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LEFT-RIGHT PROJECTIVE BIMODULES AND STABLE EQUIVALENCES OF MORITA TYPE

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Abstract. We study a connection between left-right projective bimodules and stable equivalences of Morita type for finite-dimensional associative algebras over a field. Some properties of the category of all finite-dimensional left-right projective bimodules for self-injective algebras are also given.

Introduction. Let K be a fixed field. In the representation theory of finite-dimensional associative K-algebras with identity, stable equivalences of Morita type seem to be of particular relevance. They play a substantial role in the representation theory of finite groups (see [7, 12, 15]). It was observed by Rickard that if two self-injective K-algebras are derived equivalent then they are stably equivalent of Morita type [14].

Rickard [14] showed that for two derived equivalent K-algebras their Hochschild cohomology algebras are isomorphic. Happel also proved it but did not publish. This result was generalized in [13] to self-injective K-algebras A and B which are stably equivalent of Morita type. The main trick used in the proof of this generalization rests heavily on an easy observation that there are two very small subcategories (they consist only of one Ω-orbit) which are stably equivalent; one of them is contained in the category of finite-dimensional A-bimodules, the other in the category of finite-dimensional B-bimodules. An important property of these subcategories is that they consist only of bimodules which are projective as left modules and as right modules. Following Butler and King [8] we call such bimodules left-right projective. They were first studied by Auslander and Reiten [5]. It is natural to ask whether a stable equivalence of Morita type between two finite-dimensional K-algebras A and B induces a stable equivalence between their categories of left-right projective bimodules.

One of the main aims of this paper is to give an affirmative answer to the last question. We show even more: existence of a stable equivalence between two finite-dimensional K-algebras is equivalent to existence of a special stable equivalence between the categories of left-right projective bimodules.

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In order to state our first result we need some notations. Precise definitions are in Sections 1 and 2. For a given finite-dimensional associative K-algebra A with identity 1, we denote by $\mathrm{mod}(A)$ the category of all finite-dimensional right A-modules. Its stable category modulo projectives will be denoted by $\underline{\mathrm{mod}}(A)$. The enveloping algebra A^{e} of A is the algebra $A^{\mathrm{o}} \otimes_K A$, where A^{o} is the opposite algebra to A. Furthermore, we denote by $\mathrm{lrp}(A^{\mathrm{e}})$ the full subcategory of $\mathrm{mod}(A^{\mathrm{e}})$ consisting of the left-right projective A-bimodules. Moreover, $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$ stands for the stable category of $\mathrm{lrp}(A^{\mathrm{e}})$.

Now we are able to state our first main result.

Theorem 1. *Let A, B be finite-dimensional K-algebras. Then the following conditions are equivalent*:

(1) *A and B are stably equivalent of Morita type.*

(2) *There are left-right projective bimodules ${}_AN_B$, ${}_BM_A$ such that the functor $M \otimes_A - \otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(B^{\mathrm{e}})$ induces an equivalence $F : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$ with $F(A) \cong B$ whose quasi-inverse is induced by the functor $N \otimes_B - \otimes_B M : \mathrm{lrp}(B^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{e}})$.*

The above result indicates that the subcategory $\mathrm{lrp}(A^{\mathrm{e}})$ is very important. It is of great importance to know how large this subcategory is in $\mathrm{mod}(A^{\mathrm{e}})$. Since the representation theory of the enveloping algebra A^{e} is rather complicated in general, the subcategory $\mathrm{lrp}(A^{\mathrm{e}})$ may play a crucial role in the study of stable equivalences of Morita type. Our second aim is to prove that $\mathrm{lrp}(A^{\mathrm{e}})$ is representation-infinite in case A is a self-injective K-algebra, except for some easy cases. Here, representation-infinite means that there are infinitely many pairwise nonisomorphic, indecomposable objects in $\mathrm{lrp}(A^{\mathrm{e}})$. We can even consider the case of two self-injective finite-dimensional K-algebras A and B. In this case the tensor product $A^{\mathrm{o}} \otimes_K B$ is also self-injective (see [4]). Then we show that the full subcategory $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ of left-right projective A-B-bimodules in $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B)$ is very often representation-infinite. In fact we study this problem from the Auslander–Reiten theory point of view (see [2, 3, 6]). We consider a connected component $\mathcal{C}$ (and its stable part $\mathcal{C}^{\mathrm{s}}$) of the Auslander–Reiten quiver $\Gamma_{A^{\mathrm{o}} \otimes_K B}$ of $A^{\mathrm{o}} \otimes_K B$, and indicate how many left-right projective vertices it contains. Our second main result is the following theorem.

Theorem 2. *Let A, B be two self-injective finite-dimensional K-algebras such that at least one of them is not semisimple. Let $\mathcal{C}$ be a connected component of the Auslander–Reiten quiver $\Gamma_{A^{\mathrm{o}} \otimes_K B}$ and $\mathcal{C}^{\mathrm{s}}$ its stable part. Then the following conditions are equivalent*:

(1) *There exists a vertex of $\mathcal{C}$ which is a left-right projective A-B-bimodule and is not projective.*

(2) *There exists a vertex of $\mathcal{C}^{\mathrm{s}}$ which is a left-right projective A-B-bimodule.*

(3) *Every vertex of $\mathcal{C}$ is a left-right projective A-B-bimodule.*

It is shown in Section 3 that there are also components in $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ which do not contain any left-right projective A-B-bimodule, except for the semisimple cases.

The methods we use are rather simple. In fact we need only some fundamental knowledge of Auslander–Reiten theory [2, 3, 6] and very simple facts from homological algebra.

1. Preliminaries

1.1. Let A be a finite-dimensional associative K-algebra with identity element 1. Then we denote by A^{o} the *opposite algebra*, i.e. $A^{\mathrm{o}} = A$ as K-linear spaces and the multiplication $\circ$ in A^{o} is $a_1 \circ a_2 = a_2 a_1$ for any $a_1, a_2 \in A^{\mathrm{o}}$. The *enveloping algebra* of A is $A^{\mathrm{e}} = A^{\mathrm{o}} \otimes_K A$. It is well known and easy to verify that every A-bimodule M with K acting centrally is a right A^{e}-module with the following action: $m \cdot (a \otimes a') = ama'$ for any $m \in M$ and any $a, a' \in A$. Conversely, every right A^{e}-module N is in fact an A-bimodule.

1.2. For any finite-dimensional K-algebra A, we denote by $\mathrm{Mod}(A)$ the category of all right A-modules, and by $\mathrm{mod}(A)$ its full subcategory consisting of the finite-dimensional right A-modules. Let $\mathcal{P}$ be the two-sided ideal in $\mathrm{mod}(A)$ which consists of the morphisms factorizing through projective A-modules. Then the factor category $\mathrm{mod}(A)/\mathcal{P}$ is called the *stable category* of $\mathrm{mod}(A)$ (or briefly of A). It is usually denoted by $\underline{\mathrm{mod}}(A)$. The objects of $\underline{\mathrm{mod}}(A)$ are those objects of $\mathrm{mod}(A)$ which have no nonzero projective direct summands. For $M, N \in \underline{\mathrm{mod}}(A)$, we denote by $\underline{\mathrm{Hom}}_A(M, N)$ the K-linear space of morphisms from M to N in $\underline{\mathrm{mod}}(A)$. This is the factor space $\mathrm{Hom}_A(M, N)/\mathcal{P}(M, N)$. For every $f \in \mathrm{Hom}_A(M, N)$ we denote by $\underline{f} \in \underline{\mathrm{Hom}}_A(M, N)$ its coset modulo $\mathcal{P}(M, N)$.

1.3. Two finite-dimensional K-algebras A and B are said to be *stably equivalent* if there is an equivalence of categories $\Phi : \underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(B)$. Finite-dimensional K-algebras A and B are called *stably equivalent of Morita type* provided that there is an A-B-bimodule N and a B-A-bimodule M such that:

(i) M, N are projective as left and right modules,
(ii) $M \otimes_A N \cong B \oplus \Pi$ as B-bimodules for some projective B-bimodule Π,
(iii) $N \otimes_B M \cong A \oplus \Pi'$ as A-bimodules for some projective A-bimodule Π'.

If finite-dimensional K-algebras A and B are stably equivalent of Morita type then they are stably equivalent. Indeed, if ${}_A N_B$ and ${}_B M_A$ establish a stable equivalence of Morita type between A and B then the functor $-\otimes_A N : \mathrm{mod}(A) \to \mathrm{mod}(B)$ induces an equivalence $\underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(B)$ whose quasi-inverse is induced by $-\otimes_B M : \mathrm{mod}(B) \to \mathrm{mod}(A)$.

1.4. Given two finite-dimensional K-algebras A and B, a right $A^{\mathrm{o}} \otimes_K B$-module (or equivalently an A-B-bimodule) X is called *left-right projective* if it is projective as a left A-module and as a right B-module. We denote by $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ (resp. $\mathrm{lrp}(A^{\mathrm{e}})$) the full subcategory of $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B)$ (resp. $\mathrm{mod}(A^{\mathrm{e}})$) consisting of the left-right projective bimodules.

1.5. The following easy lemma shows that the left-right projective bimodules are closed under tensor products.

Lemma. *For any finite-dimensional K-algebras A, B the following assertions hold*:

(1) *If X is a left-right projective A-bimodule and Y is a left-right projective A-B-bimodule then $X \otimes_A Y$ is a left-right projective A-B-bimodule.*

(2) *If U is a left-right projective A-B-bimodule and V is a left-right projective B-bimodule then $U \otimes_B V$ is a left-right projective A-B-bimodule.*

Proof. This was proved in [5].

1.6. The following lemma shows that tensoring left-right projective bimodules by projective ones we again obtain projective bimodules.

Lemma. *Let A, B be finite-dimensional K-algebras. Then the following assertions hold*:

(1) *If P is a projective A-B-bimodule and X is a left-right projective A-bimodule then $X \otimes_A P$ is a projective A-B-bimodule.*

(2) *If P is a projective A-B-bimodule and Y is a left-right projective B-bimodule then $P \otimes_B Y$ is a projective A-B-bimodule.*

(3) *If Q is a projective A-bimodule and V is a left-right projective A-B-bimodule then $Q \otimes_A V$ is a projective A-B-bimodule.*

(4) *If Q is a projective B-bimodule and U is a left-right projective A-B-bimodule then $U \otimes_B Q$ is a projective A-B-bimodule.*

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be complete sets of pairwise orthogonal primitive idempotents in A and B, respectively. Then the tensor products $\{e_i \otimes f_j\}_{i=1,\ldots,n;j=1,\ldots,m}$ form a complete set of pairwise orthogonal primitive idempotents in $A^{\mathrm{o}} \otimes_K B$. Thus every indecomposable projective A-B-bimodule is of the form $Ae_i \otimes f_jB$. This implies that if P is a projective A-B-bimodule then $P \cong P_1 \otimes_K P_2$ for some left projective A-module P_1 and some right projective B-module P_2. For any left-right projective A-bimodule X we have $X \otimes_A P \cong X \otimes_A P_1 \otimes_K P_2$. Since $X \otimes_A P_1$ is a left projective A-module, we conclude that $X \otimes_A P$ is a projective A-B-bimodule, which shows (1).

Assertions (2)–(4) are proved similarly; we omit the details.

1.7. Now suppose that ${}_AN_B$ and ${}_BM_A$ establish a stable equivalence of Morita type between the algebras A and B. Then we have the functors $M \otimes_A - \otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(B^{\mathrm{e}})$ and $N \otimes_B - \otimes_B M : \mathrm{lrp}(B^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{e}})$. They are not mutually inverse in general. But the next result shows that they can be mutually inverse upon passing to the stable categories $\underline{\mathrm{mod}}(A^{\mathrm{e}})$ and $\underline{\mathrm{mod}}(B^{\mathrm{e}})$.

PROPOSITION. *Let ${}_AN_B$ and ${}_BM_A$ establish a stable equivalence of Morita type between finite-dimensional K-algebras A and B. If X is a left-right projective A-bimodule which is not a projective A-bimodule then*:

(1) $N \otimes_B M \otimes_A X \otimes_A N \otimes_B M \cong X$ *in* $\underline{\mathrm{mod}}(A^{\mathrm{e}})$.
(2) $X \otimes_A N \otimes_B M \cong X$ *in* $\underline{\mathrm{mod}}(A^{\mathrm{e}})$.
(3) $N \otimes_B M \otimes_A X \cong X$ *in* $\underline{\mathrm{mod}}(A^{\mathrm{e}})$.

Proof. By the definition of stable equivalence of Morita type, $N \otimes_B M \cong A \oplus \Pi'$ as A-bimodules for some projective A-bimodule Π'. Then $N \otimes_B M \otimes_A X \otimes_A N \otimes_B M \cong (A \oplus \Pi') \otimes_A X \otimes_A (A \oplus \Pi') \cong X \oplus \Pi' \otimes_A X \oplus X \otimes_A \Pi' \oplus \Pi' \otimes_A X \otimes_A \Pi'$ in $\mathrm{mod}(A^{\mathrm{e}})$. But Lemma 1.6 shows that $\Pi' \otimes_A X \oplus X \otimes_A \Pi' \oplus \Pi' \otimes_A X \otimes_A \Pi'$ is a right projective A^{e}-module. Hence (1) follows.

A similar analysis proves (2) and (3).

1.8. Our next lemma says that in the case of self-injective algebras, dual bimodules to left-right projective bimodules are also left-right projective. Recall that an algebra A is *self-injective* if A is a right injective A-module. Some basic properties of self-injective algebras can be found in [9, 16].

LEMMA. *Let A and B be self-injective finite-dimensional K-algebras. If X is a left-right projective A-B-bimodule then* $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(X, A^{\mathrm{o}} \otimes_K B)$ *is a left-right projective B-A-bimodule.*

Proof. We have to show that $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(X, A^{\mathrm{o}} \otimes_K B)$ is a left projective B-module and a right projective A-module. To prove the latter, first observe that for every projective A-B-bimodule Q, $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(Q, A^{\mathrm{o}} \otimes_K B)$ is a projective B-A-bimodule. This follows from the well known ring isomorphism $(A^{\mathrm{o}} \otimes_K B)^{\mathrm{o}} \cong A \otimes_K B^{\mathrm{o}} \cong B^{\mathrm{o}} \otimes_K A$.

Now let X be a left-right projective A-B-bimodule which is not projective. Then there is a short exact sequence

$$0 \to L \to Q \xrightarrow{f} X \to 0$$

in $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B)$, where Q is a right projective $A^{\mathrm{o}} \otimes_K B$-module. Since X is a left projective A-module, f splits as a morphism of left A-modules.

Furthermore, we obtain the short exact sequence

$$0 \to \mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(X, A^{\mathrm{o}} \otimes_K B) \xrightarrow{f^*} \mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(Q, A^{\mathrm{o}} \otimes_K B) \\ \to \mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(L, A^{\mathrm{o}} \otimes_K B) \to 0$$

with $f^* = \mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(f, A^{\mathrm{o}} \otimes_K B)$, because $A^{\mathrm{o}} \otimes_K B$ is a self-injective K-algebra by self-injectivity of A and B. But $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(f, A^{\mathrm{o}} \otimes_K B)$ is a splittable monomorphism of right A-modules. Therefore $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(X, A^{\mathrm{o}} \otimes_K B)$ is a right projective A-module.

Similar arguments show that $\mathrm{Hom}_{A^{\mathrm{o}} \otimes_K B}(X, A^{\mathrm{o}} \otimes_K B)$ is a left projective B-module. Consequently, it is a left-right projective B-A-bimodule.

2. Stable equivalence of the subcategories of left-right projective bimodules

2.1. The aim of this section is the proof of Theorem 1. Given a finite-dimensional K-algebra A we can consider the full subcategory $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$ of $\underline{\mathrm{mod}}(A^{\mathrm{e}})$ consisting of the left-right projective A-bimodules. For two finite-dimensional K-algebras A and B we can also consider the full subcategory $\underline{\mathrm{lrp}}(A^{\mathrm{o}} \otimes_K B)$ of $\underline{\mathrm{mod}}(A^{\mathrm{o}} \otimes_K B)$ consisting of the left-right projective A-B-bimodules.

2.2. Our next proposition shows that the functor $M \otimes_A - \otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(B^{\mathrm{e}})$ from 1.7 induces an equivalence $\underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$.

PROPOSITION. *Let ${}_AN_B$, ${}_BM_A$ establish a stable equivalence of Morita type between the finite-dimensional K-algebras A and B. Then*:

(1) *There is an equivalence $F : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$.*
(2) *There is an equivalence $F_1 : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(A^{\mathrm{o}} \otimes_K B)$.*
(3) *There is an equivalence $F_2 : \underline{\mathrm{lrp}}(A^{\mathrm{o}} \otimes_K B) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$.*

Moreover, $F = F_2 \circ F_1$.

Proof. We have the functor $M \otimes_A - \otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(B^{\mathrm{e}})$ by Lemma 1.5. To prove (1) we define $F : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$ by putting $F(X) = M \otimes_A X \otimes_A N$ for every object X in $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$, and $F(\underline{f}) = \underline{1_M \otimes f \otimes 1_N}$ for every morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$. Using Lemma 1.6, an easy verification shows that a morphism $f : X \to Y$ in $\mathrm{mod}(A^{\mathrm{e}})$ between objects from $\underline{\mathrm{mod}}(A^{\mathrm{e}})$ factorizes through a right projective A^{e}-module if and only if $1_M \otimes f \otimes 1_N$ factorizes through a right projective B^{e}-module. Thus the functor F is well defined.

Now we define a functor G which is induced by $N \otimes_B - \otimes_B M : \mathrm{lrp}(B^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{e}})$ as above. Then Proposition 1.7 shows that G is a quasi-inverse of F. Therefore F is an equivalence of categories, which finishes the proof of condition (1).

For the proof of (2) we consider the functor $-\otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$. It induces the functor $F_1 : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(A^{\mathrm{o}} \otimes_K B)$ such that $F_1(X) = X \otimes_A N$ for every object X in $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$, and $F_1(\underline{f}) = \underline{f \otimes 1_N}$ for every morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{lrp}}(A^{\mathrm{e}})$. A quasi-inverse of F_1 is induced by $-\otimes_B M : \mathrm{lrp}(A^{\mathrm{o}} \otimes_K B) \to \mathrm{lrp}(A^{\mathrm{e}})$. Thus F_1 is an equivalence of categories.

For (3) we consider the functor $F_2 : \underline{\mathrm{lrp}}(A^{\mathrm{o}} \otimes_K B) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$ induced by $M \otimes_A - : \mathrm{lrp}(A^{\mathrm{o}} \otimes_K B) \to \mathrm{lrp}(B^{\mathrm{e}})$ similarly to F_1 above.

Finally, the equality $F = F_2 \circ F_1$ is clear by construction.

2.3. Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Suppose that A, B are stably equivalent of Morita type. Then there are left-right projective bimodules ${}_AN_B$, ${}_BM_A$ such that $M \otimes_A - \otimes_A N : \mathrm{lrp}(A^{\mathrm{e}}) \to \mathrm{lrp}(B^{\mathrm{e}})$ induces an equivalence $F : \underline{\mathrm{lrp}}(A^{\mathrm{e}}) \to \underline{\mathrm{lrp}}(B^{\mathrm{e}})$ by Proposition 2.2. Moreover, it is clear that $F(A) \cong B$, and a quasi-inverse to F is induced by $N \otimes_B - \otimes_B M : \mathrm{lrp}(B^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{e}})$.

Now suppose that (2) of Theorem 1 holds. As $F(A) \cong B$ there is a projective B-bimodule Π such that $M \otimes_A N \cong B \oplus \Pi$. Moreover $F^{-1}(B) \cong A$. Hence there is a projective A-bimodule Π' such that $N \otimes_B M \cong A \oplus \Pi'$ where F^{-1} is induced by $N \otimes_B - \otimes_B M : \mathrm{lrp}(B^{\mathrm{e}}) \to \mathrm{lrp}(A^{\mathrm{e}})$. Therefore the bimodules ${}_AN_B$, ${}_BM_A$ yield a stable equivalence of Morita type between A and B.

3. Left-right projective bimodules over self-injective algebras

3.1. It is an interesting problem to find how large the subcategory $\mathrm{lrp}(A^{\mathrm{e}})$ is. In this section we prove Theorem 2 and its several consequences. We also show that for a self-injective finite-dimensional K-algebra A the complement $\mathrm{mod}(A^{\mathrm{e}}) \setminus \mathrm{lrp}(A^{\mathrm{e}})$ is representation-infinite except for some trivial cases.

3.2. We recall some fundamental notions of Auslander–Reiten theory. A full exposition can be found in [2, 3, 6]. With every finite-dimensional K-algebra A we can associate its Auslander–Reiten quiver Γ_A. The vertices of Γ_A are the isomorphism classes $[X]$ of indecomposable right A-modules $X \in \mathrm{mod}(A)$. There is an arrow $[X] \to [Y]$ if and only if there is an irreducible morphism $X \to Y$. The arrow has valuation (a, b) if there is a minimal right almost split morphism $X^a \oplus Z \to Y$ where X is not a summand of Z, and a minimal left almost split morphism $X \to Y^b \oplus W$ where Y is not a summand of W. We shall not distinguish between indecomposable A-modules and the corresponding vertices of Γ_A.

It is well known that Γ_A is a translation quiver with Auslander–Reiten translation τ_A whose inverse will be denoted by τ_A^{-1}. We denote by Γ_A^{s} the stable part of Γ_A. It is obtained from Γ_A by removing the τ_A^{-1}-orbits of the projective vertices, the τ_A-orbits of the injective vertices and all arrows connected to vertices in the removed part. For a connected component $\mathcal{C}$ of Γ_A we describe its stable part $\mathcal{C}^{\mathrm{s}}$ similarly.

The translation τ_A has the following functorial interpretation. Let $D = \mathrm{Hom}_K(-, K)$ denote the usual duality. We now describe a functor $\mathrm{Tr} : \underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(A^{\mathrm{o}})$. For every $X \in \mathrm{mod}(A)$ consider its minimal projective resolution

$$\cdots \to P_1 \xrightarrow{f} P_0 \to X \to 0.$$

Applying $\mathrm{Hom}_A(-, A)$ we get the exact sequence

$$0 \to \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A) \\ \to \mathrm{coker}(\mathrm{Hom}_A(f, A)) \to 0.$$

Then we put $\mathrm{Tr}(X) = \mathrm{coker}(\mathrm{Hom}_A(f, A))$. Furthermore, for every morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{mod}}(A)$ the morphism $\mathrm{Tr}(\underline{f})$ is induced by f in the obvious way. Then $\tau_A = D\,\mathrm{Tr}$ and $\tau_A^{-1} = \mathrm{Tr}\,D$.

In what follows we freely use all properties of Auslander–Reiten sequences which can be found in [2, 3].

3.3. Our next lemma says that for self-injective algebras, the left-right projective bimodules are invariant under the Auslander–Reiten translation. This will be applied in the proof of Theorem 2.

LEMMA. *Let A, B be finite-dimensional self-injective K-algebras. If X is an indecomposable left-right projective A-B-bimodule which is not projective then $\tau_{A^{\mathrm{o}}\otimes_K B}(X)$ and $\tau^{-1}_{A^{\mathrm{o}}\otimes_K B}(X)$ are left-right projective A-B-bimodules.*

Proof. The tensor product algebra $A^{\mathrm{o}} \otimes_K B$ is self-injective. Consider an indecomposable left-right projective A-B-bimodule X which is not projective. First we show that $\mathrm{Tr}(X)$ is a left-right projective B-A-bimodule. Consider a minimal projective resolution

$$\cdots \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} X \to 0$$

in $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B)$. Then we obtain the exact sequence

$$0 \to \mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(X, A^{\mathrm{o}} \otimes_K B) \to \mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(P_0, A^{\mathrm{o}} \otimes_K B) \\ \to \mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(P_1, A^{\mathrm{o}} \otimes_K B) \to \mathrm{coker}(\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(f, A^{\mathrm{o}} \otimes_K B)) \to 0.$$

Lemma 1.8 yields that $\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(X, A^{\mathrm{o}} \otimes_K B)$ is a left-right projective B-A-bimodule. Since $\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(P_0, A^{\mathrm{o}} \otimes_K B)$ is a projective B-A-bimodule, $\ker(\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(f, A^{\mathrm{o}} \otimes_K B))$ is a left-right projective B-A-bimodule (see [5]). But $\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(P_1, A^{\mathrm{o}} \otimes_K B)$ is a projective B-A-bimodule, hence $\mathrm{coker}(\mathrm{Hom}_{A^{\mathrm{o}}\otimes_K B}(f, A^{\mathrm{o}} \otimes_K B)) = \mathrm{Tr}(X)$ is a left-right projective B-A-bimodule (see [5]).

Finally, it is obvious that the duality D sends left-right projective B-A-bimodules to left-right projective A-B-bimodules, because $A^{\mathrm{o}} \otimes_K B$ is self-injective. Thus $\tau_{A^{\mathrm{o}}\otimes_K B}(X) = D\,\mathrm{Tr}(X)$ and $\tau^{-1}_{A^{\mathrm{o}}\otimes_K B}(X) = \mathrm{Tr}\,D(X)$ are left-right projective A-B-bimodules.

3.4. Now we are able to prove Theorem 2.

Proof of Theorem 2. Suppose that there exists a vertex $X \in \mathcal{C}^{\mathrm{s}}$ which is a left-right projective A-B-bimodule. Then by the definition of $\mathcal{C}^{\mathrm{s}}$, the vertex $X \in \mathcal{C}$ is a left-right projective A-B-bimodule which is not projective. This shows that (2) implies (1).

Assume now that there exists a vertex $X \in \mathcal{C}$ which is a left-right projective A-B-bimodule and is not projective. Then by Lemma 3.3 the vertices $\tau_{A^{\mathrm{o}}\otimes_K B}(X)$, $\tau^{-1}_{A^{\mathrm{o}}\otimes_K B}(X)$ are left-right projective A-B-bimodules. Consider the Auslander–Reiten sequence

$$0 \to \tau_{A^{\mathrm{o}}\otimes_K B}(X) \to Y \to X \to 0$$

in $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B)$. Since both X and $\tau_{A^{\mathrm{o}}\otimes_K B}(X)$ are left-right projective, so is Y. Thus every indecomposable direct summand of Y is left-right projective. The same arguments show that for the Auslander–Reiten sequence

$$0 \to X \to Z \to \tau^{-1}_{A^{\mathrm{o}}\otimes_K B}(X) \to 0,$$

every indecomposable direct summand of Z is left-right projective.

Since $\mathcal{C}$ is a connected component of $\varGamma_{A^{\mathrm{o}}\otimes_K B}$, for every vertex V in $\mathcal{C}$ there is a finite walk in $\mathcal{C}$ of the form

$$V = X_n — X_{n-1} — \cdots — X_1 — X$$

where each edge — denotes either $\to$ or $\leftarrow$ and $X_n, \ldots, X_1$ are vertices in $\mathcal{C}$. We prove that V is left-right projective by induction on n. If $n = 1$ then this follows from the above considerations. Assume that for every vertex $V \in \mathcal{C}$ such that there is a walk in $\mathcal{C}$ of the form

$$V = X_m — X_{m-1} — \cdots — X_1 — X$$

with $m \le n$, V is left-right projective. Suppose that for a vertex $V \in \mathcal{C}$ there is a walk in $\mathcal{C}$ of the form

$$V = X_{n+1} — X_n — X_{n-1} — \cdots — X_1 — X.$$

We consider two cases. In the case of $X_{n+1} \to X_n$, X_n is left-right projective by the inductive assumption.

If X_n is not projective then Lemma 3.3 shows that $\tau_{A^{\mathrm{o}}\otimes_K B}(X_n)$ is left-right projective. Moreover, by Auslander–Reiten theory, X_{n+1} is a direct summand in Y, where $0 \to \tau_{A^{\mathrm{o}}\otimes_K B}(X_n) \to Y \to X_n \to 0$ is the Auslander–Reiten sequence terminating at X_n. Therefore X_{n+1} is left-right projective.

If X_n is projective then by Auslander–Reiten theory either $X_{n-1} \cong X_{n+1}$ or $X_{n+1} \to X_n \to X_{n-1}$ and $X_{n+1} \cong \tau_{A^{\mathrm{o}} \otimes_K B}(X_{n-1})$, because $A^{\mathrm{o}} \otimes_K B$ is a self-injective finite-dimensional K-algebra. If $X_{n-1} \cong X_{n+1}$ then X_{n+1} is left-right projective by the inductive assumption. If $X_{n+1} \cong \tau_{A^{\mathrm{o}} \otimes_K B}(X_{n-1})$ then X_{n+1} is left-right projective by Lemma 3.3.

In the second case we have $X_{n+1} \leftarrow X_n$. Dual arguments to those applied in the first case show that X_{n+1} is left-right projective. Consequently, we have proved inductively that every vertex in $\mathcal{C}$ is left-right projective, which shows that (1) implies (3).

The implication (3)⇒(2) is clear.

3.5. We have the following obvious consequence of Theorem 2 for finite-dimensional algebras over an algebraically closed field.

COROLLARY. *Let K be an algebraically closed field. Let A, B be self-injective nonsemisimple finite-dimensional K-algebras which are stably equivalent of Morita type. Then the category* $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ *is representation-infinite.*

Proof. We know from [11] that $A^{\mathrm{o}} \otimes_K B$ is representation-infinite and A^{e} is representation-infinite. But A is a left-right projective A-bimodule which is not projective. Then by Theorem 2 all A-bimodules in the whole connected component $\mathcal{C}$ of $\Gamma_{A^{\mathrm{e}}}$ containing an indecomposable direct summand of A are left-right projective. Thus $\operatorname{lrp}(A^{\mathrm{e}})$ is representation-infinite, because $\operatorname{mod}(A^{\mathrm{e}})$ is representation-infinite (see [11]). Furthermore, Theorem 1 yields that $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-infinite.

3.6. There is another natural problem. Suppose that we have an infinite component $\mathcal{C}$ of $\Gamma_{A^{\mathrm{o}} \otimes_K B}$, where A and B are two self-injective finite-dimensional K-algebras. Is it true that this component is the only one all of whose vertices are left-right projective A-B-bimodules? The following proposition indicates how to obtain another such component.

PROPOSITION. *Let A, B be self-injective finite-dimensional K-algebras. Let $\mathcal{C}$ be a connected component of the Auslander–Reiten quiver $\Gamma_{A^{\mathrm{o}} \otimes_K B}$ which contains a projective A-B-bimodule P. Then the following conditions are equivalent*:

(1) *Every vertex of $\mathcal{C}$ is a left-right projective A-B-bimodule.*

(2) *Every vertex of $\mathcal{C}^{\mathrm{s}}$ is a left-right projective A-B-bimodule.*

(3) *$\mathcal{C}^{\mathrm{s}}$ contains a left-right projective A-B-bimodule.*

(4) $\operatorname{top}(P)$ *is left-right projective.*

(5) *Every vertex of the connected component $\mathcal{D}$ containing* $\operatorname{top}(P)$ *is a left-right projective A-B-bimodule.*

Proof. The implications (1)⇒(2) and (2)⇒(3) are clear.

Assume that $\mathcal{C}^{\mathrm{s}}$ contains a left-right projective A-B-bimodule. Then Theorem 2 shows that every vertex of $\mathcal{C}$ is a left-right projective A-B-bimodule. But $\mathrm{rad}(P) \in \mathcal{C}$. Moreover, it is obvious that $\Omega^{-1}_{A^{\mathrm{o}}\otimes_K B}(\mathrm{rad}(P)) \cong \mathrm{top}(P)$, where $\Omega_{A^{\mathrm{o}}\otimes_K B} : \underline{\mathrm{mod}}(A^{\mathrm{o}}\otimes_K B) \to \underline{\mathrm{mod}}(A^{\mathrm{o}}\otimes_K B)$ is Heller's loop-space functor [10] which is an equivalence for any self-injective algebra. Since $\mathrm{rad}(P)$ is a left-right projective A-B-bimodule which is not projective and $A^{\mathrm{o}} \otimes_K B$ is a self-injective K-algebra, it follows that $\Omega^{-1}_{A^{\mathrm{o}}\otimes_K B}(\mathrm{rad}(P)) \cong \mathrm{top}(P)$ is a left-right projective A-B-bimodule. Thus (3) implies (4).

The implication (4)$\Rightarrow$(5) follows from Theorem 2.

Assume now that every vertex of the connected component $\mathcal{D}$ containing $\mathrm{top}(P)$ is a left-right projective A-B-bimodule. If $\mathrm{top}(P)$ is a projective A-B-bimodule then $\mathrm{top}(P) = P$ and $\mathcal{C} = \mathcal{D}$. If $\mathrm{top}(P)$ is not projective then $\mathrm{rad}(P) \cong \Omega_{A^{\mathrm{o}}\otimes_K B}(\mathrm{top}(P))$ is a left-right projective A-B-bimodule which is not projective. Thus Theorem 2 yields that every vertex of $\mathcal{C}$ is a left-right projective A-B-bimodule, which finishes our proof.

3.7. The following lemma shows when the category of left-right projective bimodules is representation-finite. We consider only connected finite-dimensional K-algebras, that is, algebras which cannot be viewed as direct products of some lower-dimensional K-algebras.

LEMMA. *Let A, B be self-injective finite-dimensional K-algebras which are connected. Then the following conditions are equivalent*:

(1) $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ *is representation-finite.*

(2) *Every A-B-bimodule in* $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ *is projective.*

Proof. Assume that $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-finite. Suppose to the contrary that there exists an indecomposable A-B-bimodule X which is left-right projective and nonprojective. If A and B are semisimple then so is $A^{\mathrm{o}} \otimes_K B$ and (2) is clear. If either A or B is not semisimple then by Theorem 2 every A-B-bimodule in the connected component $\mathcal{C}$ of $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ containing X is left-right projective. Since $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-finite, $\mathcal{C}$ must be a finite component, and so $\mathcal{C} = \Gamma_{A^{\mathrm{o}}\otimes_K B}$ by a theorem of Auslander [1]. Thus every simple right $A^{\mathrm{o}} \otimes_K B$-module is left-right projective. Therefore every simple right $A^{\mathrm{o}} \otimes_K B$-module is of the form $P_1 \otimes_K P_2$, where P_1 is a left projective A-module and P_2 is a right projective B-module. Since $P_1 \otimes_K P_2$ is simple, the modules P_1, P_2 are simple. Consequently, every left simple A-module is projective and every right simple B-module is projective. Hence A and B are semisimple K-algebras and all A-B-bimodules in $\mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ are projective. Thus (1) implies (2). The converse implication is obvious by finite-dimensionality of the algebras considered.

3.8. Finally we want to show that $\mathrm{lrp}(A^{\mathrm{o}}\otimes_K B)$ is representation-infinite if and only if its complement $\mathrm{mod}(A^{\mathrm{o}} \otimes_K B) \setminus \mathrm{lrp}(A^{\mathrm{o}} \otimes_K B)$ is.

Proposition. *Let A, B be self-injective finite-dimensional K-algebras which are stably equivalent of Morita type and connected. Then the following conditions are equivalent*:

(1) $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ *is representation-infinite.*

(2) *There exists an infinite connected component of the Auslander–Reiten quiver $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ with all vertices being left-right projective A-B-bimodules.*

(3) *There exists an infinite connected component of $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ which contains at most finitely many left-right projective A-B-bimodules*, *and each of them is projective.*

(4) $\operatorname{mod}(A^{\mathrm{o}} \otimes_K B) \setminus \operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ *is representation-infinite.*

Proof. Suppose that $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-infinite. Then there exists an indecomposable left-right projective A-B-bimodule which is not projective. Thus by Theorem 2 all vertices of the connected component $\mathcal{C}$ of $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ containing this bimodule are left-right projective A-B-bimodules. This component is infinite, because otherwise $A^{\mathrm{o}} \otimes_K B$ is representation-finite, which contradicts our assumption that $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-infinite. Consequently, we have proved the implication (1)⇒(2).

Now suppose that there exists an infinite connected component of $\Gamma_{A^{\mathrm{o}}\otimes_K B}$ with all vertices being left-right projective A-B-bimodules. Consider a connected component $\mathcal{D}$ which contains a simple A-B-bimodule S. Since $A^{\mathrm{o}} \otimes_K B$ is a connected finite-dimensional K-algebra (because A, B are), the bimodule S cannot be left-right projective. Indeed, if it were then $S \cong T \otimes_K L$, where T is a left simple A-module and L is a right simple B-module. Since S is left-right projective, T is a simple projective A-module. Since A is self-injective and connected, A is simple. The same arguments show that B is simple. Thus $\operatorname{mod}(A^{\mathrm{o}} \otimes_K B)$ is representation-finite. Hence $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-finite, contrary to our assumption.

The implication (3)⇒(4) is obvious by Proposition 3.6.

If $\operatorname{mod}(A^{\mathrm{o}} \otimes_K B) \setminus \operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-infinite then A and B cannot be semisimple. Since A, B are stably equivalent of Morita type, there is a left-right projective A-B-bimodule ${}_A N_B$ which yields this equivalence. This bimodule is an indecomposable nonprojective object in $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$. Therefore $\operatorname{lrp}(A^{\mathrm{o}} \otimes_K B)$ is representation-infinite by Theorem 2.

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