COLLOQUIUM MATHEMATICUM

VOL. 92 2002 NO. 2

M-BASES IN SPACES OF CONTINUOUS FUNCTIONS ON ORDINALS

BY

ONDŘEJ F. K. KALENDA (Praha)

Abstract. We prove, among other things, that the space $C[0,\omega_2]$ has no countably norming Markushevich basis. This answers a question asked by G. Alexandrov and A. Plichko.

1. Introduction. A *Markushevich basis* (or briefly, an *M-basis*) in a Banach space X is an indexed family $(x_\alpha, f_\alpha)_{\alpha\in A}$ of pairs from $X \times X^*$ satisfying the following three conditions:

(i) $f_\alpha(x_\alpha) = 1$, $f_\alpha(x_\beta) = 0$ for $\alpha \neq \beta$;

(ii) $\overline{\operatorname{span}\{x_\alpha : \alpha \in A\}} = X$;

(iii) $\forall x \in X \setminus \{0\}\ \exists \alpha \in A : f_\alpha(x) \neq 0$.

By a classical theorem of Markushevich (see e.g. [HHZ, Theorem 272]) any separable Banach space admits an M-basis. M-bases with additional properties are also an important tool in studying the structure of non-separable Banach spaces and are (obviously) closely related to existence of certain types of embeddings of the dual. For example, a Banach space X is weakly compactly generated (resp. weakly K-analytic, weakly countably determined, weakly Lindelöf determined) if and only if X admits an M-basis $(x_\alpha, f_\alpha)_{\alpha\in A}$ such that the set $\{0\} \cup \{x_\alpha : \alpha \in A\}$ is weakly compact (resp. weakly K-analytic, weakly K-countably determined, weakly Lindelöf). Another scale of M-bases is obtained if we ask in how strong sense condition (iii) holds. Let us remark that this condition is equivalent to the assumption that

$$S_0 = \operatorname{span}\{f_\alpha : \alpha \in A\} = \{f \in X^* : \{\alpha \in A : f(x_\alpha) \neq 0\} \text{ is finite}\}$$

is weak* dense in X^*. If S_0 is even norm dense, the M-basis is called *shrinking*. It is well known [F, Theorem 8.3.3 and following remarks] that X has

2000 *Mathematics Subject Classification*: 46B15, 46B04, 46B26.

Key words and phrases: countably norming Markushevich bases, bounded projectional resolution, ordinal segment.

This work was partially supported by grants GAUK 277/2001, GAUK 160/1999 and MSM 113200007.

a shrinking M-basis if and only if it is simultaneously weakly compactly generated and Asplund. The M-basis is called *norming* (resp. 1-*norming*) if S_0 is norming (resp. 1-norming). Recall that a linear subspace $Y \subset X^*$ is called C-*norming* where $C \geq 1$ if

$$\|x\| \leq C \sup\{|f(x)| : f \in Y,\ \|f\| \leq 1\}$$

for all $x \in X$. This is equivalent, due to the Hahn–Banach separation theorem, to

$$\overline{B_{X^*} \cap Y}^{w^*} \supset \frac{1}{C} B_{X^*}.$$

Further, Y is called *norming* if it is C-norming for some $C \geq 1$.

A weakening of (1-)norming M-bases are countably (1-)norming ones, studied for example by A. Plichko [P1], [P2]. An M-basis $(x_\alpha, f_\alpha)_{\alpha \in A}$ is called *countably norming* (resp. *countably* 1-*norming*) if the subspace

$$S = \{f \in X^* : \{\alpha \in A : f(x_\alpha) \neq 0\} \text{ is countable}\}$$

is norming (resp. 1-norming). A. Plichko [P1] proved that any Banach space X with a countably norming M-basis admits a *bounded projectional resolution of the identity* (briefly *BPR*), i.e. a long sequence $(P_\alpha : 0 \leq \alpha \leq \kappa)$ of linear projections on X, where $\kappa = \operatorname{dens} X$, with the following properties:

(i) $P_0 = 0$ and $P_\kappa = \mathrm{Id}_X$;
(ii) $\sup_{\alpha < \kappa} \|P_\alpha\| < \infty$;
(iii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $0 \leq \alpha \leq \beta \leq \kappa$;
(iv) $\operatorname{dens} P_\alpha X \leq \aleph_0 \cdot \operatorname{card} \alpha$ for $0 \leq \alpha \leq \kappa$;
(v) $P_\alpha X = \overline{\bigcup_{\beta < \alpha} P_\beta X}$ if $\alpha \leq \kappa$ is limit.

There are several interesting problems on relations between various types of M-bases. Such an open problem is, for example, whether any weakly compactly generated space has a norming M-basis. There are also some known results. A recent result of G. Alexandrov and A. Plichko [AP, Theorem 1] says that any space with a countably norming M-basis has a *strong M-basis*, i.e. an M-basis $(x_\alpha, f_\alpha)_{\alpha \in A}$ such that

$$x \in \overline{\operatorname{span}\{x_\alpha : f_\alpha(x) \neq 0\}}$$

for any $x \in X$. This is done by transfinite induction, using the above mentioned result on BPR and a theorem of Terenczi [T] saying that any separable Banach space has a strong M-basis. They also show [AP, Theorem 2] that the space $C[0, \omega_1]$ has no norming M-basis while the canonical M-basis is countably 1-norming and strong. An example of a Banach space with a strong M-basis but without a countably norming one is given by A. Plichko and D. Yost [PY, Section 7]. This space even does not admit any BPR.

In the present paper we show that the space $C[0, \omega_2]$ has no countably norming M-basis. Notice that the canonical M-basis of this space is strong and that this space admits a BPR. This answers a question posed in [AP].

We will also use an alternative description of the spaces having a countably norming M-basis using the notion of a Σ-subspace. A subspace $S \subset X^*$ is called a *Σ-subspace* of X^* if there is a linear one-to-one weak* continuous mapping $T : X^* \to \mathbb{R}^\Gamma$ such that $S = T^{-1}(\Sigma(\Gamma))$, where

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

A space X is called a *Plichko* (resp. *C-Plichko*) *space* if X^* admits a norming (C-norming, respectively) Σ-subspace. By [K1, Theorem 4.16], X is a Plichko space if and only if it has a countably norming M-basis.

2. Main results. Our main result is the following theorem.

THEOREM 1. *The space $C[0, \omega_2]$ has no countably norming M-basis, i.e. it is not a Plichko space.*

This theorem answers a question posed by G. Alexandrov and A. Plichko [AP]. In fact we prove something more.

THEOREM 2. *Let $\kappa \geq \omega_2$ be a regular cardinal and η an ordinal such that $\kappa \leq \eta < \kappa \cdot \omega$. Then $C[0, \eta]$ is not a Plichko space.*

However, we do not know whether Theorem 2 can be generalized to arbitrary ordinals $\eta \geq \omega_2$. We discuss related questions in the final section of the paper.

3. Auxiliary results. Let us fix an uncountable ordinal η and put

$$X = \{x \in C[0, \eta] : x(0) = 0\}.$$

Then X is isometric to $C[0, \eta]$ and the dual X^* can be represented as the space of all finite signed Radon measures on $[0, \eta]$ vanishing at 0, equipped with the total variation norm. For $\alpha \in [0, \eta]$ and $x \in X$ put

$$(P_\alpha x)(\gamma) = \begin{cases} x(\gamma), & \gamma \leq \alpha, \\ x(\alpha), & \gamma > \alpha. \end{cases}$$

Then $P_0 = 0$ and P_α is a norm one projection for $\alpha > 0$.

The following lemma is an analogue of [AP, Proposition 2].

LEMMA 1. *Let $(\alpha_\gamma : 0 \leq \gamma \leq \omega_1 \cdot \omega)$ be a family of ordinals with the following properties*:

(i) $\alpha_0 = 0$, $\alpha_{\omega_1 \cdot \omega} \leq \eta$;
(ii) $\alpha_\gamma < \alpha_\delta$ *whenever* $0 \leq \gamma < \delta \leq \omega_1 \cdot \omega$;
(iii) $\alpha_\lambda = \sup_{\gamma < \lambda} \alpha_\gamma$ *if $\lambda \leq \omega_1 \cdot \omega$ is a limit ordinal.*

Then the subspace

$$S = \{\mu \in X^* : \{\gamma < \omega_1 \cdot \omega : \mu\restriction(P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X \neq 0\} \text{ is countable}\}$$

is not norming.

Proof. Let us first remark that

$$(*)\qquad \mu \in S,\, n \in \mathbb{N} \;\Rightarrow\; \mu[\alpha_{\omega_1 \cdot n}, \eta] = 0.$$

Indeed, suppose $\mu[\alpha_{\omega_1\cdot n}, \eta] \neq 0$. Then, as μ is supported by a countable set, there is $\beta < \alpha_{\omega_1\cdot n}$ such that $\mu\restriction(\beta, \alpha_{\omega_1\cdot n}) = 0$. By (iii) there is $\delta < \omega_1 \cdot n$ such that $\alpha_\delta > \beta$. Let $\gamma \in [\delta, \omega_1 \cdot n)$ be arbitrary. Then

$$\langle \mu, \chi_{(\alpha_\gamma,\eta]}\rangle = \mu(\alpha_\gamma, \eta] = \mu(\alpha_\gamma, \alpha_{\omega_1\cdot n}) + \mu[\alpha_{\omega_1\cdot n}, \eta] = \mu[\alpha_{\omega_1\cdot n}, \eta] \neq 0,$$

and thus $\mu\restriction(P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X \neq 0$ (since $\chi_{(\alpha_\gamma,\eta]} \in (P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X$), therefore $\mu \notin S$.

Further, for $n \in \mathbb{N}$ put

$$x_n = \sum_{k=1}^{n} \frac{k}{n}\, \chi_{(\alpha_{\omega_1\cdot k}, \alpha_{\omega_1\cdot(k+1)}]} + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n}\, \chi_{(\alpha_{\omega_1\cdot k}, \alpha_{\omega_1\cdot(k+1)}]}.$$

Then $x_n \in X$, $\|x_n\| = 1$ and for every $\mu \in S$ we have

$$\begin{aligned}
\langle \mu, x_n\rangle &= \sum_{k=1}^{n} \frac{k}{n}\, \mu(\alpha_{\omega_1\cdot k}, \alpha_{\omega_1\cdot(k+1)}] + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n}\, \mu(\alpha_{\omega_1\cdot k}, \alpha_{\omega_1\cdot(k+1)}] \\
&= \sum_{k=1}^{n} \frac{k}{n}\, (\mu(\alpha_{\omega_1\cdot k}, \eta] - \mu(\alpha_{\omega_1\cdot(k+1)}, \eta]) \\
&\quad + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n}\, (\mu(\alpha_{\omega_1\cdot k}, \eta] - \mu(\alpha_{\omega_1\cdot(k+1)}, \eta]) \\
&= \sum_{k=1}^{n} \frac{k}{n}\, (\mu(\{\alpha_{\omega_1\cdot(k+1)}\}) - \mu(\{\alpha_{\omega_1\cdot k}\})) \\
&\quad + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n}\, (\mu(\{\alpha_{\omega_1\cdot(k+1)}\}) - \mu(\{\alpha_{\omega_1\cdot k}\})) \\
&= \sum_{k=2}^{n+1} \frac{k-1}{n}\, \mu(\{\alpha_{\omega_1\cdot k}\}) - \sum_{k=1}^{n} \frac{k}{n}\, \mu(\{\alpha_{\omega_1\cdot k}\}) \\
&\quad + \sum_{k=n+2}^{2n} \frac{2n+1-k}{n}\, \mu(\{\alpha_{\omega_1\cdot k}\}) - \sum_{k=n+1}^{2n-1} \frac{2n-k}{n}\, \mu(\{\alpha_{\omega_1\cdot k}\}) \\
&= -\frac{1}{n} \sum_{k=1}^{n} \mu(\{\alpha_{\omega_1\cdot k}\}) + \frac{1}{n} \sum_{k=n+1}^{2n} \mu(\{\alpha_{\omega_1\cdot k}\}) \le \frac{1}{n}\, \|\mu\|,
\end{aligned}$$

and so S is not norming. ∎

Next we give a generalization of [K3, Lemma 2]. Let us recall that a subset A of a topological space X is *countably closed* if $\overline{C} \subset A$ whenever $C \subset A$ is countable.

LEMMA 2. *Let E be a Banach space, $S \subset E^*$ a Σ-subspace and $Z \subset E^*$ a weak* countably closed linear subspace. If $Z \cap S$ is norming, then $S \subset Z$.*

Proof. Suppose that $Z \cap S$ is norming. Then there is $c > 0$ such that $cB_{E^*} \subset \overline{Z \cap S \cap B_{E^*}}^{w^*}$. Let $\xi \in S \cap cB_{E^*}$. Since $\xi \in \overline{Z \cap S \cap B_{E^*}}^{w^*}$ there is a sequence $\xi_n \in Z \cap S \cap B_{E^*}$ such that $\xi_n \xrightarrow{w^*} \xi$ (see [K1, Lemma 1.6]). As Z is weak* countably closed we get $\xi \in Z$. Hence $S \cap cB_{E^*} \subset Z \cap cB_{E^*}$, and thus $S \subset Z$. ∎

LEMMA 3. *Let E be a Plichko Banach space and $Z \subset E^*$ be a weak* dense weak* countably closed linear subspace of finite codimension. Then there is a norming Σ-subspace S of E^* satisfying $S \subset Z$.*

Proof. As E is a Plichko space, there is a norming Σ-subspace $S_0 \subset E^*$. Let $T_0 : E^* \to \mathbb{R}^\Gamma$ be a linear one-to-one weak* continuous mapping such that $S_0 = T_0^{-1}(\Sigma(\Gamma))$.

First let us show that $S_0 \cap Z$ is relatively weak* closed in S_0. Put $Y = \overline{S_0 \cap Z}^{w^*}$. Then Y has finite codimension. Indeed, as S_0 is norming, for any $\xi \in E^*$ there is a bounded net $\xi_\alpha \in S_0$ weak* converging to ξ. Further, $S_0 \cap Z$ is a norm closed subspace of S_0 with finite codimension, so it is complemented in S_0; let Q denote a bounded projection of S_0 onto $S_0 \cap Z_0$. The net $(\mathrm{Id}_{S_0} - Q)\xi_\alpha$ is a bounded net in the finite-dimensional space $\operatorname{Ker} Q$, so passing to a subnet, we can suppose it is norm convergent. Denote the limit by θ. Then $\theta \in \operatorname{Ker} Q$. Moreover, $Q\xi_\alpha \xrightarrow{w^*} \xi - \theta$, hence $\xi - \theta \in Y$. Therefore $E^* = Y + \operatorname{Ker} Q$, and so Y is of finite codimension.

The space Y can be canonically identified with $(E/(S_0 \cap Z)_\perp)^*$, hence we can consider it as a dual space. By the previous paragraph $(S_0 \cap Z)_\perp$ has finite dimension, so $S_0 \cap Y$ is a norming Σ-subspace of Y by [K1, Theorem 4.36(iii)]. Further, $S_0 \cap Z$ is weak* dense in Y and of finite codimension in $S_0 \cap Y$, so it is norming by [K3, Lemma 1]. Now it follows by Lemma 2 that $S_0 \cap Y \subset Z$, in other words $S_0 \cap Z$ is weak* closed in S_0.

Let n denote the codimension of Y in E^*. Then there are $e_1, \dots, e_n \in E$ linearly independent such that $Y = \operatorname{span}\{e_1, \dots, e_n\}^\perp$. The vectors $e_1, \dots, e_n$ are linearly independent and weak* continuous as functionals on E^*. Further, S_0 is weak* dense, hence $e_1 \restriction S_0, \dots, e_n \restriction S_0$ are also linearly independent. Thus we can choose $\xi_1, \dots, \xi_n \in S_0$ such that $\xi_i(e_j) = 0$ for $i \neq j$ and $\xi_i(e_i) = 1$. In the same way we can choose $\theta_1, \dots, \theta_n \in Z$ such that $\theta_i(e_j) = 0$ for $i \neq j$ and $\theta_i(e_i) = 1$.

Let $T : E^* \to \mathbb{R}^{\Gamma \cup \{1,\dots,n\}}$ be defined by

$$T(\xi)(\gamma) = \begin{cases} T_0(\xi)(\gamma) - \displaystyle\sum_{j=1}^n T_0(\theta_j)\langle \xi, e_j\rangle, & \gamma \in \Gamma, \\ \langle \xi, e_\gamma\rangle, & \gamma = 1, \dots, n, \end{cases}$$

and put $S = T^{-1}(\Sigma(\Gamma \cup \{1, \ldots, n\}))$. As T is linear, weak* continuous and one-to-one, S is a Σ-subspace of E^*. Further, S contains $S_0 \cap Y = S_0 \cap Z$ and $\theta_1, \ldots, \theta_n$. Next we will show that $S \cap Z$ is weak* dense in E^*. Let $x \in (S \cap Z)_\perp$. Put $y = x - \sum_{j=1}^n \langle \xi_j, x\rangle e_j$. Then $y \in (S_0 \cap Z)_\perp$ and $\xi_j(y) = 0$ for every j. Hence $y \in (S_0)_\perp$ (as clearly $S_0 = \operatorname{span}((S_0 \cap Z) \cup \{\xi_1, \ldots, \xi_n\})$ and so $y = 0$. It follows that $0 = \langle \theta_j, y\rangle = \langle \xi_j, x\rangle$. Therefore $x = 0$.

So, $S \cap Z$ is weak* dense and has finite codimension in $(S \cap Z) + S_0$. The latter is norming and hence also $S \cap Z$ is norming by [K3, Lemma 1]. Now clearly S is norming and, by Lemma 2, $S \subset Z$. ■

LEMMA 4. *Let E be a Banach space, and $\kappa = \operatorname{dens} E$ be an uncountable regular cardinal. Suppose that $(Q^i_\alpha : 0 \le \alpha \le \kappa)$, $i = 1, 2$, are two BPR's on E. Put $S_i = \bigcup_{\alpha<\kappa} (Q^i_\alpha)^* E^*$. If $S_1 \cap S_2$ is norming, then $S_1 = S_2$.*

Proof. Let M^i_α be a dense subset of $(Q^i_{\alpha+1} - Q^i_\alpha)E$ of cardinality at most $\operatorname{card} \alpha$. Put $M^i = \bigcup_{\alpha<\kappa} M^i_\alpha$. Then M^i is linearly dense in E and it can be easily checked that

$$S_i = \{\xi \in E^* : \operatorname{card}\{m \in M^i : \langle \xi, m\rangle \ne 0\} < \kappa\},$$

so S_i is a Σ_κ-subspace of E^*, in terms of [K1, Remark 4.12]. As $S_1 \cap S_2$ is norming, there is $c > 0$ such that $cB_{E^*} \subset \overline{S_1 \cap S_2 \cap B_{E^*}}^{w^*}$. Let $\xi \in S_1 \cap cB_{E^*}$. Since $\xi \in \overline{S_1 \cap S_2 \cap B_{E^*}}^{w^*}$ there is $C \subset S_1 \cap S_2 \cap B_{E^*}$ with $\operatorname{card} C < \kappa$ and $\xi \in \overline{C}^{w^*}$ (see [K1, Lemma 1.19(ii)]). Further, $C \subset S_2$, so $\overline{C}^{w^*} \subset S_2$ (see [K1, Lemma 1.19(i)]), and so $\xi \in S_2$. Hence $S_1 \cap cB_{E^*} \subset S_2 \cap cB_{E^*}$. By interchanging the roles of S_1 and S_2 we get the inverse inclusion, hence $S_1 = S_2$. ■

The following lemma is a generalization of [P, Lemmas 1 and 2].

LEMMA 5. *Let E be a Banach space, and $\kappa = \operatorname{dens} E$ be an uncountable regular cardinal. Suppose that $(Q^i_\alpha : 0 \le \alpha \le \kappa)$, $i = 1, 2$, are two BPR's on E such that $\bigcup_{\alpha<\kappa} (Q^1_\alpha)^* E^* = \bigcup_{\alpha<\kappa} (Q^2_\alpha)^* E^*$. Then $\{\alpha < \kappa : Q^1_\alpha = Q^2_\alpha\}$ is a closed unbounded subset of $[0, \kappa)$.*

Proof. Let us first remark that

$$Q^1_\alpha = Q^2_\alpha \;\Leftrightarrow\; Q^1_\alpha E = Q^2_\alpha E \;\&\; (Q^1_\alpha)^* E^* = (Q^2_\alpha)^* E^*,$$

hence the set in question is clearly closed. Further note that for $i = 1, 2$ we have

$$(*) \qquad \forall \alpha < \kappa\ \exists \beta \in (\alpha, \kappa) : \quad Q^i_\alpha E \subset Q^{3-i}_\beta E,$$

$$(**) \qquad \forall \alpha < \kappa\ \exists \beta \in (\alpha, \kappa) : \quad (Q^i_\alpha)^* E^* \subset (Q^{3-i}_\beta)^* E^*.$$

To show $(*)$ let $D \subset Q^i_\alpha E$ be a dense subset of cardinality at most $\operatorname{card} \alpha$. For any $d \in D$ there is $\beta_d \in (\alpha, \kappa)$ such that $d \in Q^{3-i}_{\beta_d} E$ (by property (v) of a BPR). Put $\beta = \sup_{d \in D} \beta_d$. Then $\beta < \kappa$ as κ is regular. And clearly

$Q^i_\alpha E \subset Q^{3-i}_\beta E$. This proves (∗). The proof of (∗∗) is completely analogous, we only need to use the fact that the weak* density of $(Q^i_\alpha)^* E^*$ is at most $\operatorname{card} \alpha$ and the equality from the assumptions instead of property (v) of a BPR.

Finally, choose $\alpha < \kappa$ arbitrary. By an obvious induction we can construct $\beta^i_n < \kappa$, $i = 1, \ldots, 4$, $n \in \mathbb{N}$, such that $\beta^1_1 > \alpha$, $Q^1_\alpha E \subset Q^2_{\beta^1_1} E$ and

(a) $\beta^2_n > \beta^1_n$, $Q^2_{\beta^1_n} E \subset Q^1_{\beta^2_n} E$;
(b) $\beta^3_n > \beta^2_n$, $(Q^2_{\beta^2_n})^* E^* \subset (Q^1_{\beta^3_n})^* E^*$;
(c) $\beta^4_n > \beta^3_n$, $(Q^1_{\beta^3_n})^* E^* \subset (Q^2_{\beta^4_n})^* E^*$;
(d) $\beta^1_{n+1} > \beta^4_n$, $Q^1_{\beta^4_n} E \subset Q^2_{\beta^1_{n+1}} E$.

If we put $\beta = \sup\{\beta^i_n : i = 1, \ldots, 4,\ n \in \mathbb{N}\}$, then $\beta < \kappa$ and $Q^1_\beta = Q^2_\beta$. This completes the proof. ■

4. Proof of the main results. Theorem 1 is an immediate consequence of Theorem 2, so we prove the latter. Fix a regular cardinal $\kappa \geq \omega_2$. If $\eta \in [\kappa, \kappa \cdot \omega)$ then there is $n \in \mathbb{N}$ such that $\eta \in [\kappa \cdot n, \kappa \cdot (n+1))$. Then clearly $[0, \eta]$ is homeomorphic to $[0, \kappa \cdot n]$. Hence we can suppose that $\eta = \kappa \cdot n$.

Let $\widetilde{X} = \{x \in C[0, \eta] : x(0) = 0\}$ and for $\alpha \in [0, \kappa]$ and $x \in \widetilde{X}$ put

$$(\widetilde{P}_\alpha x)(\kappa \cdot k + \gamma) = \begin{cases} x(\kappa \cdot k + \gamma), & \gamma \leq \alpha, \\ x(\kappa \cdot k + \alpha), & \gamma > \alpha; \end{cases} \qquad k = 0, \ldots, n-1,\ \gamma \in (0, \kappa].$$

Further put $\widetilde{P}_0 = 0$. Then the family $(\widetilde{P}_\alpha : 0 \leq \alpha \leq \kappa)$ forms a BPR on $\widetilde{X}$ such that

$$Z = \bigcup_{\alpha < \kappa} P^*_\alpha \widetilde{X}^* = \{\mu \in \widetilde{X}^* : \mu(\{\kappa \cdot k\}) = 0,\ k = 1, \ldots, n\}.$$

Let us remark that Z is a 1-norming weak* countably closed subspace of $\widetilde{X}^*$ with finite codimension.

If $\widetilde{X}$ is a Plichko space there is, due to Lemma 3, a norming Σ-subspace S of $\widetilde{X}^*$ satisfying $S \subset Z$. By [K2, Lemma 5] there is $M \subset \widetilde{X}$ linearly dense such that

$$S = \{\mu \in \widetilde{X}^* : \{m \in M : \langle \mu, m \rangle \neq 0\} \text{ is countable}\}.$$

Further, by [K1, Lemma 4.19] there is a BPR $(Q_\alpha : \alpha \leq \kappa)$ on $\widetilde{X}$ such that $M \subset \bigcup_{\alpha < \kappa} (Q_{\alpha+1} - Q_\alpha) \widetilde{X}$. Then clearly $S \subset \bigcup_{\alpha < \kappa} Q^*_\alpha \widetilde{X}^*$, and so

$$\bigcup_{\alpha < \kappa} Q^*_\alpha \widetilde{X}^* = \bigcup_{\alpha < \kappa} \widetilde{P}^*_\alpha \widetilde{X}^*$$

by Lemma 4. It follows from Lemma 5 that there is a family of ordinals $(\alpha_\gamma : 0 \leq \gamma \leq \omega_1 \cdot \omega)$ satisfying conditions (i)–(iii) of Lemma 1 such that

$\widetilde{P}_{\alpha_\gamma} = Q_{\alpha_\gamma}$ for each γ. Put

$$S' = \{\mu \in \widetilde{X}^* : \{\gamma < \omega_1 \cdot \omega : \mu\restriction(\widetilde{P}_{\alpha_{\gamma+1}} - \widetilde{P}_{\alpha_\gamma})\widetilde{X} \neq 0\} \text{ is countable}\}.$$

Then clearly $S \subset S'$, and so S' is norming. Let $C \geq 1$ be such that S' is C-norming.

Further, put $X = \{x \in C[0,\kappa] : x(0) = 0\}$ and define P_α as at the beginning of Section 3 (with κ instead of η). Consider X canonically embedded to $\widetilde{X}$. Let $x \in X$ be arbitrary. As S' is C-norming, there is $\xi' \in S'$ such that $\|\xi'\| \leq C+1$ and $\langle \xi', x\rangle = 1$. Put $\xi = \xi'\restriction X$. Then $\|\xi\| \leq C+1$ and $\langle \xi, x\rangle = 1$. Moreover,

$$\xi \in S'' = \{\mu \in X^* : \{\gamma < \omega_1 \cdot \omega : \mu\restriction(P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X \neq 0\} \text{ is countable}\}.$$

Indeed, let $\xi\restriction(P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X \neq 0$, i.e. there is $y \in (P_{\alpha_{\gamma+1}} - P_{\alpha_\gamma})X$ such that $\langle \xi, y\rangle \neq 0$. Then also $y \in (\widetilde{P}_{\alpha_{\gamma+1}} - \widetilde{P}_{\alpha_\gamma})\widetilde{X}$. So $\xi'\restriction(\widetilde{P}_{\alpha_{\gamma+1}} - \widetilde{P}_{\alpha_\gamma})\widetilde{X} \neq 0$. Hence $\xi \in S''$ as $\xi' \in S'$. Therefore S'' is norming, which contradicts Lemma 1. ∎

5. Open problems. As remarked above, we do not know the answer to the following question.

PROBLEM 1. *Suppose that $C[0,\eta]$ is a Plichko space. Is then necessarily $\eta < \omega_2$?*

One of the reasons why we do not know the answer is that the following problem is open (cf. [K1, Question 4.45]).

PROBLEM 2. *Is any (complemented) subspace of a Plichko space again Plichko?*

A partial answer to Problem 2 is given in [K3]. Another related concrete question is the following.

PROBLEM 3. *Is $C[0,\omega_2]$ isomorphic to a subspace of a Plichko space? Or even, is there an equivalent norm on $C[0,\omega_2]$ such that the dual unit ball is a Valdivia compactum?*

REFERENCES

[AP] G. Alexandrov and A. Plichko, *On connection between strong and norming Markushevich bases in nonseparable Banach spaces*, to appear.

[F] M. Fabian, *Gâteaux Differentiability of Convex Functions and Topology: Weak Asplund Spaces*, Wiley-Interscience, New York, 1997.

[HHZ] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, lecture notes, Matfyzpress, Praha, 1996.

[K1] O. Kalenda, *Valdivia compact spaces in topology and Banach space theory*, Extracta Math. 15 (2000), 1–85.

[K2] O. Kalenda, *A characterization of weakly Lindelöf determined Banach spaces*, Serdica Math. J., to appear.

[K3] —, *Note on Markushevich bases in subspaces and quotients of Banach spaces*, Bull. Polish Acad. Sci. Math., to appear.

[P1] A. N. Plichko, *On projective resolutions of the identity operator and Markushevich bases*, Dokl. Akad. Nauk SSSR 263 (1982), 543–546 (in Russian); English transl.: Soviet Math. Dokl. 25 (1982), 386–389.

[P2] —, *Projective resolutions*, *Markushevich bases and equivalent norms*, Mat. Zametki 34 (1983), 719–726 (in Russian).

[PY] A. N. Plichko and D. Yost, *Complemented and uncomplemented subspaces of Banach spaces*, Extracta Math. 15 (2000), 335–371.

[T] P. Terenczi, *Every separable Banach space has a bounded strong norming biorthogonal system which is also a Steinitz basis*, Studia Math. 111 (1994), 207–222.

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: kalenda@karlin.mff.cuni.cz

Received 29 May 2001 (4070)