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DECOMPOSITIONS OF LOCAL RIGID ACD GROUPS

BY

ADOLF MADER (Honolulu, HI) and OTTO MUTZBAUER (Würzburg)

Abstract. We study direct decompositions of extensions of rigid completely decomposable groups by finite primary groups. These decompositions are unique and can be found by finite procedures. By passing to certain quotients the determination of the direct decompositions is made more efficient.

1. Introduction. In [MM00] we studied decompositions of rigid local almost completely decomposable (briefly, ACD) groups whose regulator quotient is a direct sum of cyclic p-groups all of the same order. There was a complete and definite algorithmic procedure for determining the indecomposable direct decomposition of such a group. In this paper we deal with the case of arbitrary primary regulator quotients.

Recall that a *type* τ is an isomorphism class of rational groups, and a *rational group* is an additive subgroup of the rationals containing $\mathbb{Z}$. We abuse notation and additionally use τ for a representative of the class τ. The groups $G(\tau)$, $G^\sharp(\tau)$ are the usual (pure) *type subgroups* of G. A group G is *rigid* if $\operatorname{rk}(G(\tau)/G^\sharp(\tau)) \leq 1$ for every type τ and its *critical typeset* $\mathrm{T}_{\mathrm{cr}}(G) = \{\tau : G(\tau)/G^\sharp(\tau) \neq 0\}$ is an anti-chain. For a subgroup H of a torsion-free group G, the symbol H_*^X denotes the purification of H in G. An almost completely decomposable X is *p-local* for a prime p if $X/\mathrm{R}(X)$ is a (finite) p-group where $\mathrm{R}(X)$ is the regulator of X. For a rigid almost completely decomposable group X the *regulator* is simply $\mathrm{R}(X) = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(X)} X(\varrho)$ and $X(\tau) \cong \tau$.

Let X be a rigid p-local almost completely decomposable group and $A = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\varrho$ a completely decomposable subgroup such that $p^d X \subset A$ for some d. It was shown in [MM00] that X has a unique indecomposable decomposition and that finding it amounts to finding the partition

$$(1.1)\qquad \mathrm{T}_{\mathrm{cr}}(X) = T_1 \cup \ldots \cup T_n \quad \text{such that} \quad X = \bigoplus_{i=1}^{n} \Big(\bigoplus_{\varrho \in T_i} A_\varrho \Big)_*^X$$

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and each summand is indecomposable. The equality of X with the direct sum in (1.1) is equivalent to an index equality, namely,

$$[X : A] = \prod_{i=1}^{n} \Big[\Big(\bigoplus_{\varrho \in T_i} A_\varrho \Big)_*^X : \Big(\bigoplus_{\varrho \in T_i} A_\varrho \Big) \Big]. \tag{1.2}$$

In order to compute the indices in (1.2) we assume that the group X is given in the standard description developed in [BM98] (see below). The so-called Purification Lemma (Lemma 2.4) then provides a criterion for testing partitions of $\mathrm{T}_{\mathrm{cr}}(X)$ for corresponding to direct decompositions. In principle, this produces the decomposition in a finite number of steps, but it is inefficient and can be improved. We will call a partition (1.1) a *d-partition* if (1.2) is satisfied.

2. Preliminaries. All groups in this paper are abelian, and the torsion-free groups all have finite rank. The expression $\mathbb{M}_{k\times r}(S)$ denotes the set of $k \times r$ matrices with entries in the set S. The set S is usually a ring, in particular the ring of integers $\mathbb{Z}$ will occur and its quotient ring $\mathbb{Z}/e\mathbb{Z}$, but S may also be an abelian group G. When $r = 1$ we write $s^{\downharpoonleft} \in \mathbb{M}_{k\times 1}(S)$. Similarly, when $k = 1$ we write $\overrightarrow{s} \in \mathbb{M}_{1\times r}(S)$. Frequently we will need to deal with submatrices of a matrix and we will use the following notation. Let $M \in \mathbb{M}_{k\times r}(S)$. Then $M[i\lfloor]$ for $1 \le i \le k$ denotes the submatrix of M consisting of its ith row; $M[\lfloor j]$ for $1 \le j \le r$ denotes the submatrix of M consisting of its jth column; $M[i\lfloor j]$ is the entry of M in the ith row and jth column; $M[\alpha\lfloor]$ for $\alpha \subset \{1, \ldots, k\}$ denotes the submatrix of M formed by the rows with index in α; $M[\lfloor\beta]$ for $\beta \subset \{1, \ldots, r\}$ denotes the submatrix of M formed by the columns with index in β; $M[\alpha\lfloor\beta]$ for $\alpha \subset \{1, \ldots, k\}$ and $\beta \subset \{1, \ldots, r\}$ denotes the submatrix of M formed by deleting all rows with index not listed in α and all columns with index not listed in β.

For background on almost completely decomposable groups we refer the reader to the survey article [Mad95] or the monograph [Mad00].

Throughout, X denotes an almost completely decomposable group, and A a completely decomposable subgroup of finite index in X. The completely decomposable group A can be written as $A = \tau_1 v_1 \oplus \ldots \oplus \tau_r v_r$ (r for rank) where the τ_i are rational groups. We call $\mathcal{V} = \{v_1, \ldots, v_r\}$ a *conditioned basis* of A. Since X contains A as a subgroup of finite index, in other words, since X is a *finite essential extension* of A, there is a positive integer e such that $eX \subset A$. In the following we will restrict to the case where e is a p-power, say $e = p^d$. In this case a p-divisible critical type τ_i creates a p-divisible direct summand $\tau_i v_i$ of X and these summands are uninteresting for most purposes. We therefore assume that the groups under consideration are *p-reduced*, meaning that there are no non-trivial p-divisible subgroups. In this situation $\mathcal{V} = \{v_1, \ldots, v_r\}$ may be assumed to be a *p-basis*, which means that $\gcd^A(p, v_i) = 1$ for each i, or, equivalently, $1/p \notin \tau_i$. Write

$$X = A + \mathbb{Z}x_1 + \ldots + \mathbb{Z}x_k, \quad \text{where} \quad p^{d_i}x_i = a_i \in A.$$

Given a conditioned basis $\mathcal{V}$ of A we can write

$$a_i = m_{i1}v_1 + \ldots + m_{ir}v_r,$$

and we obtain a *coordinate matrix* $M = [m_{ij}]$ such that $a^\downharpoonleft = Mv^\downharpoonleft$. It was shown in [BM98] (see also [Mad00, Chapter 11]) that $\mathcal{V}$ can be chosen such that M is an integral matrix, $M \in \mathbb{M}_{k\times r}(\mathbb{Z})$, and in addition $\mathcal{V}$ may be chosen to be a p-basis of A. Each row of the coordinate matrix M determines and corresponds to a generator of X, namely the generator

$$x_i = p^{-d_i}(m_{i1}v_1 + \ldots + m_{ir}v_r),$$

and we call $\operatorname{supp}(i) = \{\tau_j : m_{ij} \not\equiv 0 \bmod p^{d_i}\}$ the *support* of i or of x_i or of the ith row $M[i\lfloor]$ of M. We can write

$$x_i = p^{-d_i}\sum\{m_{ij}v_j : \tau_j \in \operatorname{supp}(i)\} + \sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\},$$

where $\sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\} \in A$. Thus x_i can be replaced by $x_i - \sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\}$, so that we may assume without loss of generality that $\tau_j \notin \operatorname{supp}(i)$ if and only if $m_{ij} = 0$.

The standard description of an almost completely decomposable group is

$$\begin{aligned}
&X = A + \overrightarrow{\mathbb{Z}}N^{-1}Mv^\downharpoonleft, \qquad A = \tau_1v_1 \oplus \ldots \oplus \tau_rv_r,\\
&\mathcal{V} = \{v_1, \ldots, v_r\} \text{ is a } p\text{-basis of } A, \text{ i.e., } \gcd^A(p, v_i) = 1,\\
&N = \operatorname{diag}(p^{d_1}, \ldots, p^{d_k}) \quad \text{ with } 1 \le d_1 \le \ldots \le d_k =: d,\\
&M \in \mathbb{M}_{k\times r}(\mathbb{Z}), \qquad \gcd^A(N, Mv^\downharpoonleft) = I.
\end{aligned} \tag{2.3}$$

Under these assumptions $\gcd^A(N, Mv^\downharpoonleft) = \operatorname{gcld}(N, M)$ and greatest common divisor can be computed by column reduction of the augmented matrix $[N|M]$ ([BM98, Theorem 3.3], [Mad00, Section 11.2]).

The Purification Lemma ([BM98, Lemma 4.1], [Mad00, Lemma 11.4.1]) will be a convenient and necessary tool.

LEMMA 2.4 (Purification Lemma). *Assume that $A = B \oplus C$ is an arbitrary torsion-free abelian group of arbitrary rank, $a^\downharpoonleft = b^\downharpoonleft + c^\downharpoonleft$, where $b^\downharpoonleft \in B^\downharpoonleft$ and $c^\downharpoonleft \in C^\downharpoonleft$. Let*

$$X = A + \overrightarrow{\mathbb{Z}}N^{-1}a^\downharpoonleft$$

be a finite essential extension of A where $\gcd^A(N, a^\downharpoonleft) = I$. Let $N_C = \gcd^A(N, b^\downharpoonleft)$ and $N_B = \gcd^A(N, c^\downharpoonleft)$. Then N_C, N_B are non-singular and the following hold.

1. *$B_* = B_*^X$ is a finite essential extension of B and*

$$B_* = B + \overrightarrow{\mathbb{Z}}N_B^{-1}b^\downharpoonleft, \quad \gcd^A(N_B, b^\downharpoonleft) = I.$$

2. *X/B_* is a finite essential extension of $(B_* \oplus C)/B_* \cong C$. Specifically, writing $N = N_B M_B$, and $c^\downharpoonleft = N_B c'^\downharpoonleft$,*

$$\frac{X}{B_*} \cong C + \vec{\mathbb{Z}} M_B^{-1} c'^\downharpoonleft \subset \mathbb{Q}A, \quad \gcd^A(M_B, c'^\downharpoonleft) = I.$$

The isomorphism is induced by the projection $\mathbb{Q}B \oplus \mathbb{Q}C \to \mathbb{Q}C$ that maps the generic element

$$x + B_* = (b_x + c_x + \vec{\alpha} N^{-1}(b^\downharpoonleft + c^\downharpoonleft)) + B_*, \qquad b_x \in B,\ c_x \in C,\ \vec{\alpha} \in \vec{\mathbb{Z}}$$

to

$$c_x + \vec{\alpha} N^{-1} c^\downharpoonleft = c_x + \vec{\alpha} M_B^{-1} c'^\downharpoonleft.$$

The Purification Lemma implies that (1.1) is a d-partition of X if and only if

$$(2.5) \qquad \begin{aligned} &(\det N_1) \ldots (\det N_n) = \det N \quad \text{where} \\ &N_i = \mathrm{gcld}(N, M[\downharpoonleft\{1, \ldots n\} - T_i]). \end{aligned}$$

It was verified in [MMN00] that X has a standard description where

$$(2.6) \qquad M = [E|F] \in \mathbb{M}_{k\times r}(\mathbb{Z})$$

with

$$E = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ m_{21} & 1 & 0 & \ldots & 0 & 0 \\ m_{31} & m_{32} & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{k-1,1} & m_{k-1,2} & m_{k-1,3} & \ldots & 1 & 0 \\ m_{k,1} & m_{k,2} & m_{k,3} & \ldots & m_{k,k-1} & 1 \end{bmatrix}.$$

In the uniform case, i.e. when $d_1 = \ldots = d_k = d$, the matrix E may be assumed to be the identity matrix. If E is the identity matrix, then each row of M determines an indecomposable fully invariant subgroup of X that is a cyclic extension of its regulator, and using these subgroups one can find the decomposition of the group X.

The general case is more complicated and is approached inductively by passing to certain quotient groups. Examples illustrate the gains of this method over the crude approach.

3. Decomposition of rigid local ACD groups. The argument that worked for uniform groups breaks down when the quotient X/A is allowed to be an arbitrary p-group. In this case the simplest available form of the coordinate matrix is (2.6). The support of the last row of M (the kth generator of X) may well be all of $\mathrm{T}_{\mathrm{cr}}(X)$ and the corresponding purification all of X, so that no information is gained. This is in contrast with the uniform case where the purification of $\mathrm{supp}(k)$ was a cyclic extension of

$\sum\{\tau_j v_j : \tau_j \in \operatorname{supp}(k)\}$ with known indecomposable decomposition. In order to make the proof more accessible, we give some examples. A technical lemma will clarify the form of the matrix greatest common divisors that are basic to the arguments.

LEMMA 3.7.

$$\operatorname{gcld}\left(\begin{bmatrix} p^{d_1} & 0 & \dots & 0 \\ * & p^{d_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & p^{d_k} \end{bmatrix}, \begin{bmatrix} m_{11} & \dots & m_{1r} \\ \dots & \dots & \dots \\ m_{k1} & \dots & m_{kr} \end{bmatrix}\right) = \begin{bmatrix} p^{d'_1} & 0 & \dots & 0 \\ * & p^{d'_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & p^{d'_k} \end{bmatrix}$$

with $d'_i \leq d_i$.

Proof. Set

$$N = \begin{bmatrix} p^{d_1} & 0 & \dots & 0 \\ * & p^{d_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & p^{d_k} \end{bmatrix}.$$

The greatest common left divisor D is a subdiagonal matrix provided it is computed by column reduction as usual. The identity

$$N = DL \tag{3.8}$$

shows that D and L are both non-singular. Hence $L = D^{-1}N$ (in $\mathbb{M}_k(\mathbb{Q})$) is subdiagonal as a product of subdiagonal matrices. The claim is now clear from (3.8). ∎

The following immediate observation will also be used.

LEMMA 3.9. *Let $N \in \mathbb{M}_k(\mathbb{Z})$ and $M \in \mathbb{M}_{k\times r}(\mathbb{Z})$. Suppose that M is partitioned as $M = [M_1|M_2]$. Then*

$$\operatorname{gcld}(N, M) = \operatorname{gcld}(\operatorname{gcld}(N, M_1), \operatorname{gcld}(N, M_2)).$$

This means in practice that a greatest common divisor can be computed in steps.

We are now ready for the examples.

EXAMPLE 3.10. *Let $A = \tau_1 v_1 \oplus \ldots \oplus \tau_{12} v_{12}$ and $X = A + \overrightarrow{\mathbb{Z}} N^{-1} M v^{\downarrow}$ where*

$$N = \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 \\ 0 & 0 & 0 & 0 & p^6 \end{bmatrix},$$

$$M = \left[\begin{array}{ccccc:ccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right].$$

1. $\mathrm{T}_{\mathrm{cr}}(X) = \{\tau_1, \tau_2, \tau_6, \tau_8, \tau_{11}\} \cup \{\tau_3, \tau_4, \tau_5, \tau_7, \tau_9, \tau_{10}, \tau_{12}\}$ *is not a d-partition.*

2. $\mathrm{T}_{\mathrm{cr}}(X) = \{\tau_1, \tau_2, \tau_4, \tau_5, \tau_6, \tau_8, \tau_9, \tau_{10}, \tau_{11}\} \cup \{\tau_3, \tau_7, \tau_{12}\}$ *is a d-partition.*

Proof. 1. In order to compute the index of $A_1 = \tau_1 v_1 \oplus \tau_2 v_2 \oplus \tau_6 v_6 \oplus \tau_8 v_8 \oplus \tau_{11} v_{11}$ in its purification we must column reduce the augmented matrix (the first line contains the column labels and is not relevant to the computation)

$$\tag{3.11} \begin{array}{c}
\begin{array}{ccccc|ccccccc}
 & & & & & \tau_3 & \tau_4 & \tau_5 & \tau_7 & \tau_9 & \tau_{10} & \tau_{12}
\end{array} \\
\left[\begin{array}{ccccc|ccccccc}
p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & p^4 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & p^4 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & p^6 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}.$$

The echelon form of the columns labeled τ_3, τ_4, τ_5 implies that the column reduced echelon form (omitting columns of zeros) looks like

$$\begin{bmatrix}
* & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

The first row of (3.11) only contains one non-zero entry that will remain in the column reduced form. The second row contains a 1 that is used to make all other entries in this row equal to zero. The resulting reduced matrix (and greatest common divisor) is

$$\begin{bmatrix}
p^2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

By the Purification Lemma, $[(A_1)_*^X : A_1] = p^2$.

Similarly, the index of $A_2 = \tau_3 v_3 \oplus \tau_4 v_4 \oplus \tau_5 v_5 \oplus \tau_7 v_7 \oplus \tau_9 v_9 \oplus \tau_{10} v_{10} \oplus \tau_{12} v_{12}$ in its purification is obtained by column reduction of the augmented matrix

$$\begin{array}{c} \begin{array}{ccccc|ccccc} & & & & & \tau_1 & \tau_2 & \tau_6 & \tau_8 & \tau_{11} \end{array} \\ \left[\begin{array}{ccccc|ccccc} p^2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & p^4 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^6 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

The echelon form of the columns labeled τ_1, τ_2 implies that the column reduced echelon form (omitting columns of zeros) looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * \end{bmatrix}.$$

The third row only contains one non-zero entry, p^4, that will remain in the column reduced form. The fourth row contains a 1 that can be used to make all other entries in this row equal to zero. The resulting reduced matrix (and greatest common divisor) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & p^4 \end{bmatrix}.$$

By the Purification Lemma, $[(A_2)_*^X : A_2] = p^8$. Since $p^2 \cdot p^4 \cdot p^4 \neq [X : A]$, the partition under consideration is not a d-partition.

2. In order to compute the index of

$$B_1 = \tau_1 v_1 \oplus \tau_2 v_2 \oplus \tau_4 v_4 \oplus \tau_5 v_5 \oplus \tau_6 v_6 \oplus \tau_8 v_8 \oplus \tau_9 v_9 \oplus \tau_{10} v_{10} \oplus \tau_{11} v_{11}$$

in its purification we must column reduce the augmented matrix

$$\begin{array}{c} \begin{array}{ccccc|ccc} & & & & & \tau_3 & \tau_7 & \tau_{12} \end{array} \\ \left[\begin{array}{ccccc|ccc} p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^6 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

The reduced matrix (and greatest common divisor) is

$$\begin{bmatrix} p^2 & 0 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 \\ 0 & 0 & 0 & 0 & p^6 \end{bmatrix}.$$

By the Purification Lemma, $[(B_1)_*^X : B_1] = p^2 \cdot p^4 \cdot p^4 \cdot p^6$.

Similarly, the index of $B_2 = \tau_3 v_3 \oplus \tau_7 v_7 \oplus \tau_{12} v_{12}$ in its purification is obtained by column reduction of the augmented matrix

$$\left[\begin{array}{ccccc|ccccccccc} & & & & & \tau_1 & \tau_2 & \tau_4 & \tau_5 & \tau_6 & \tau_8 & \tau_9 & \tau_{10} & \tau_{11} \\ p^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & p^4 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & p^6 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

The echelon form of the columns labeled $\tau_1, \tau_2, \tau_4, \tau_5$ implies that the column reduced echelon form (omitting columns of zeros) looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The third row only contains one non-zero entry, p^4, that will remain in the column reduced form. The resulting reduced matrix (and greatest common divisor) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By the Purification Lemma, $[(B_2)_*^X : B_2] = p^4$. Since $(p^2 \cdot p^4 \cdot p^4 \cdot p^6) \cdot p^4 = [X : A]$ the partition under consideration is a d-partition. ∎

We now change Example 3.10 slightly and argue with ad hoc arguments that it is an indecomposable group. We will develop a more systematic way later and consider the same example again.

EXAMPLE 3.12. *Let* $A = \tau_1 v_1 \oplus \ldots \oplus \tau_{12} v_{12}$ *and* $X = A + \overrightarrow{\mathbb{Z}} N^{-1} M v^{\downharpoonleft}$ *where*

$$N = \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 \\ 0 & 0 & 0 & 0 & p^6 \end{bmatrix},$$

$$M = \left[\begin{array}{ccccc:ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Then X is indecomposable.

Proof. Let $\mathrm{T_{cr}}(X) = S_1 \cup S_2$ be a d-partition with $\tau_1 \in S_1$. We need to show that $S_1 = \mathrm{T_{cr}}(X)$ and $S_2 = \emptyset$. Set $A_1 = \bigoplus\{\tau_i v_i : \tau_i \in S_1\}$ and $A_2 = \bigoplus\{\tau_i v_i : \tau_i \in S_2\}$. Along the way we will assume that certain types belong to S_1 and certain other types to S_2 and derive contradictions. Each time the indices $[(A_i)_*^X : A_i] = \det N_i$ need to be checked (see (2.5)) and this is done by computing the matrices $N_i = \mathrm{gcld}(N, M_i)$ where M_i is the submatrix of M obtained by deleting the columns corresponding to the types in S_i. The column belonging to the type τ_j will interchangeably be referred to as the column j or the column τ_j.

Suppose first that $\tau_1 \in S_1$ and $\tau_2 \in S_2$. Then ($\xrightarrow{c}$ denotes column reduction)

$$\begin{array}{c} \begin{array}{ccccc|cc} & & & & & \tau_2 & \end{array} \\ \left[\begin{array}{ccccc|cc} p^2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 0 & \dots \end{array}\right] \end{array} \xrightarrow{c} \begin{array}{c} \begin{array}{ccccc|c} & \tau_2 & & & & \end{array} \\ \left[\begin{array}{ccccc|c} p^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & \dots \end{array}\right] \end{array} \xrightarrow{c} N_1,$$

$$\begin{array}{c} \begin{array}{ccccc|cc} & & & & & \tau_1 & \end{array} \\ \left[\begin{array}{ccccc|cc} p^2 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 0 & \dots \end{array}\right] \end{array} \xrightarrow{c} \begin{array}{c} \begin{array}{ccccc|c} \tau_1 & & & & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -p^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & \dots \end{array}\right] \end{array} \xrightarrow{c} N_2.$$

The second lines of the intermediate greatest common divisors already show that (2.5) cannot hold. The conclusion is that $\tau_2 \in S_1$.

Assume next that $\tau_1, \tau_2 \in S_1$ and $\tau_5 \in S_2$. The type τ_9 must belong either to S_1 or S_2. We will see that neither is possible and conclude that $\tau_5 \in S_1$. In fact, assume first that $\tau_1, \tau_2, \tau_9 \in S_1$ and $\tau_5 \in S_2$. Then

$$\begin{array}{c} \begin{array}{ccccc|cc} & & & & & \tau_5 & \end{array} \\ \left[\begin{array}{ccccc|cc} p^2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 1 & \dots \end{array}\right] \end{array} \xrightarrow{c} \left[\begin{array}{ccccc|c} p^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \end{array}\right] \xrightarrow{c} N_1,$$

$$\begin{array}{c} \begin{array}{ccccc|cccc} & & & & & \tau_1 & \tau_2 & \tau_9 & \end{array} \\ \left[\begin{array}{ccccc|cccc} p^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 0 & 0 & 1 & \dots \end{array}\right] \end{array} \xrightarrow{c}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \end{array}\right] \xrightarrow{c} N_2.$$

so that (2.5) is violated. On the other hand assume that $\tau_1, \tau_2 \in S_1$ and $\tau_5, \tau_9 \in S_2$. Then

$$\begin{array}{c} \begin{array}{ccccc|ccc} & & & & & \tau_5 & \tau_9 & \end{array} \\ \left[\begin{array}{ccccc|ccc} p^2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 1 & 1 & \dots \end{array}\right] \end{array} \xrightarrow{c} \left[\begin{array}{ccccc|c} p^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \end{array}\right] \xrightarrow{c} N_1,$$

$$\begin{array}{c} \begin{array}{ccccc|ccc} & & & & & \tau_1 & \tau_2 & \end{array} \\ \left[\begin{array}{ccccc|ccc} p^2 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & 0 & 0 & \dots \end{array}\right] \end{array} \xrightarrow{c} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & \dots \end{array}\right] \xrightarrow{c} N_2.$$

so that (2.5) is again violated.

We now know that $\tau_1, \tau_2, \tau_5 \in S_1$, and the inclusion of τ_5 in S_1 implies that $\tau_4 \in S_1$ just as the inclusion of τ_1 in S_1 implied the inclusion of τ_2. If

$\tau_3 \in S_2$, then τ_{10} can neither belong to S_1 nor to S_2. For example, assume that $\tau_1, \tau_2, \tau_4, \tau_5 \in S_1$ and $\tau_3, \tau_{10} \in S_2$. Then

$$\begin{array}{c} \begin{array}{ccccccccc} & & & & & & \tau_3 & \tau_{10} & \end{array} \\ \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 & \vert & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & \vert & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \vert & 1 & 1 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \vert & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & p^6 & \vert & 0 & 0 & \dots \end{bmatrix} \end{array} \xrightarrow{c} \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 & \vert & \dots \\ 0 & p^4 & 0 & 0 & 0 & \vert & \dots \\ 0 & 0 & 1 & 0 & 0 & \vert & \dots \\ 0 & 0 & 0 & 1 & 0 & \vert & \dots \\ 0 & 0 & 0 & 1 & p^6 & \vert & \dots \end{bmatrix} \xrightarrow{c} N_1$$

$$\begin{array}{c} \begin{array}{cccccccccc} & & & & & & \tau_1 & \tau_2 & \tau_4 & \tau_5 \end{array} \\ \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 & \vert & 1 & 0 & 0 & 0 & \dots \\ 0 & p^4 & 0 & 0 & 0 & \vert & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & p^4 & 0 & 0 & \vert & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & p^4 & 0 & \vert & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & p^6 & \vert & 0 & 0 & 1 & 1 & \dots \end{bmatrix} \end{array} \xrightarrow{c}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \vert & \dots \\ 0 & 1 & 0 & 0 & 0 & \vert & \dots \\ 0 & 0 & p^4 & 0 & 0 & \vert & \dots \\ 0 & 0 & 0 & 1 & 0 & \vert & \dots \\ 0 & 0 & 0 & 0 & 1 & \vert & \dots \end{bmatrix} \xrightarrow{c} N_2.$$

This violates (2.5).

It has now been shown that $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \in S_1$ and evidently $S_2 = \emptyset$. ■

Luckily, the first support supp(1) produces a pure, fully invariant, indecomposable subgroup as in [MM00].

Lemma 3.13. *Let $X = A + \vec{\mathbb{Z}} N^{-1} M v^{\downharpoonleft}$ be as before with N in Smith Normal Form and M as in* (2.6). *Then*

$$\begin{aligned} Y &= \Big(\bigoplus \{\varrho v_\varrho : \varrho \in \operatorname{supp}(1)\} \Big)_*^X \\ &= \Big(\bigoplus \{\varrho v_\varrho : \varrho \in \operatorname{supp}(1)\} \Big) + \mathbb{Z} p^{-d_1} \sum \{ m_{1\varrho} v_\varrho : \varrho \in \operatorname{supp}(1)\}. \end{aligned}$$

Moreover, Y is a pure, fully invariant, and indecomposable subgroup of X.

Proof. To apply the Purification Lemma we must compute

$$\operatorname{gcld}(N, M[1, \dots, k \downharpoonleft \mathrm{T}_{\mathrm{cr}}(X) - \operatorname{supp}(1)]).$$

To do this the augmented matrix

$$\left[\begin{array}{ccc:cccccc:ccc}
p^{d_1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots \\
0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & \dots & m_{2j} & \dots \\
0 & \dots & 0 & m_{32} & 1 & \dots & 0 & 0 & \dots & m_{3j} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & m_{k-1,2} & m_{k-1,3} & \dots & 1 & 0 & \dots & m_{k-1,j} & \dots \\
0 & \dots & p^{d_k} & m_{k,2} & m_{k,3} & \dots & m_{k,k-1} & 1 & \dots & m_{kj} & \dots
\end{array}\right]$$

must be reduced to column echelon form and this clearly results in the matrix

$$\begin{bmatrix}
p^{d_1} & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 0 & 1
\end{bmatrix}$$

and the claim follows from the Purification Lemma. Clearly

$$Y = \Big(\bigoplus \{X(\varrho) : \varrho \in \operatorname{supp}(1)\} \Big)_*^X$$

and therefore Y is pure and fully invariant. By [MM00, Lemma 3.3] the subgroup Y is indecomposable. ∎

The indecomposable fully invariant subgroup Y may be a summand of X, but this can easily be tested as follows.

Proposition 3.14. *Let X be a rigid p-local almost completely decomposable group given in the form of* (2.3) *and* (2.6). *Then the fully invariant pure subgroup*

$$Y = \Big(\bigoplus \{\varrho v_\varrho : \varrho \in \operatorname{supp}(1)\} \Big)_*^X$$

is a direct summand of X if and only if

$$m_{21} \equiv 0 \bmod p^{d_2 - d_1}, \; m_{31} \equiv 0 \bmod p^{d_3 - d_1}, \dots, m_{k1} \equiv 0 \bmod p^{d_k - d_1},$$

and for each $j \in \operatorname{supp}(1)$,

$$m_{2j} \equiv m_{1j} m_{21} \bmod p^{d_2}, \qquad m_{3j} \equiv m_{1j} m_{31} \bmod p^{d_3}, \dots,$$
$$m_{kj} \equiv m_{1j} m_{k1} \bmod p^{d_k}.$$

Proof. We have seen (Lemma 3.13) that $[Y : (\bigoplus\{\varrho v_\varrho : \varrho \in \operatorname{supp}(1)\})] = p^{d_1}$. Hence Y is a summand of X if and only if

$$\Big[\Big(\bigoplus_{\varrho \notin \operatorname{supp}(1)} \varrho v_\varrho \Big)_*^X : \Big(\bigoplus_{\varrho \notin \operatorname{supp}(1)} \varrho v_\varrho \Big) \Big] = p^{d_2} \dots p^{d_k}.$$

This index is obtained by column reduction of the matrix (with $j \in \operatorname{supp}(1)$)

$$\begin{bmatrix} p^{d_1} & \dots & 0 & \vdots & 1 & \vdots & \dots & m_{1j} & \dots \\ 0 & \dots & 0 & \vdots & m_{21} & \vdots & \dots & m_{2j} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \vdots & m_{k-1,1} & \vdots & \dots & m_{k-1,j} & \dots \\ 0 & \dots & p^{d_k} & \vdots & m_{k1} & \vdots & \dots & m_{kj} & \dots \end{bmatrix} \tag{3.15}$$

which must reduce to (omitting zero columns)

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ * & p^{d_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & 0 & \dots & 0 \\ * & 0 & \dots & p^{d_k} \end{bmatrix}. \tag{3.16}$$

If we take advantage of the 1 in the first row of (3.15), the matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 \dots & 0 & \vdots & 0 & \dots & 0 & \dots \\ m_{21} & p^{d_2} & 0 \dots & 0 & \vdots & p^{d_1} m_{21} & \dots & m_{2j} - m_{1j} m_{21} & \dots \\ m_{31} & 0 & p^{d_3} \dots & 0 & \vdots & p^{d_1} m_{31} & \dots & m_{3j} - m_{1j} m_{31} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-1,1} & 0 & 0 \dots & 0 & \vdots & p^{d_1} m_{k-1,1} & \dots & m_{k-1,j} - m_{1j} m_{k-1,1} & \dots \\ m_{k1} & 0 & 0 \dots & p^{d_k} & \vdots & p^{d_1} m_{k1} & \dots & m_{kj} - m_{1j} m_{k1} & \dots \end{bmatrix} \tag{3.17}$$

Assume first that Y is in fact a direct summand of X. If any entry other than the first of the second row of (3.17) is not divisible by p^{d_2}, then the greatest common divisor cannot be (3.16) and Y is not a summand contrary to assumptions. Hence the first equations of the claim are established. Now the second column of (3.17) can be used to reduce the matrix to

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \vdots & 0 & \dots & 0 & \dots \\ m_{21} & p^{d_2} & 0 & \dots & 0 & \vdots & 0 & \dots & 0 & \dots \\ m_{31} & 0 & p^{d_3} & \dots & 0 & \vdots & p^{d_1} m_{31} & \dots & m_{3j} - m_{1j} m_{31} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_{k-1,1} & 0 & 0 & \dots & 0 & \vdots & p^{d_1} m_{k-1,1} & \dots & m_{k-1,j} - m_{1j} m_{k-1,1} & \dots \\ m_{k1} & 0 & 0 & \dots & p^{d_k} & \vdots & p^{d_1} m_{k1} & \dots & m_{kj} - m_{1j} m_{k1} & \dots \end{bmatrix}.$$

The argument continues in the same way and establishes the claim. Conversely, if the conditions on the entries of M are satisfied, then the matrix (3.15) reduces to the form (3.16) and Y is a summand of X. ∎

If Y is indeed a summand, then the unique complementary summand X' of smaller rank will be treated as its predecessor X. If Y is not a direct summand of X, then we can reduce the rank of the problem by factoring out Y. The following evident lemma contains the relevant facts.

LEMMA 3.18. *Let X be a rigid p-local almost completely decomposable group and $A = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(A)} \varrho v_\varrho$ a completely decomposable subgroup of finite index in X.*

1. *Let T be a subset of $\mathrm{T}_{\mathrm{cr}}(X)$ and $Y = (\bigoplus_{\varrho \in T} \varrho v_\varrho)^X_*$. Then Y is pure and fully invariant in X.*

2. *Assume that Y is indecomposable but not a direct summand of X. Also let $X = X_1 \oplus \ldots \oplus X_n$ be the indecomposable decomposition of X. Then $\mathrm{T}_{\mathrm{cr}}(X/Y) = \mathrm{T}_{\mathrm{cr}}(X) - \mathrm{T}_{\mathrm{cr}}(Y)$ and without loss of generality $Y \subset X_n$. Also*

$$X/Y \cong X_1 \oplus \ldots \oplus X_{n-1} \oplus X_n/Y$$

with indecomposable summands $X_1, \ldots, X_{n-1}$ and $X_n/Y \neq 0$. Hence X/Y has at least n indecomposable summands. In particular, if X/Y is indecomposable, then so is X.

3. *If $\mathrm{T}_{\mathrm{cr}}(X/Y) = T_1 \cup \ldots \cup T_m$ is the indecomposable d-partition of X/Y, then $m \geq n$ and without loss of generality*

$$\mathrm{T}_{\mathrm{cr}}(X_i) = T_i \quad \text{for } i = 1, \ldots, n-1, \qquad \mathrm{T}_{\mathrm{cr}}(X_n) = \mathrm{T}_{\mathrm{cr}}(Y) \cup T_n \cup \ldots \cup T_m.$$

Thus if Y is not a summand of X and X/Y is indecomposable, then X is indecomposable and the matter is settled. Otherwise, in order to continue the process with X/Y, it is necessary to obtain a description of the quotient group X/Y that has the same special properties as the description of X. The Purification Lemma provides the answer.

LEMMA 3.19. *Let X be an ACD group given in the form* (2.3) *with $M = [E|F]$ as in* (2.6). *Let $Y = (\bigoplus\{\varrho v_\varrho : \varrho \in \mathrm{supp}(1)\})^X_*$. Then*

$$X/Y \cong X_1 = \bigoplus\{\varrho v_\varrho : \varrho \notin \mathrm{supp}(1)\} + \overrightarrow{\mathbb{Z}}(N_1)^{-1} M_1 {v_1}^{\downharpoonleft},$$

where

$$N_1 = \mathrm{diag}(p^{d_2}, \ldots, p^{d_k}), \qquad M_1 = [E_1|F_1],$$

with

$$E_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ m_{32} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ m_{k2} & m_{k3} & \ldots & 1 \end{bmatrix},$$

F_1 is obtained from F by deleting the first row and all columns that are indexed by types in $\mathrm{supp}(1)$, *and ${v_1}^{\downharpoonleft}$ is obtained from $v^{\downharpoonleft}$ by deleting all component elements v_ϱ with $\varrho \in \mathrm{supp}(1)$.*

Proof. The matrix $\operatorname{gcld}(N, M[\downharpoonleft \mathrm{T}_{\mathrm{cr}}(X) - \operatorname{supp}(1)])$ has been found in the proof of Lemma 3.13 to be $\operatorname{diag}(p^{d_1}, 1, \ldots, 1)$ and the description of X/Y is a straightforward application of the Purification Lemma. ∎

According to Lemma 3.19 finding the description of the group $X_1 \cong X/Y$ could not be simpler: We obtain the description of X_1 by deleting the first row of the matrices involved and certain columns that correspond to types in the support of the last row. The description of X_1 is automatically again of the desired form and the induction can proceed. We illustrate this with the previous Example 3.12.

EXAMPLE 3.20. *Let*

$$A = \tau_1 v_1 \oplus \ldots \oplus \tau_{12} v_{12} \quad \textit{and} \quad X = A + \vec{\mathbb{Z}} N^{-1} M v^{\downharpoonleft}$$

where

$$N = \begin{bmatrix} p^2 & 0 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 \\ 0 & 0 & 0 & 0 & p^6 \end{bmatrix},$$

$$M = \left[\begin{array}{ccccc:ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Then X is indecomposable.

Proof. The first row of M shows that

$$\mathrm{T}_{\mathrm{cr}}(Y_1) = \operatorname{supp}(1) = \{\tau_1, \tau_6, \tau_8, \tau_{11}\}.$$

We note that Y_1 is not a summand of $X = X_1$ since

$$\left[\begin{array}{ccccc|cccc} & & & & & \tau_1 & \tau_6 & \tau_8 & \tau_{11} \\ p^2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & p^4 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^6 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & p^4 \end{bmatrix}.$$

The coordinate matrix M_2 of $X_2 \cong X_1/Y_1$ is obtained by dropping the columns belonging to $\mathrm{T}_{\mathrm{cr}}(Y_1) = \operatorname{supp}(1)$ and the first row. We obtain

$$M_2 = \begin{array}{c} \begin{array}{cccc:cccc} \tau_2 & \tau_3 & \tau_4 & \tau_5 & \tau_7 & \tau_9 & \tau_{10} & \tau_{12} \end{array} \\ \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

By inspecting the first row of M_2 we find that $\mathrm{T_{cr}}(Y_2) = \{\tau_2, \tau_9, \tau_{12}\}$. We note that Y_2 is not a summand of X_2 since

$$\begin{array}{c} \begin{array}{cccc|ccc} & & & & \tau_2 & \tau_9 & \tau_{11} \end{array} \\ \left[\begin{array}{cccc|ccc} p^4 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & p^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^6 & 0 & 1 & 0 \end{array}\right] \end{array} \xrightarrow{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p^4 & 0 & 0 \\ 0 & 0 & p^4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The coordinate matrix M_3 of $X_3 \cong X_2/Y_2$ is obtained by dropping the columns belonging to $\mathrm{T_{cr}}(Y_2)$ and the first row. We obtain

$$M_3 = \begin{array}{c} \begin{array}{ccc:cc} \tau_3 & \tau_4 & \tau_5 & \tau_7 & \tau_{10} \end{array} \\ \left[\begin{array}{ccc:cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

By inspecting the first row of M_3 we find that $\mathrm{T_{cr}}(Y_3) = \{\tau_3, \tau_7, \tau_{10}\}$. We note that Y_3 is not a summand of X_3 since

$$\begin{array}{c} \begin{array}{ccc|ccc} & & & \tau_3 & \tau_7 & \tau_{10} \end{array} \\ \left[\begin{array}{ccc|ccc} p^4 & 0 & 0 & 1 & 1 & 1 \\ 0 & p^4 & 0 & 0 & 0 & 1 \\ 0 & 0 & p^6 & 0 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^6 \end{bmatrix}.$$

The coordinate matrix M_4 of $X_4 \cong X_3/Y_3$ is obtained by dropping the columns belonging to $\mathrm{T_{cr}}(Y_3)$ and the first row. We obtain

$$M_4 = \begin{array}{c} \begin{array}{cc:} \tau_4 & \tau_5 \end{array} \\ \left[\begin{array}{cc:} 1 & 0 \\ 1 & 1 \end{array}\right] \end{array}.$$

By inspecting the first row of M_4 we find that $\mathrm{T_{cr}}(Y_4) = \{\tau_4\}$. We note that Y_4 is not a summand of X_4 since

$$\begin{array}{c} \begin{array}{cc|c} & & \tau_4 \end{array} \\ \left[\begin{array}{cc|c} p^4 & 0 & 1 \\ 0 & p^6 & 1 \end{array}\right] \end{array} \xrightarrow{c} \begin{bmatrix} 1 & 0 \\ 1 & p^4 \end{bmatrix}.$$

The group $X_5 \cong X_4/Y_4$ is now indecomposable. By Lemma 3.18 it follows successively that X_4, X_3, X_2, and finally $X = X_1$ are indecomposable. ■

The improved procedure is now clear. We assume that the almost completely decomposable group X is given in the form of (2.3) and (2.6). Set $X_1 = X$ and let $Y_1 = (\bigoplus\{\varrho v_\varrho : \varrho \in \mathrm{supp}(1)\})_*^{X_1}$. Inductively, let

$$X_{i+1} \cong X_i/Y_i \quad \text{and} \quad Y_{i+1} = \Big(\bigoplus\{\varrho v_\varrho : \varrho \in \mathrm{supp}(1)\}\Big)_*^{X_{i+1}},$$

where supp(1) is now the support of the first row of the standard description of X_{i+1} as determined by Lemma 3.19. The ranks of the groups X_i are decreasing. Suppose that the indecomposable decomposition of X_{i+1} has been found, and $\mathrm{T}_{\mathrm{cr}}(X_{i+1}) = T_1 \cup \ldots \cup T_n$ is the indecomposable d-partition. If Y_i was a summand of X_i, then $\mathrm{T}_{\mathrm{cr}}(X_i) = \mathrm{T}_{\mathrm{cr}}(Y_i) \cup T_1 \cup \ldots \cup T_n$ is the indecomposable d-partition of X_i. If Y_i was not a summand of X_i, then one of the n partitions

$$\mathrm{T}_{\mathrm{cr}}(X_i) = T_1 \cup \ldots \cup [\mathrm{T}_{\mathrm{cr}}(Y_i) \cup T_j] \cup \ldots \cup T_n, \quad j = 1, \ldots, n,$$

will be the indecomposable d-partition of X_i. It must be tested which one is correct.

Example 3.20 suggests the following proposition.

PROPOSITION 3.21. *If, for each i, either Y_i is not a summand of X_i or else $Y_i = X_i$, then X is indecomposable.*

Proof. The ranks of the X_i are decreasing and no Y_i is a summand until $Y_n = X_n$ for some n. Then inductively $X_n, X_{n-1}, \ldots, X_1 = X$ are all indecomposable. ■

An easy example shows that the converse of Proposition 3.21 does not hold.

EXAMPLE 3.22. *Let $A = \tau_1 v_1 \oplus \tau_2 v_2 \oplus \tau_3 v_3 \oplus \tau_4 v_4$ and $X = A + \overrightarrow{\mathbb{Z}} N^{-1} M v^{\downharpoonleft}$ where*

$$N = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

Then $Y_1 = (\tau_1 v_1 \oplus \tau_4 v_4)_*^X = (\tau_1 v_1 \oplus \tau_4 v_4) + \mathbb{Z}\frac{1}{p}(v_1 + v_4)$ is not a summand of $X_1 = X$, but $X_2 = (\tau_2 v_2 \oplus \tau_3 v_3) + \mathbb{Z}\frac{1}{p} v_2 + \mathbb{Z}\frac{1}{p} v_3 = Y_2 \oplus \mathbb{Z}\tau_3 \frac{1}{p} v_3$ where $Y_2 = \tau_2 \frac{1}{p} v_2$. Thus Y_2 is a summand of X_2 and $Y_2 \neq X_2$ while X is indecomposable.

Proof. The uniform group X is indecomposable by [MM00, Theorem 4.7]. ■

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Department of Mathematics
University of Hawaii
2565 The Mall
Honolulu, HI 96822, U.S.A.
E-mail: adolf@math.hawaii.edu

Mathematisches Institut
Universität Würzburg
Am Hubland
D-970074 Würzburg, Germany
E-mail: mutzbauer@mathematik.uni-wuerzburg.de

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