define a sequence $\mathbf{b} = \{b_k\}$ by

$$b_k = \begin{cases} T^k \delta(x) & \text{if } m-N \leq |k| \leq m-1, \\ T^k v_0(x) & \text{if } m \leq |k| \leq m+N, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that, for each $m \in [-M, M]$, we have $\{b_k\} \in l_1$. Therefore, as before,

$$E_N^2 \subset \Big\{ x : \inf_n \sum_{i=-n}^{n} \varphi(i) b_{i+m} < -\lambda \Big\} = \Big\{ x : \sup_n \sum_{i=-n}^{n} \varphi(i) b_{-i+m} > \lambda \Big\}.$$

Consequently, we observe that

$$\begin{aligned}E_N &= \Big\{x : \sup_{1\le n\le N}\Big|\sum_{i=-n}^{n}\varphi(i)T^m f_i(x)\Big| > \lambda\Big\}\\ &\subset \Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\cup\Big\{x : \inf_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)b_{i+m} < -\lambda\Big\}\\ &= \Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\cup\Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)b_{-i+m} > \lambda\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(E_N) &= \frac{1}{2M+1}\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\Big|\sum_{i=-n}^{n}\varphi(i)T^m f_i(x)\Big| > \lambda\Big\}\\ &\le \frac{1}{2M+1}\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\\ &\quad + \frac{1}{2M+1}\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)b_{-i+m} > \lambda\Big\}.\end{aligned}$$

Now, by Fubini's theorem, if card stands for the counting measure on $\mathbb{Z}$,

$$\begin{aligned}&\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\\ &\qquad\le (\text{card}\times\mu)\Big\{(m,x) : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\\ &\qquad\le \int\Big|\Big\{m : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)a_{i+m} > \lambda\Big\}\Big|\,d\mu\\ &\qquad\le \int\Big(\frac{C_1}{\lambda}\sum_{j\in\mathbb{Z}}|a_j|\Big)\,d\mu \quad \text{(by Theorem A)}\\ &\qquad\le \frac{C_1}{\lambda}\sum_{-N-M}^{N+M}\int|a_j|d\mu \le \frac{C_1}{\lambda}\sum_{-N-M}^{N+M}\|\delta\|_1\\ &\qquad= \frac{2C_1(M+N)}{\lambda}\,\|\delta\|_1\end{aligned}$$

for some constant C_1. In the same fashion, we also find that, for some constant C_2,

$$\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\sum_{i=-n}^{n}\varphi(i)b_{-i+m} > \lambda\Big\} \le \frac{2C_2(M+N)}{\lambda}\,\|\delta\|_1.$$

Thus,

$$\mu(E_N) \leq \frac{2(C_1 + C_2)(M + N)}{\lambda(2M + 1)} \|\delta\|_1.$$

Letting $M \to \infty$ and $C = C_1 + C_2$, we obtain

$$\mu(E_N) \leq \frac{C}{\lambda} \|\delta\|_1. \blacksquare$$

COROLLARY 2.2 ([Ç2]) *Let $F = \{f_i\} \subset L_1$ be a symmetric, bounded T-admissible process with exact dominant δ. Then, for any $\lambda > 0$, there exists a constant C such that*

$$\mu\left\{x : \sup_{N\geq 1} \left| \sum_{i=-N}^{N}{}' \frac{f_i(x)}{i} \right| > \lambda \right\} \leq \frac{C}{\lambda} \|\delta\|_1,$$

where $\sum_{i=-k}^{\prime k}$ means the sum without the $i = 0$ term.

REMARK. In [Ç1] the condition on the purely subadditive part was used in proving the weak $(1,1)$ maximal inequality for the ergodic Hilbert transform of bounded symmetric admissible processes. Hence, Theorem 2.1 generalizes the maximal inequality in [Ç1] as well as the maximal inequalities in [P] and in [AM].

THEOREM 2.3. *Let $F \subset L_1$ be a strongly bounded symmetric T-admissible process relative to an invertible measure preserving transformation T, and φ be an odd singular kernel. Then*

$$S^{\varphi}F(x) = \lim_n \sum_{i=-n}^{n} \varphi(i) f_i(x) \quad \text{exists a.e.}$$

Proof. By the existence of the discrete ergodic singular transform for additive processes [AM], we can assume without loss of generality that $f_i \geq 0$ for each $i \in \mathbb{Z}$. For a fixed $k \geq 0$, define $g_i^k(x) = f_i(x)$ for $0 \leq |i| \leq k$ and

$$g_i^k(x) = \begin{cases} T^{i-k} f_k(x) & \text{for } i > k, \\ T^{-i+k} f_{-k}(x) & \text{for } -i > k. \end{cases}$$

Thus, $g_i^k(x) \leq f_i(x)$ for all $i \in \mathbb{Z}$ and $k \geq 0$. Also,

$$0 \leq f_i(x) - g_i^k(x) \leq \begin{cases} T^i(\delta - v_k)(x) & \text{if } |i| > k, \\ 0 & \text{if } |i| \leq k. \end{cases}$$

From the properties of the sequence $\{v_k\}$, we know that $\|\delta - v_k\|_p \downarrow 0$ as $k \to \infty$. The process $G = \{g_i^k\}$ is T-additive (except for the terms $|i| \leq k$). In particular, since $g_i^k = T^i(T^{-k} f_k) = T^i v_k$ for all $|i| \geq k$, we have

$$\sum_{i=-n}^{n} \varphi(i) g_i^k = \sum_{i=-n}^{n} \varphi(i) T^i v_k + \sum_{i=-k}^{k} \varphi(i)(f_i - T^i v_k),$$

therefore $\lim_n \sum_{i=-n}^{n} \varphi(i) g_i^k$ exists a.e. Now,

$$\sum_{i=-n}^{n}{}' \frac{f_i(x)}{i} - \sum_{i=-n}^{n}{}' \frac{g_i^k}{i} = \sum_{i=-n}^{n}{}' \frac{s_i}{i},$$

where

$$s_i(x) = \begin{cases} 0 & \text{for } 1 \le |i| \le k, \\ f_i(x) - T^{i-k} f_k(x) & \text{for } i > k, \\ f_{-i}(x) - T^{-i+k} f_{-k}(x) & \text{for } -i > k. \end{cases}$$

Since

$$\begin{aligned} Ts_i &= T(f_i - T^{i-k} f_k) \le f_{i+1} - T^{i+1-k} f_k = s_{i+1} && \text{for } i > k, \\ T^{-1} s_{-i} &= T^{-1}(f_{-i} - T^{-i+k} f_{-k}) \\ &\le f_{-i-1} - T^{-i-1+k} f_{-k} = s_{-i-1} && \text{for } -i > k, \end{aligned}$$

the process $S = \{s_i\}$ is T-admissible. From the construction, S is bounded with exact dominant $\delta - v_k$. Letting $f^* = \limsup_n \sum_{i=-n}^{n} \varphi(i) f_i$ and $f_* = \liminf_n \sum_{i=-n}^{n} \varphi(i) f_i$, we observe that $0 \le f^* - f_* \le 2|f^* - g_k^*|$, where $g_k^* = \lim_n \sum_{i=-n}^{n} \varphi(i) g_i^k$. Therefore, if $E = \{x : f^*(x) - f_*(x) > \lambda\}$, then

$$E \subset \Big\{ x : \limsup_n \Big| \sum_{i=-n}^{n} \varphi(i) s_i(x) \Big| > \lambda/2 \Big\}.$$

From Theorem 2.1 it follows that, for some constant $C > 0$,

$$\mu(E) \le \frac{C}{\lambda} \, \|\delta - v_k\|_1.$$

By letting $k \to \infty$, we obtain $\mu(E) = 0$. Thus $S^\varphi F(x) = \lim_n S_n^\varphi F(x)$ exists a.e. ■

COROLLARY 2.4 ([Ç$_2$]). *Let $F = \{f_i\} \subset L_1$ be a symmetric, bounded T-admissible process relative to an invertible measure preserving transformation T. Then*

$$\lim_n \sum_{i=-n}^{n}{}' \frac{f_i(x)}{i} \quad \textit{exists a.e.}$$

REMARKS. 1. Corollary 2.4 does not require any additional condition on the subadditive part of the process F, hence it generalizes Theorem 3.3 in [Ç$_1$].

2. Since (X, Σ, μ) is a probability space, we have $L_p \subset L_1$ and the strong boundedness in L_p implies strong boundedness in L_1. Consequently, the assertion of Theorem 2.3 (and Corollary 2.4) is also valid for strongly bounded symmetric superadditive processes $F \subset L_p, 1 < p < \infty$.

3. The symmetry condition cannot be removed [Ç$_1$].

3. Further comments. The ordinary ergodic averages $n^{-1}\sum_{k=0}^{n-1} T^k f$ are special cases of the averages of the form $r_n^{-1}\sum_{k=0}^{r_n-1} T^{v_n+k} f$ (known as the *moving averages*), where $\{(v_n, r_n)\} \subset \mathbb{Z}\times\mathbb{Z}$ is a sequence with $r_n \to \infty$. By the moving averages theorem of A. Bellow, R. Jones and J. Rosenblatt [BJR], if the sequence $\{(v_n, r_n)\}$ satisfies the *cone condition* then the moving averages converge a.e. for every $f \in L_1$. Observing the interplay between the ordinary ergodic averages and the ergodic Hilbert transform (via Abel's summation by parts formula), one might ask about the existence of the ergodic Hilbert transform along sequences satisfying the cone condition, that is, the a.e. existence of $\lim_n {\sum'}_{k=-r_n}^{r_n} \frac{T^{v_n+k} f(x)}{k}$. It turns out that the answer is negative, as the following example shows. The original version of this example is due to R. Jones [J].

EXAMPLE 3.1. Let (X, μ, T) be a dynamical system where $X=\{-1,0,1\}$, with $\mu(\{j\}) = 1/3$ for $j = -1, 0, 1$, and $T(-1) = 0$, $T(0) = 1$, $T(1) = -1$. Let the function f be defined as $f(-1) = f(1) = 0$ and $f(0) = 1$. Then ${\sum'}_{i=-\infty}^{\infty} \frac{T^i f(0)}{i} = 0$ and

$$\sum_{i=-\infty}^{\infty}{}' \frac{T^i f(-1)}{i} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \left(\frac{1}{7}-\frac{1}{8}\right) + \cdots \neq 0.$$

Take any sequence $\{(v_n, r_n)\}$ satisfying the cone condition and such that $v_n, r_n \to \infty$. Assume that there are infinitely many $v_n \equiv 0 \pmod 3$ and infinitely many $v_n \equiv 1 \pmod 3$ (for otherwise, replace v_n by $v_n \pm 1$). Let $A = \{n : v_n \equiv 0 \pmod 3\}$ and $B = \{n : v_n \equiv 1 \pmod 3\}$. Then for $n \in A$,

$$\lim_n \sum_{i=-r_n}^{r_n}{}' \frac{T^{v_n+i} f(-1)}{i} = \lim_n \sum_{i=-r_n}^{r_n}{}' \frac{T^i f(-1)}{i} \to \sum_{i=-\infty}^{\infty}{}' \frac{T^i f(-1)}{i} \neq 0.$$

On the other hand, for $n \in B$,

$$\begin{aligned}\lim_n \sum_{i=-r_n}^{r_n}{}' \frac{T^{v_n+i} f(-1)}{i} &= \lim_n \sum_{i=-r_n}^{r_n}{}' \frac{T^{1+i} f(-1)}{i} \\ &= \lim_n \sum_{i=-r_n}^{r_n}{}' \frac{T^i f(0)}{i} \to \sum_{i=-\infty}^{\infty}{}' \frac{T^i f(0)}{i} = 0.\end{aligned}$$

Consequently, $\lim_n {\sum'}_{i=-r_n}^{r_n} \frac{T^{v_n+i} f(-1)}{i}$ fails to exist. Using Rokhlin tower techniques, one can construct such examples in the non-atomic case as well.

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