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UNIVERSAL CONTAINER FOR PACKING RECTANGLES

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Abstract. The aim of the paper is to find a rectangle with the least area into which each sequence of rectangles of sides not greater than 1 with total area 1 can be packed.

Introduction. Let R be a rectangle and let (R_n) be a finite or infinite sequence of rectangles. We say that (R_n) *can be packed* into R if there exist rigid motions σ_i such that the sets $\sigma_i R_i$, where $i = 1, 2, \ldots$, have pairwise disjoint interiors and are subsets of R. A packing is *translative* if all the motions are translations. By *parallel translative packing* we mean a translative packing in which each side of R_i is parallel to a side of R for $i = 1, 2, \ldots$

There are many questions concerning packing sequences of squares, rectangles or convex bodies (see for example [1], [2], [5]). By *universal container* we mean a rectangle into which each sequence of rectangles of sides no longer than 1 with total area 1 can be packed. The aim of the paper is to find a *least universal container*, i.e. a universal container with the least area (cf. [4]). Some theorems and conjectures concerning least universal containers for parallel translative packing, translative packing and for the usual packing are given.

1. Parallel translative packing. By $a \times b$ we mean a rectangle such that one of its sides, of length a, is parallel to the first coordinate axis and the other side has length b. The area of C will be denoted by $|C|$.

LEMMA. *A rectangle of side lengths* 1 *and* 2 *is a universal container for parallel translative packing.*

Proof. Let R be a rectangle of side lengths 1 and 2. Moreover let (R_n) be a sequence of rectangles of side lengths not greater than 1, whose total area is equal to 1, and let each side of R_i be parallel to a side of R for $i = 1, 2, \ldots$ We can assume that R_i is of the form $w_i \times h_i$ for $i = 1, 2, \ldots$, where $h_1 \geq h_2 \geq \ldots$ and that $R = \{(x, y); 0 \leq x \leq 2, 0 \leq y \leq 1\}$.

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The method of packing (R_n) into R is similar to the method from [3]. First we will assign to each R_i two numbers a_i and d_i, then the motion σ_i is defined by the condition

$$\sigma_i R_i = \{(x,y);\ a_i \le x \le a_i + w_i,\ d_i \le y \le d_i + h_i\}.$$

The numbers a_i, d_i are determined as follows. We begin with $d_1 = 0$ and $a_1 = 0$. Assume that $i > 1$. Put $S_j = \{(x,y);\ 0 \le x \le 2,\ y = d_j\}$ and $R'_j = \{(x,y);\ a_j \le x \le a_j + w_j,\ d_j \le y < d_j + h_j\}$ for $j = 1, \ldots, i-1$. If the intersection of $\bigcup_{j<i} R'_j$ with S_{i-1} is empty or if it is a segment of length not greater than $2 - w_i$, we put $d_i = d_{i-1}$. In the opposite case, we put $d_i = 1 - h_i$ provided $d_{i-1} = 0$, and $d_i = d_{i-1} - h_i$ if $d_{i-1} \ne 0$. If $\bigcup_{j<i} R'_j \cap S_i = \emptyset$, then $a_i = 0$. Otherwise this intersection is a segment $[0, s_i]$. In this case we put $a_i = s_i$. Let n_1 be the smallest integer such that $d_{n_1} > 0$. We stop the packing process if $a_z > 2 - w_z$ (see Fig. 1) or if $d_z < h_{n_1}$ (see Fig. 2) for a rectangle R_z with $z \ge n_1$.

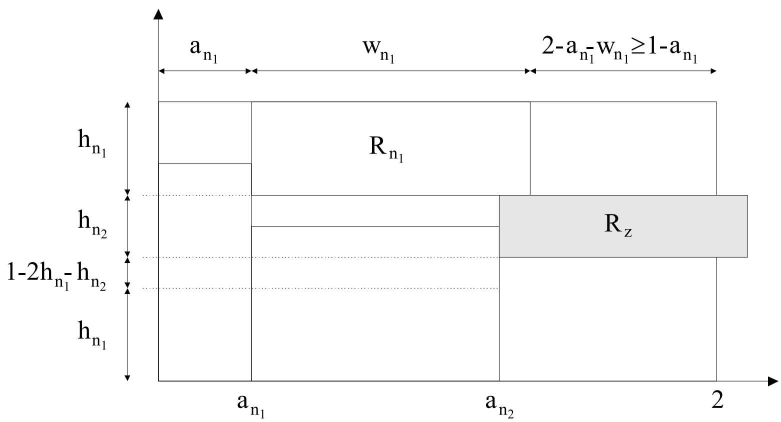

Fig. 1

We show that if (R_n) cannot be packed into R by the method described above, i.e. if there exists a rectangle R_z which terminates the packing process, then $\sum_{j=1}^{z} |R_j| > 1$, which is a contradiction.

Denote by $R_{n_1}, \ldots, R_{n_m}$, where $n_1 > \ldots > n_m$, all rectangles from among $R_2, \ldots, R_z$ with $d_{n_i} \ne d_{n_i - 1}$ for $i = 1, \ldots, m$. Obviously, $n_m = z$.

Observe that if $d_z \ge h_{n_1}$ (see Fig. 1, where $m = 2$), we have

$$\sum_{j=1}^{n_1} |R_j| > \sum_{i=1}^{m-1} a_{n_i} h_{n_{i+1}} + 2h_{n_1} + \Big(1 - 2h_{n_1} - \sum_{i=2}^{m} h_{n_i}\Big).$$

(For $m = 1$ the last sum is taken to be zero.) Moreover, if $m \ge 2$, then

$$\sum_{j=n_k+1}^{n_{k+1}} |R_j| > (1 - a_{n_k}) h_{n_{k+1}}$$

for $k = 1, \ldots, m-1$. (If $a_{n_k} > 1$, we have the obvious inequality that the area of rectangles is greater than zero.) Consequently, $\sum_{j=1}^{z} |R_j| > 1$.

Assume that $d_z < h_{n_1}$. If $m = 1$, then obviously $\sum_{j=1}^{z} |R_j| > 1$.

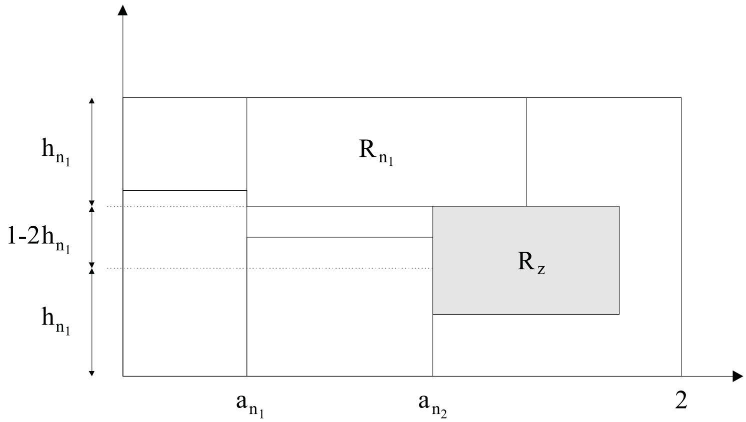

Fig. 2

For $m \geq 2$ (see Fig. 2, where $m = 2$) we have

$$\sum_{j=1}^{n_1} |R_j| > \sum_{i=1}^{m-2} a_{n_i} h_{n_{i+1}} + 2h_{n_1} + a_{n_{m-1}} \Big(1 - 2h_{n_1} - \sum_{i=2}^{m-1} h_{n_i}\Big),$$

$$\sum_{j=n_{m-1}+1}^{n_m} |R_j| > (1 - a_{n_{m-1}}) \Big(1 - 2h_{n_1} - \sum_{i=2}^{m-1} h_{n_i}\Big),$$

and

$$\sum_{j=n_k+1}^{n_{k+1}} |R_j| > (1 - a_{n_k}) h_{n_{k+1}}$$

for $k = 1, \ldots, m-2$, provided $m \geq 3$. Consequently, $\sum_{j=1}^{z} |R_j| > 1$. ∎

THEOREM 1. *The least universal container for parallel translative packing has side lengths* 1 *and* 2.

Proof. For simplicity, consider only the rectangles with sides parallel to the axes. Observe that no rectangle of type $1 \times b$ or $c \times 2$ is a universal container if $b < 2$ and $c < 1$. The reason is that a square of side 1 cannot be packed into $c \times 1$; also two rectangles $\frac{4-b}{4} \times \frac{2}{4-b}$ cannot be parallel translative packed into $1 \times b$. By the Lemma we see that 1×2 and 2×1 are universal containers. Thus, to end the proof it is sufficient to show that no rectangle $a \times \frac{2}{a}$ for $0 < a < 1$ is a universal container. Let

$$\varepsilon = \frac{-a^2 + 3a - 2}{2a}.$$

The total area of the rectangles $1 \times \left(\frac{2}{a} - 1 + \varepsilon\right)$ and $(a - 1 + \varepsilon) \times 1$ is equal to 1. It is easy to see that these rectangles cannot be parallel translative packed into $a \times \frac{2}{a}$. ■

2. Translative packing. Obviously, the sides of a universal container R for translative packing are not smaller than $\sqrt{2}$: consider one square of side 1 with diagonals parallel to sides of R.

THEOREM 2. *The area of a least universal container for translative packing is not smaller than* $2.3673\ldots$

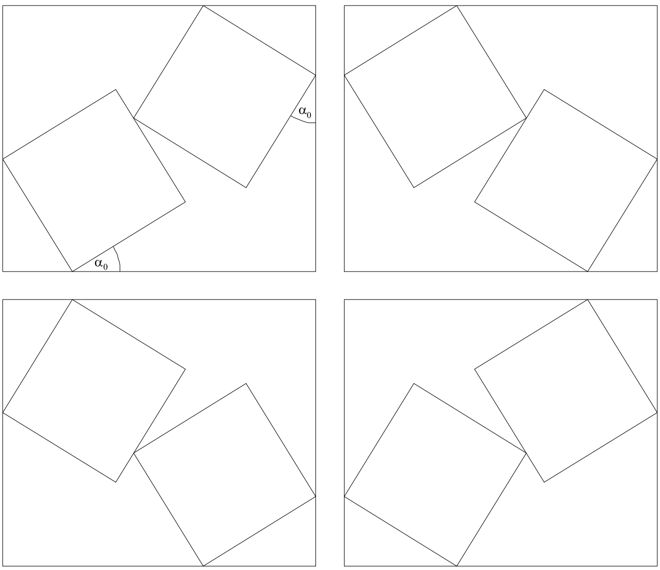

Fig. 3

Proof. Let $t_0 = 0.3699\ldots$ be the solution of the equation

(1) $$-80t^6 + 108t^4 + 8\sqrt{2}\,t^3 - 54t^2 + 5 = 0.$$

Moreover, let $p = 3t_0 + (1 - t_0^2 - \sqrt{2}t_0)(1/2 - t_0^2)^{-1/2} = 1.6739\ldots$ Consider two squares $S_1(\alpha)$ and $S_2(\alpha)$ of side length $\frac{1}{2}\sqrt{2}$ such that no side of $S_1(\alpha)$ is parallel to a side of $S_2(\alpha)$ and that the angle $\alpha \in (0, \pi/4)$ between a side of $S_1(\alpha)$ and the first coordinate axis is equal to the angle between a side of $S_2(\alpha)$ and the second coordinate axis. Let $\alpha_0 = \arcsin(t_0\sqrt{2})$. We show that the rectangle $\sqrt{2} \times p$, of area $2.3673\ldots$, is the rectangle of the least area, from among all rectangles of type $q \times s$, where $q \geq \sqrt{2}$, $s \geq \sqrt{2}$, into which $S_1(\alpha_0)$ and $S_2(\alpha_0)$ can be packed.

Let us explain the choice of p and t_0. A simple computation shows that $S_1(\alpha)$ and $S_2(\alpha)$ can be packed into a rectangle

$$\sqrt{2} \times \left(3t + \frac{1 - t^2 - \sqrt{2}t}{\sqrt{0.5 - t^2}} \right),$$

where $t = \frac{1}{2}\sqrt{2}\sin\alpha$. It is easy to show that the second side of this rectangle is maximal if $t = 0.3699\ldots$ satisfies (1).

Observe that there are four possible packings of $S_1(\alpha_0)$ and $S_2(\alpha_0)$ into $\sqrt{2} \times p$ (see Fig. 3) and that there is no rectangle $\sqrt{2} \times p_0$, where $p_0 < p$, into which $S_1(\alpha_0)$ and $S_2(\alpha_0)$ can be packed. To end the proof it remains to show that if $q \times s$, where $q, s \geq \sqrt{2}$, is a rectangle into which $S_1(\alpha_0)$ and $S_2(\alpha_0)$ can be packed, then $qs \geq p\sqrt{2}$.

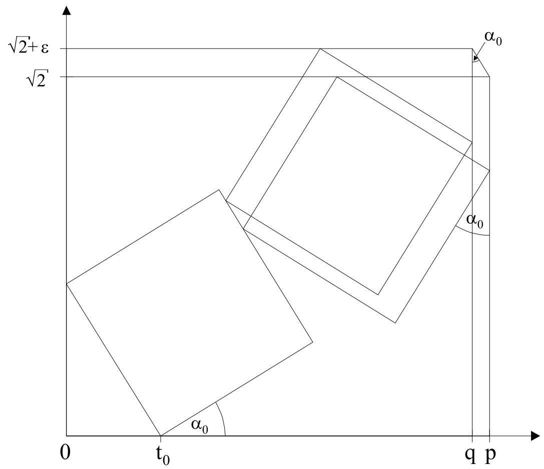

Fig. 4

It is sufficient to consider the case when $q \geq s$ (see Fig. 4). Let $s = \sqrt{2}+\varepsilon$ for $\varepsilon > 0$. We can assume that $\varepsilon \leq \sqrt{p\sqrt{2}} - \sqrt{2}$, because in the opposite case $qs \geq s^2 > p\sqrt{2}$. It is easy to see that $q \geq p - \varepsilon \tan\alpha_0$. Consequently, $qs \geq -\varepsilon^2 \tan\alpha_0 + \varepsilon\big(p - \sqrt{2}\tan\alpha_0\big) + p\sqrt{2} > p\sqrt{2}$. ■

The author conjectures that the rectangle $\sqrt{2} \times 1.6739\ldots$ is a least universal container for translative packing.

3. Usual packing. In [4] it is shown that the rectangle $\sqrt{2} \times \frac{2\sqrt{3}}{3}$ is a least universal container for packing squares. The "worst" packing of two and three equal squares is presented in Fig. 5.

The author conjectures that this rectangle is also a least universal container for packing rectangles. Unfortunately, by using the packing method similar to that from [3] we can only prove that each rectangle $a \times \frac{2}{a}$, for $1 \leq a \leq 2$, is a universal container for the usual packing of rectangles.

Remark. In this paper we find universal containers of the shape of a rectangle. Instead of rectangles we can consider compact convex bodies.

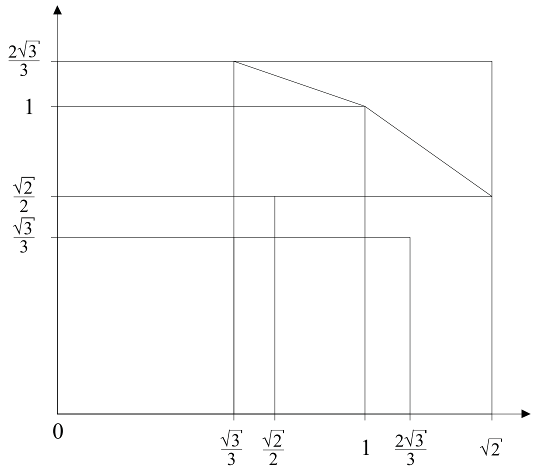

Fig. 5

Denote by s the least positive number such that there exists a compact convex set S, with area s, into which each sequence of rectangles of sides not greater than 1 and of total area 1 can be packed. It seems that s is equal to the area of the hexagon of vertices $(0,0)$, $(\sqrt{2},0)$, $(\sqrt{2},\sqrt{2}/2)$, $(1,1)$, $(\sqrt{3}/3, 2\sqrt{3}/3)$, $(0, 2\sqrt{3}/3)$ (see Fig. 5). The same question of finding a least compact convex container can be posed for translative packing.

REFERENCES

[1] H. T. Croft, K. J. Falconer and R. K. Guy, *Unsolved Problems in Geometry*, Springer, 1991.

[2] H. Groemer, *Covering and packing by sequences of convex sets*, in: Discrete Geometry and Convexity, Ann. New York Acad. Sci. 440, 1985, 262–278.

[3] J. Januszewski, *Packing rectangles into the unit square*, Geom. Dedicata 81 (2000), 13–18.

[4] D. Kleitman and M. Krieger, *An optimal bound for two dimensional bin packing*, in: 16th Annual Symp. on Foundations of Comput. Sci. (Berkeley, CA, 1975), IEEE Computer Science, Long Beach, CA, 1975, 163–168.

[5] J. W. Moon and L. Moser, *Some packing and covering theorems*, Colloq. Math. 17 (1967), 103–110.

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