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ON THE BOFFA ALTERNATIVE

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Abstract. Let G^* denote a nonprincipal ultrapower of a group G. In 1986 M. Boffa posed a question equivalent to the following one: *if G does not satisfy a positive law, does G^* contain a free nonabelian subsemigroup?* We give the affirmative answer to this question in the large class of groups containing all residually finite and all soluble groups, in fact, all groups considered in traditional textbooks on group theory.

Preliminaries. Let $\mathcal{F}$ be a free semigroup generated by $x_1, x_2, \ldots$ We say that an n-tuple of elements $g_1, \ldots, g_n$ in a group G satisfies a *nontrivial positive relation* $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ if u, v are different words in $\mathcal{F}$ and the equality $u(g_1, \ldots, g_n) = v(g_1, \ldots, g_n)$ holds. A group G satisfies the law $u = v$ if the equality holds under each substitution of elements in G for the generators x_i. We note that every n-variable positive law implies a nontrivial two-variable positive law if we replace each variable x_i by xy^i.

NOTATION. We say that G is an *$\mathcal{F}$-group* (respectively an *$\mathcal{F}^*$-group*) if G (resp. G^*) contains a free nonabelian subsemigroup; we say that G is a *no$\mathcal{F}$-group* (resp. a *no$\mathcal{F}^*$-group*) if G (resp. G^*) does not contain a free nonabelian subsemigroup.

Since G^* is the image of a cartesian power G^ω under the congruence defined by a nonprincipal ultrafilter, the group G is embedded into G^* via $\operatorname{diag} G^\omega$. Hence every $\mathcal{F}$-group is an $\mathcal{F}^*$-group. To see that the class of $\mathcal{F}^*$-groups is wider, we note that in a no$\mathcal{F}^*$-group G every pair of elements satisfies a two-variable relation from some finite set $\{u_i = v_i : i = 1, \ldots, n\}$ (see Lemma 2 below), and hence G satisfies a nontrivial (left-normed) commutator law $[u_1 v_1^{-1}, \ldots, u_n v_n^{-1}]$. So infinite torsion groups are no$\mathcal{F}$-groups, while by ([12], IV, Thm. 2.6) they do not satisfy any law and hence are $\mathcal{F}^*$-groups.

Question (Q) in [1] suggests the following

BOFFA ALTERNATIVE. *A group either satisfies a positive law or is an $\mathcal{F}^*$-group.*

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Since every nilpotent group satisfies a positive law [7], every extension of a nilpotent group by a group of finite exponent also does, and hence it trivially satisfies the Boffa alternative. So finitely generated soluble groups ([8], Thms. 4.7, 4.12), elementary amenable groups [3], finitely generated linear groups over any field [10] satisfy the Boffa alternative, because they are all nilpotent-by-(finite expotent). Every relatively free group satisfies the Boffa alternative because every positive relation on free generators provides a positive law.

It is not known whether every group satisfies the alternative. In this paper we present a large class of groups satisfying the Boffa alternative. We need the following

Notation. Let $\mathfrak{N}_c$ denote the variety of all nilpotent groups of nilpotency class c, $\mathfrak{S}_n$ the variety of all soluble groups of solubility class n, and $\mathfrak{B}_k$ the restricted Burnside variety of exponent k, that is, the variety generated by all finite groups of exponent k. It follows from the positive solution of the Restricted Burnside Problem (see [11]) that all groups in $\mathfrak{B}_k$ are locally finite of exponent dividing k.

To recall the definition of the class $\mathcal{C}$ introduced in [2], we denote by Δ_1 the class of groups contained in finite products of varieties $\mathfrak{V}_1 \cdots \mathfrak{V}_m$, where $\mathfrak{V}_i$ is either $\mathfrak{S}_n$ or $\mathfrak{B}_k$ for various n, k. Now we define inductively: $\Delta_{n+1} = \{\text{groups locally in } \Delta_n\} \cup \{\text{groups residually in } \mathcal{C} = \Delta_n\}$.

The class $\mathcal{C}$ is defined as the union $\mathcal{C} = \bigcup_n \Delta_n$.

Among all laws satisfied in $G \in \mathfrak{N}_c\mathfrak{B}_k$ there is a two-variable positive law of the form $u_c(x,y) = u_c(y,x)$, where $u_1(x,y) = x^k y^k$, and $u_c(x,y) = u_{c-1}(x,y)u_{c-1}(y,x)$ (cf. [7]). For groups with certain properties the parameters c, k can be bounded as functions of some n. Then we denote such a law by $M_{ck}(n)$. We need the following

Lemma 1 (cf. [9], p. 52, Thms. A′, B). *Let G be a two-generator residually finite group, and let there exist n such that for every n-element subset $S \subseteq G$, the inequality $|S^n| < n^n$ holds. Then G satisfies the law $M_{ck}(n)$, where c, k depend on n only.* ■

Properties of no$\mathcal{F}^*$-groups. By ([1], Thm. 3), G is a no$\mathcal{F}^*$-group if and only if there exists a finite set $\mathfrak{A}$ of nontrivial two-variable positive relations such that every pair of elements in G satisfies at least one of them.

We denote by $\mathfrak{L}_n$ the set of all nontrivial two-variable positive relations of length n, that is,

$$\mathfrak{L}_n = \{u(x,y) = v(x,y);\ |u(x,y)| = |v(x,y)| = n\},$$

where $|u(x,y)|$ is the length of the word $u(x,y)$.

The following is clear: a relation $u = v$ implies both $uv = vu$ and $xu = xv$; a relation $uvw = w$ implies $uv = vu$. Now it can be easily shown that for any finite set $\mathfrak{A}$ of nontrivial two-variable positive relations there exists n such that if every pair of elements in G satisfies at least one relation in $\mathfrak{A}$, then every pair of elements in G satisfies at least one relation in $\mathfrak{L}_n$. So we obtain

LEMMA 2 (cf. [1], Thm. 3). *A group G is a no$\mathcal{F}^*$-group if and only if there exists n such that every pair of elements in G satisfies at least one relation in $\mathfrak{L}_n$.* ■

From the above lemma it follows that for $N \lhd H \subseteq G$, every pair of elements in H/N also satisfies at least one relation in $\mathfrak{L}_n$ and hence H/N is a no$\mathcal{F}^*$-group.

Now we need to show that if every pair of elements in G satisfies at least one relation in $\mathfrak{L}_n$, then every residually finite subgroup in G and every residually finite quotient of G satisfy the positive law $M_{ck}(n)$, where the parameters c, k depend on n only.

LEMMA 3. *Let G be a no$\mathcal{F}^*$-group and $N \lhd H \subseteq G$. If H/N is locally residually finite, then there exists n such that H/N satisfies the law $M_{ck}(n)$.*

Proof. If h_1, h_2 are elements in H/N, then by assumption, the subgroup $\mathrm{gp}(h_1, h_2)$ is residually finite. By Lemma 2, there exists n such that every pair of elements in G, and hence in H/N, satisfies a relation in $\mathfrak{L}_n$. We consider any set $S = \{s_1, \ldots, s_n\}$ in $\mathrm{gp}(h_1, h_2)$. The elements s_1, s_2 satisfy some relation $u(x, y) = v(x, y)$ in $\mathfrak{L}_n$, so the elements $u(s_1, s_2)$ and $v(s_1, s_2)$ belong to in S^n, and are equal. Thus we have $|S^n| < n^n$, and by Lemma 1, the group $\mathrm{gp}(h_1, h_2)$ satisfies the law $M_{ck}(n)$. So the whole section H/N satisfies the law $M_{ck}(n)$, as required. ■

COROLLARY 1. *If G is a no$\mathcal{F}^*$-group, then there exists n such that every soluble subgroup H in G satisfies the law $M_{ck}(n)$.*

Proof. By Lemma 2, there exists n such that every pair of elements in G (and hence in H) satisfies a relation in $\mathfrak{L}_n$. Let h_1, h_2 be in H. Then the subgroup $\mathrm{gp}(h_1, h_2)$ is a finitely generated, soluble no$\mathcal{F}$-group. Now by ([8], Thms. 4.7, 4.12), $\mathrm{gp}(h_1, h_2)$ is an extension of a nilpotent group by a finite group, and hence by [4], is residually finite. So by Lemma 3, the elements h_1, h_2 satisfy $M_{ck}(n)$. Since the same is true for any h_1, h_2 in H, we conclude that H satisfies the law $M_{ck}(n)$. ■

The main result. To show that every group in the class $\mathcal{C}$ satisfies the Boffa alternative, it is enough to prove that every no$\mathcal{F}^*$-group in the class $\mathcal{C}$ satisfies a positive law.

We start with a no$\mathcal{F}^*$-group in a product of a restricted Burnside variety and a soluble variety.

LEMMA 4. *If G is a* no$\mathcal{F}^*$*-group in $\mathfrak{B}_e\mathfrak{S}_d$ for some e, d, then G satisfies a positive law and $G \in \mathfrak{N}_c\mathfrak{B}_k$ for some c, k.*

Proof. By Lemma 2, there exists n such that every pair of elements in G satisfies a relation in $\mathfrak{L}_n$. We show that G satisfies the law $M_{ck}(n)$. If H is a finitely generated subgroup in G, then H is a no$\mathcal{F}$-group and by ([6], Corollary 3), all its derived subgroups are finitely generated. Since by assumption $H \in \mathfrak{B}_e\mathfrak{S}_d$, we get $H^{(d)} \in \mathfrak{B}_e$. So $H^{(d)}$ is a finitely generated group in $\mathfrak{B}_e$ and hence is finite. The centralizer Z of the finite subgroup $H^{(d)}$ in H must have a finite index in H ([5], 3.1.4), and hence is finitely generated ([5], 14.3.2). Moreover, Z is soluble, because $1 = [H^{(d)}, Z] \supseteq [Z^{(d)}, Z^{(d)}] = Z^{(d+1)}$.

The finitely generated soluble no$\mathcal{F}$-group Z is, by [8], nilpotent-by-finite. So H, as a finite extension of Z, is also nilpotent-by-finite. Now G is locally nilpotent-by-finite and hence by [4], locally residually finite. So by Lemma 3, G satisfies the positive law $M_{ck}(n)$. By ([2], Theorem B) it follows that G is in $\mathfrak{N}_c\mathfrak{B}_k$ for some c, k, as required. ■

THEOREM 1. *If G is in the class $\mathcal{C}$, then either G satisfies a positive law or G is an $\mathcal{F}^*$-group.*

Proof. It is enough to show that if $G \in \mathcal{C}$ is a no$\mathcal{F}^*$-group then G satisfies a positive law. By Lemma 2 we assume that there exists n such that every pair of elements in $G \in \mathcal{C}$ satisfies a relation in $\mathfrak{L}_n$. We shall prove that G satisfies the law $M_{ck}(n)$.

Let our group G be in Δ_1. Then $G \in \mathfrak{V}_1 \cdots \mathfrak{V}_m$, where each variety $\mathfrak{V}_i$ is either $\mathfrak{S}_d$ or $\mathfrak{B}_e$ for some d, e. Since the product of varieties is associative, we replace, by Lemma 4, (starting from the right) every pair of the type $\mathfrak{B}\mathfrak{S}$ by some pair of the type $\mathfrak{N}\mathfrak{B}$, and find that G belongs to a soluble-by-restricted Burnside variety $\mathfrak{S}_k\mathfrak{B}_e$ for some k, e. Then G is a locally soluble-by-finite no$\mathcal{F}$-group and by [8], is locally nilpotent-by-finite. Hence G is locally residually finite and then by Lemma 3, satisfies the law $M_{ck}(n)$.

So: *If in $G \in \Delta_1$ every pair of elements satisfies a relation in $\mathcal{L}_n$, then G satisfies the law $M_{ck}(n)$.*

We now consider a group G in Δ_2. Since $\Delta_2 = \{$groups locally in $\Delta_1\} \cup \{$groups residually in $\Delta_1\}$, there are two cases.

(i) If G is locally in Δ_1, and if K is a finitely generated subgroup in G, then $K \in \Delta_1$ and hence, as we have proved above, satisfies the law $M_{ck}(n)$. Since the same is true for every finitely generated subgroup $K \subseteq G$, we conclude that G satisfies the positive law $M_{ck}(n)$.

(ii) If G is residually in Δ_1, then G is a subcartesian product of its finite quotients $G/N \in \Delta_1$. By assumption G and hence G/N are no$\mathcal{F}^*$-groups. Hence, as we have proved, each G/N satisfies the law $M_{ck}(n)$. So G, as a subcartesian product of G/N's, satisfies the same law.

It follows that every no$\mathcal{F}^*$-group G in Δ_2 satisfies a positive law. Then by ([2], Theorem B), there exist s, t such that $G \in \mathfrak{N}_s\mathfrak{B}_t$, and hence G is in Δ_1, which ends the induction. So if G is an $\mathcal{F}^*$-group in the class $\mathcal{C}$, then G is in Δ_1 and, as we have proved, satisfies a positive law. ∎

By ([2], Theorem B) it follows that every group $G \in \mathcal{C}$ either is in $\mathfrak{N}_c\mathfrak{B}_k$ for some c, k or is an $\mathcal{F}^*$-group.

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