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GIBBS MEASURES IN A MARKOVIAN CONTEXT AND DIMENSION

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Abstract. The main goal is to use Gibbs measures in a markovian matrices context and in a more general context, to compute the Hausdorff dimension of subsets of $[0,1[$ and $[0,1[^2$. We introduce a parameter t which could be interpreted within thermodynamic framework as the variable conjugate to energy. In some particular cases we recover the Shannon–McMillan–Breiman and Eggleston theorems. Our proofs are deeply rooted in the properties of non-negative irreducible matrices and large deviations techniques as introduced by Ellis.

Introduction. We consider a finite Markov chain with (p_{ij}) as the matrix of transition probabilities and a family (M_t) indexed by a real parameter t, which may be interpreted within thermodynamic framework as the variable conjugate to energy, defined by $(M_t) = (p_{ij}^t)$ (here t is naturally an exponent). Let $c(t)$ be the logarithm of the Perron–Frobenius eigenvalue of the matrix (p_{ij}^t). We are going to deal with the paths $(i_0, i_1, \ldots)$ of the chain for which the limit of $n^{-1} \log p_{i_0 i_1} \ldots p_{i_{n-2} i_{n-1}}$ is finite. The function c' establishes a one-to-one mapping between $\mathbb{R}$ and the set of the above limits.

We give a family of Gibbs measures γ_t supported for each t by the set of chain's paths whose limit is $c'(t)$. As an application, we compute the Hausdorff dimension of some subsets of $[0,1[$ and $[0,1[^2$ and we extend the results to a multivariate setting. In some particular cases, we recover the theorems of Shannon [6] and Eggleston [3].

Statements of results. Let b be an integer greater than or equal to 2 and $E = \{0, 1, \ldots, b-1\}$. $E^{\mathbb{N}}$ is the set of sequences of elements belonging to E, endowed with the product topology. For each $n \geq 1$ let π_n be the projection

$$E^{\mathbb{N}} \to E^n, \quad (a_0, a_1, \ldots) \mapsto (a_0, \ldots, a_{n-1}).$$

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The topology in $E^{\mathbb{N}}$ is generated by the cylinders $\pi_n^{-1}(a_0, \dots, a_{n-1})$ which we sometimes denote by $a = (a_0, \dots, a_{n-1})$ for simplicity.

Let $M = (p_{ij})_{i,j=0,\dots,b-1}$ be an irreducible markovian matrix. We define a family $(M_t)_{t\in\mathbb{R}}$ of matrices where

$$(M_t)_{i,j} = \begin{cases} p_{ij}^t & \text{if } p_{ij} > 0, \\ 0 & \text{if } p_{ij} = 0. \end{cases}$$

If $\varrho(t)$ is the Perron–Frobenius eigenvalue of the matrix M_t then the function $c(t) = \log \varrho(t)$ is convex, non-increasing [7] and analytic. (A basic fact is that $\varrho(t)$ is an eigenvalue of multiplicity 1 which ensures the last property.)

We denote by $x(t)$ and $y(t)$ respectively non-negative right and left eigenvectors corresponding to $\varrho(t)$.

We recall that if λ is an eigenvalue of an arbitrary $n \times n$ complex matrix A, then we define a *right* (respectively *left*) *eigenvector* associated with λ as any non-zero (column) vector x in $\mathbb{C}^n$ which satisfies $Ax = \lambda x$ (respectively $x^T A = \lambda x^T$, where x^T is the transpose of x).

It is well known that for non-negative irreducible square matrices, both the left and right Perron–Frobenius eigenvectors are positive. Obviously, we can normalize the vectors $x(t)$ and $y(t)$ so that $\langle x(t), y(t)\rangle = y(t)^T x(t)$ will be equal to $\varrho(t)$.

For each real t, we define a Borel probability measure γ_t on $E^{\mathbb{N}}$ by setting for the cylinder $a = (a_0, \dots, a_{n-1})$,

$$\gamma_t(a) = \frac{y(t)_{a_0} p_a^t x(t)_{a_{n-1}}}{\varrho(t)^n}$$

where

$$p_a^t = p_{a_0 a_1}^t \dots p_{a_{n-2} a_{n-1}}^t.$$

Note that the condition of projectivity

$$\gamma_t(a_0, \dots, a_{n-1}) = \sum_{k\in E} \gamma_t(a_0, \dots, a_{n-1}, k)$$

is satisfied, which is sufficient to define a measure γ_t on $E^{\mathbb{N}}$.

The main result is the following:

Theorem 1. *For $t \in \mathbb{R}$ set*

$$E_{c'(t)} = \{(a_0, a_1, \dots) \in E^{\mathbb{N}} \mid \lim_{n\to\infty} n^{-1} \log p_{a_0 a_1} \dots p_{a_{n-2} a_{n-1}} = c'(t)\}.$$

Then $\gamma_t(E_{c'(t)}) = 1$.

If we apply this result to compute the Hausdorff dimension of sets of elements from $[0, 1[$ defined by their base b expansion we get the following theorem:

THEOREM 2. *Under the same hypotheses, the Hausdorff dimension of the set*

$$\Big\{x = \sum_{j\ge 1} a_{j-1} b^{-1} \in [0,1[\;\Big|\; \lim_{n\to\infty} n^{-1} \log p_{a_0a_1} \dots p_{a_{n-2}a_{n-1}} = c'(t)\Big\}$$

is equal to

$$\frac{-tc'(t) + c(t)}{\log b}.$$

EXAMPLE 1. Let $(p_0, p_1, \dots, p_{b-1})$ be a probability on E. For the probability transitions $p_{ij} = p_j$, we get $c(t) = \log \sum_{k\in E} p_k^t$. In the particular case where $t = 1$ we recover the Shannon–McMillan–Breiman theorem [6].

EXAMPLE 2. Consider the base 3 expansion of elements from $[0,1[$, fix p in $]0,1[$, let $q = 1-p$ and let the markovian matrix

$$\begin{pmatrix} 0 & p & q \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

give the probability transitions:

$$p_{00} = 0, \quad p_{01} = p, \quad p_{02} = q, \quad p_{10} = 1, \quad p_{11} = 0, \ \dots$$

The largest eigenvalue of M_t is $\varrho(t) = (p^t + q^t)^{1/2}$ and the relevant Hausdorff dimension is given for each real t by

$$\frac{-t(p^t \log p + q^t \log q) + (p^t + q^t)\log(p^t + q^t)}{2(p^t + q^t)\log 3}.$$

In the particular case where $p = 1/2$, we get $c'(t) = -(\log 2)/2$ independently of t, and the Hausdorff dimension is $(\log 2)/(2\log 3)$.

Proof of Theorem 1. Let a be a cylinder of length n. For $n \ge 1$ set

$$W_n : E^{\mathbb{N}} \to \mathbb{R}, \qquad (a_0, a_1, \dots) \mapsto \log p_{a_0a_1} \dots p_{a_{n-2}a_{n-1}}.$$

Then

$$(*) \qquad 0 < (\inf_{k,l\in E} y(t)_k x(t)_l) \frac{p_a^t}{\varrho(t)^n} \le \gamma_t(a) \le \frac{p_a^t}{\varrho(t)^n} (\sup_{k,l\in E} y(t)_k x(t)_l).$$

According to the terminology of Ellis [4] the partition function of the states $(E^{\mathbb{N}}, \gamma_{t_0}, (W_n)_{n\ge 1})$ at level n is the function

$$Z_{t_0,n}(t) = \sum_{a\in E^n} \gamma_{t_0}(a) p_a^t = \int e^{tW_n(a)}\, d\gamma_{t_0}(a).$$

We define the Gibbs measure of parameter t at level n as follows:

$$\gamma_{n,t} = Z_{t_0,n}(t)^{-1} e^{tW_n} \gamma_{t_0}.$$

Using inequality $(*)$ for $t+t_0$ and the definition of γ_{t_0}, it follows that

$$0 < \frac{\inf_{k,l\in E} y(t+t_0)_k x(t+t_0)_l}{y(t_0)_{a_0} x(t_0)_{a_{n-1}}}\, \gamma_{t_0}(a)\, \frac{p_a^t \varrho(t_0)^n}{\varrho(t+t_0)^n} \le \gamma_{t+t_0}(a)$$

$$\gamma_{t_0}(a)\, \frac{p_a^t \varrho(t_0)^n}{\varrho(t+t_0)^n} \cdot \frac{\sup_{k,l\in E} y(t+t_0)_k x(t+t_0)_l}{y(t_0)_{a_0} x(t_0)_{a_{n-1}}}.$$

Now, it is easily seen by summing over all $a \in E^n$ and then taking the logarithms of all terms that

$$n^{-1} \log Z_{t_0,n}(t) = \log \varrho(t+t_0) - \log \varrho(t_0) + O(1/n).$$

By analogy with (2.27) of Ellis [4], the function $c_{t_0}(t) = \log \varrho(t+t_0) - \log \varrho(t_0)$ is called the *free energy function* of the states $(E^{\mathbb{N}}, \gamma_{t_0}, (W_n)_{n\ge1})$; the Legendre–Fenchel transform of c_{t_0}, the *entropy function* of the same states is, by definition, the function

$$I_{t_0}(x) = \sup_{t\in\mathbb{R}} (tx - c_{t_0}(t)).$$

Using the theorem of Gärtner–Ellis [4, II.6.1], [2, p. 45], we get the first large deviations inequality, for a closed subset F of $\mathbb{R}$:

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{W_n}{n}(\gamma_{t_0})\{F\} \le -\inf_{x\in F} I_{t_0}(x),$$

where $\frac{W_n}{n}(\gamma_{t_0})$ stands for the distribution of $\frac{W_n}{n}$.

Here the function c_{t_0} is differentiable and $I_{t_0}(c'_{t_0}(t)) = tc'_{t_0}(t) - c_{t_0}(t)$. Since $c_{t_0}(0) = 0$, $I_{t_0}(x) \ge 0$ and $I_{t_0}(c'_{t_0}(0)) = 0$, it follows that I_{t_0} attains its infimum exactly at the point $c'_{t_0}(0) = c'(t_0)$. The probability measures $\frac{W_n}{n}(\gamma_{t_0})$ converge exponentially to $c'(t_0)$, in the sense that for any open set U containing $c'(t_0)$, there exist $r > 0$ and n_0 such that for any $n \ge n_0$,

$$\frac{W_n}{n}(\gamma_{t_0})\{U^{\mathrm{c}}\} \le e^{-nr}.$$

The following lemma yields the proof of Theorem 1:

LEMMA 1. *Let $X_n : (\Omega, \mathcal{B}, \mu) \to \mathbb{R}^d$ be a sequence of random vectors and (ν_n) a sequence of positive numbers tending to infinity such that $\sum_{n\ge1} e^{-r\nu_n} < \infty$ for each $r > 0$. If $(X_n(\mu))$ converges exponentially in a compact set K, then the set of points $w \in \Omega$ such that $(X_n(w))_{n\ge1}$ is bounded and has all its boundary points in K, is of measure* 1.

Proof. Let U be an open set containing K. By the hypotheses,

$$\sum_{n\ge0} \mu(X_n \in U^{\mathrm{c}}) < \infty.$$

Applying the Borel–Cantelli lemma we get

$$\mu(\limsup_{n\to\infty}(X_n \in U^{\mathrm{c}})) = \mu\Big(\bigcap_{p\ge 1}\bigcup_{n\ge p}(X_n \in U^{\mathrm{c}})\Big) = 0.$$

Equivalently, the following set is of measure 1:

$$\bigcap_{\substack{K\subset U\\ U\text{ open}}}\bigcup_{p\ge 1}\bigcap_{n\ge p}(X_n \in U).$$

It remains to prove the equality

$$\bigcap_{\substack{K\subset U\\ U\text{ open}}}\bigcup_{p\ge 1}\bigcap_{n\ge p}(X_n \in U) = \{w \in \Omega \mid \text{the boundary points of } X_n(w) \text{ are still in } K\}.$$

Let a be an element of the left-hand side set. If $v \notin K$, then there exist disjoint neighborhoods U and V of K and v respectively; consequently, v cannot be a boundary point of the sequence $X_n(a)$.

Conversely, suppose that there exists an open set U containing K such that $a \notin \bigcup_{p\ge 1}\bigcap_{n\ge p}(X_n \in U)$, therefore there exists a subsequence of $(X_n(a))$ lying in U^{c}.

Proof of Theorem 2. Let $C : [0,1[\ \to E^{\mathbb{N}}$ be a mapping which assigns to each $t \in [0,1[$ its digit in the base b expansion. We apply the following lemma:

LEMMA 2. *Let c be a mapping of a set X into $E^{\mathbb{N}}$ and $c^{-1}(\mathcal{B})$ the inverse image of the σ-algebra $\mathcal{B}$ of $E^{\mathbb{N}}$. Then $\nu \mapsto c(\nu)$ establishes a one-to-one mapping between the measures ν on $(X, c^{-1}(\mathcal{B}))$ and the subset of measures μ on $(E^{\mathbb{N}}, \mathcal{B})$ satisfying $\mu(F) = 0$ for each closed subset F of $E^{\mathbb{N}}$ such that $F \cap c(X) = \emptyset$.*

In the present case, the b-adic intervals $\{c^{-1}(a) \mid a \in E^n\}$ generate the Borel subsets of the interval $[0,1[$ so that γ_t is a measure on $[0,1[$ if and only if $\gamma_t(a_0a_1\ldots(b-1)(b-1)\ldots) = 0$.

The last equality is indeed true: obviously it is enough to show that $p^t_{b-1,b-1} < \varrho(t)$. Observe that

$$p^t_{b-1,b-1}x(t)_{b-1} + \sum_{j\ne b-1} p^t_{b-1,j}x(t)_j = \varrho(t)x(t)_{b-1}$$

and the sum $\sum_{j\ne b-1} p^t_{b-1,j}x(t)_j$ is positive because the matrix M_t is irreducible.

We write $\delta(a) = 1/b^n$ for the length of the b-adic interval $c^{-1}(a)$. Using the theorem of Billingsley [1, p. 129] for the sets

$$B_t = \left\{(a_0, a_1, \ldots) \in E^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty}\frac{\log\gamma_t(a_0,\ldots,a_{n-1})}{\log\delta(a_0,\ldots,a_{n-1})} = -\frac{c(t) - tc'(t)}{\log b}\right\}$$

and

$$A_t = \{(a_0, a_1, \dots) \in E^{\mathbb{N}} \mid \lim_{n\to\infty} n^{-1} \log p(a_0, \dots, a_{n-1}) = c'(t)\}$$

yields the Hausdorff dimension in the statement of Theorem 2.

Multivariate case. We wish to extend Theorem 1 to the multivariate case. We consider a mapping $f : E^2 \to \mathbb{R}^d$ and introduce a family $(M_t)_{t\in\mathbb{R}^d}$ of matrices, where

$$(M_t)_{i,j} = \begin{cases} e^{\langle t, f(i,j)\rangle} & \text{if } f(i,j) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We suppose that all these matrices are irreducible; $\varrho(t)$ is the Perron–Frobenius eigenvalue of M_t. Right and left eigenvectors, respectively $x(t)$ and $y(t)$, corresponding to $\varrho(t)$ satisfy the condition $\langle x(t), y(t)\rangle = \varrho(t)$. For $t \in \mathbb{R}^d$, the function $c(t) = \log \varrho(t)$ is convex and analytic.

With the above notations, for each n we set

$$W_n : E^{\mathbb{N}} \to \mathbb{R}^d, \qquad W_n(a_0, a_1, \dots) = \sum_{i=0}^{n-2} f(a_i, a_{i+1}).$$

For each $t \in \mathbb{R}^d$ we define a probability measure γ_t on $E^{\mathbb{N}}$ by setting, for a cylinder $a = (a_0, \dots, a_{n-1})$,

$$\gamma_t(a) = \frac{y(t)_{a_0} e^{\langle t, W_n(a)\rangle} x(t)_{a_{n-1}}}{\varrho(t)^n}.$$

By the same arguments we get:

THEOREM 3. *Set*

$$E_{c'(t)} = \{(a_0, a_1, \dots) \in E^{\mathbb{N}} \mid \lim_{n\to\infty} n^{-1} W_n(a_0, \dots, a_{n-1}) = c'(t)\}.$$

Then $\gamma_t(E_{c'(t)}) = 1$.

THEOREM 4. *The Hausdorff dimension of the set*

$$\Big\{x = \sum_{j\geq 1} a_{j-1} b^{-j} \in [0,1[\,\Big|\, \lim_{n\to\infty} n^{-1} W_n(a_0, \dots, a_{n-1}) = c'(t)\Big\}$$

is equal to

$$\frac{-\langle t, c'(t)\rangle + c(t)}{\log b}.$$

EXAMPLE 3. Take for $f(i,j)$ the vector e_i, $0 \leq i \leq b-1$, from the canonical basis of $\mathbb{R}^b$. For $t = (t_0, \dots, t_{b-1})$ it is easily seen that $\varrho(t) =$

$\sum_{k=0}^{b-1} e^{t_k}$, $y(t) = (1, \ldots, 1)$ and

$$x(t) = \begin{pmatrix} e^{t_0} \\ \vdots \\ e^{t_{b-1}} \end{pmatrix}.$$

The measure γ_t is given, for a cylinder $(a_0, \ldots, a_{n-1}) \in E^n$, by

$$\gamma_t(a_0, \ldots, a_{n-1}) = \frac{e^{t_{a_0}} \ldots e^{t_{a_{n-1}}}}{(\sum_{k=0}^{b-1} e^{t_k})^n}.$$

The function c' establishes a one-to-one mapping of $\mathbb{R}^d$ into the set of positive probability measures on $E = \{0, \ldots, b-1\}$. If we set $c'(t) = (p_0, \ldots, p_{b-1})$ we can compute the Hausdorff dimension of

$$E_{c'(t)} = \{(a_0, a_1, \ldots) \in E^{\mathbb{N}} \mid \lim_{n\to\infty} n^{-1} W_n(a_0, \ldots, a_{n-1}) = c'(t)\},$$

which is the set of sequences $(a_0, a_1, \ldots)$ such that the relative frequency of occurrences of symbols $0, \ldots, b-1$ converges respectively to $p_0, \ldots, p_{b-1}$. This dimension is equal to

$$\frac{-\langle t, c'(t)\rangle + c(t)}{\log b} = \frac{-\sum_{k=0}^{b-1} p_k \log p_k}{\log b}.$$

We thus recover the Eggleston theorem.

Example 4. Consider the mapping

$$c : [0, 1[\to (E \times E)^{\mathbb{N}}, \qquad (x, y) \mapsto ((\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots),$$

where the sequences $(\alpha_0, \alpha_1, \ldots)$ and $(\beta_0, \beta_1, \ldots)$ are the base b digits in the expansions of x and y respectively. The inverse image of a cylinder $((\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ is a b-adic square generating the Borel sets of $[0, 1[^2$.

Consider a mapping $f : (E \times E)^2 \to \mathbb{R}^d$ and a family $(M_t)_{t\in\mathbb{R}^d}$ of matrices defined as usual by

$$(M_t)_{(\alpha,\beta),(\alpha',\beta')} = \begin{cases} e^{\langle t, f((\alpha,\beta),(\alpha',\beta'))\rangle} & \text{if } f((\alpha,\beta),(\alpha',\beta')) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The aim is to prove that the Hausdorff dimension of the set of points (x, y) in $[0, 1[^2$ such that $x = \sum_{j\geq 1} \alpha_{j-1} b^{-j}$, $y = \sum_{j\geq 1} \beta_{j-1} b^{-j}$ and

$$\lim_{n\to\infty} n^{-1} W_n((\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{n-1}, \beta_{n-1})) = c'(t)$$

is equal to

$$\frac{-\langle t, c'(t)\rangle + c(t)}{2 \log b}.$$

Obviously, it is enough to show that the measures γ_t satisfy the condition of Lemma 2: $\gamma_t(F) = 0$ for each closed set F such that $F \cap c([0, 1[^2) = \emptyset$.

LEMMA 3. *If* $f((b-1,\beta),(b-1,\beta'))=0$ *when* $\beta\neq\beta'$ *and* $f((\alpha,b-1),(\alpha',b-1))=0$ *when* $\alpha\neq\alpha'$, *then the last condition is satisfied.*

Proof. Each closed set F disjoint from $c([0,1[^2)$ is contained in a countable union of sets of the form

$$\{(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)\}\times E^{\mathbb{N}} \quad \text{or}$$
$$E^{\mathbb{N}}\times\{(\beta_0,\dots,\beta_{l-1},b-1,b-1,\dots)\}.$$

It is enough to show that for each sequence of digits $(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)$,

$$\gamma_t(\{(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)\}\times E^{\mathbb{N}})=0.$$

Because

$$\begin{aligned}\gamma_t(&\{(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)\}\times E^{\mathbb{N}})\\ &=\inf_{n\geq 0}\gamma_t(\{(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)\}\times E^{n}),\end{aligned}$$

we have to show that $\gamma_t(\{(\alpha_0,\dots,\alpha_{k-1},b-1,b-1,\dots)\}\times E^{n})$ converges to 0 as $n\to\infty$. But

$$\begin{aligned}\gamma_t(&(\alpha_0,\dots,\alpha_{k-1},b-1,\dots,b-1)(\beta_0,\dots,\beta_{n-1}))\\ &=\gamma_t((\alpha_0,\beta_0),\dots,(\alpha_{k-1},\beta_{k-1}),(b-1,\beta_k),\dots(b-1,\beta_{n-1}))\end{aligned}$$

is zero unless $\beta_{k-1}=\beta_k=\ldots=\beta_{n-1}$. Set

$$C_k=\sum_{(\beta_0,\dots,\beta_{k-1})} y(t)_{(\alpha_0,\beta_0)}e^{\langle t,f((\alpha_0,\beta_0),(\alpha_1,\beta_1))\rangle}\dots e^{\langle t,f((\alpha_{k-1},\beta_{k-1}),(b-1,\beta_k))\rangle}.$$

Then

$$\begin{aligned}\gamma_t(&\{(\alpha_0,\dots,\alpha_{k-1},b-1,\dots,b-1)\}\times E^{n})\\ &=\varrho(t)^{-k}C_k\sum_{\beta\in E}\varrho(t)^{k-n}e^{(n-k)\langle t,f((b-1,\beta),(b-1,\beta))\rangle}x(t)_{(b-1,\beta_{n-1})}.\end{aligned}$$

From the properties of the matrix it follows that $e^{\langle t,f((b-1,\beta),(b-1,\beta))\rangle}<\varrho(t)$ and this completes the proof.

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