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ON DITTMAR'S APPROACH TO THE BELTRAMI EQUATION

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Abstract. We recall an old result of B. Dittmar. This result permits us to obtain an existence theorem for the Beltrami equation and some other results as a direct consequence of Moser's classical estimates for elliptic operators.

1. Introduction. Over twenty years ago Dittmar [4] found a simple proof of the existence of homeomorphic solutions of the Beltrami equation (in $\mathbb{C}$):

$$\frac{\partial\varphi}{\partial\overline{z}} = \mu\frac{\partial\varphi}{\partial z}, \qquad \mu \in C_0^3(\mathbb{C}),\ \sup_{z\in\mathbb{C}}|\mu(z)| =: \|\mu\|_\infty \le k < 1.$$

This result of Dittmar was completely forgotten. All proofs of the solvability of the Beltrami equation, published recently, were based on the classical method, invented by Bojarski ([2], [3], see also [9]). This method uses the Beurling–Ahlfors transform.

The aim of the present note is to recall Dittmar's result and extend his method to obtain a complete alternative proof of the existence of homeomorphic solutions of the Beltrami equation in the general case of $\mu \in L^\infty(\mathbb{C})$, $\|\mu\|_\infty \le k < 1$.

This proof will use only Moser's estimates ([11], [12], see also [8], Ch. 9) for *strongly* elliptic homogeneous differential equations of the second order.

Finally we shall use this approach to get new proofs of some basic facts from the theory of quasiregular functions.

2. The Beltrami equation. We start by recalling Dittmar's proof:

Let $\mu \in C_0^3(\mathbb{C})$, $\|\mu\|_\infty = \sup_{z\in\mathbb{C}}|\mu(z)| \le k < 1$ and $\mu = \mu_1 + i\mu_2$. Set $E = |1+\mu|^2$, $F = 2\mu_2$, $G = |1-\mu|^2$ and $W = 1-|\mu|^2$.

Dittmar introduces a second order differential operator

$$\Delta_\mu = \frac{\partial}{\partial y}\left(\frac{E}{W}\frac{\partial}{\partial y} - \frac{F}{W}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{G}{W}\frac{\partial}{\partial x} - \frac{F}{W}\frac{\partial}{\partial y}\right) \qquad (z = x+iy).$$

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The operator Δ_μ is strongly elliptic on $\mathbb{R}^2$, since

$$\frac{1-k}{1+k}(\xi_1^2+\xi_2^2) \le \frac{G}{W}\xi_1^2 + \frac{E}{W}\xi_2^2 - \frac{2F}{W}\xi_1\xi_2 \le \frac{1+k}{1-k}(\xi_1^2+\xi_2^2).$$

Assume now that $\operatorname{supp}\mu \subset\subset B(0,R_0)$. Let H be the real Hilbert space of functions $\varphi \in W_2^1(\mathbb{C})$ for which $\int_{\partial B(0,R_0)} \varphi\, d\sigma = 0$, equipped with the scalar product $\langle \varphi, \psi\rangle = \int_{\mathbb{C}} \nabla_\mu(\varphi,\psi)\, dV$, where

$$\nabla_\mu(\varphi,\psi) = \frac{E}{W}\frac{\partial\varphi}{\partial y}\cdot\frac{\partial\psi}{\partial y} + \frac{G}{W}\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\psi}{\partial x} - \frac{F}{W}\left(\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\psi}{\partial x}\right).$$

One can define two functions:

$$J = \frac{1}{16W}\left[2F\left(\frac{\partial E}{\partial x} - \frac{\partial G}{\partial x} - 2\frac{\partial F}{\partial y}\right) + \left(2\frac{\partial F}{\partial x} - \frac{\partial E}{\partial y} + \frac{\partial G}{\partial y}\right)(G-E-4)\right],$$

$$I = \frac{1}{16W}\left[2F\left(\frac{\partial G}{\partial y} - \frac{\partial E}{\partial y} - 2\frac{\partial F}{\partial x}\right) + \left(2\frac{\partial F}{\partial y} + \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}\right)(E-G-4)\right].$$

Dittmar observed that if u satisfies the equation

$$(*) \qquad \Delta_\mu u = -\left(\frac{\partial J}{\partial y} + \frac{\partial I}{\partial x}\right),$$

then the differential form

$$\omega = \left(\frac{F}{W}\frac{\partial u}{\partial x} - \frac{E}{W}\frac{\partial u}{\partial y} - J\right)dx + \left(\frac{G}{W}\frac{\partial u}{\partial x} - \frac{F}{W}\frac{\partial u}{\partial y} + I\right)dy$$

is closed and therefore the function $v(z) = \int_{z_0}^{z} \omega$ is well defined, and the function $\Psi(z) = u(z) + iv(z)$ satisfies the inhomogeneous Beltrami equation $\partial\Psi/\partial\overline{z} = \mu\partial\Psi/\partial z + \partial\mu/\partial z$. Hence the form $\omega_1 = e^{\Psi}d\overline{z} + \mu e^{\Psi}dz$ is d-closed. The function $\Phi(z) = \int_{z_0}^{z}\omega_1$ is well defined and satisfies the Beltrami equation $\partial\Phi/\partial\overline{z} = \mu\partial\Phi/\partial z$. Moreover Φ is a local homeomorphism of class C^2, since $\partial\Phi/\partial\overline{z} = e^{\Psi} \ne 0$.

To solve the equation $(*)$ Dittmar takes the following linear continuous functional on H:

$$L(f) = \int_C f\left(\frac{\partial J}{\partial y} + \frac{\partial I}{\partial x}\right) dV.$$

By the Riesz representation theorem there exists $u \in H$ such that $L(f) = \langle u, f\rangle$ for every $f \in H$ and thus u satisfies $(*)$. Since Δ_μ is strongly elliptic and $\mu \in C_0^3(\mathbb{C})$, we have $u \in C^2(\mathbb{C})$. One can now define v, Ψ and Φ as above. Observe that u and v are in $W_2^1(\mathbb{C})$ and that Ψ and Φ are holomorphic outside $B(0,R_0)$. Since $\Psi \in W_1^2(\mathbb{C})$, it is holomorphic also at ∞. Let $c = \lim_{z\to\infty}\Psi(z)$. Hence $\lim_{z\to\infty}\Phi'(z) = e^c \ne 0$. The mapping Φ extends to a local homeomorphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and hence it is a homeomorphism of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ by the monodromy theorem, $\Phi(\infty) = \infty$.

At this point Dittmar's original proof ends. For the general case he just gives a reference to the Lehto–Virtanen book [10]. We shall, however, proceed further and write $\Phi = f + ig$. Since Φ satisfies the Beltrami equation and is of class C^2, the functions f and g satisfy the μ-Cauchy–Riemann equations

$$\frac{\partial g}{\partial x} = \frac{F}{W}\frac{\partial f}{\partial x} - \frac{E}{W}\frac{\partial f}{\partial y}, \qquad \frac{\partial g}{\partial y} = \frac{G}{W}\frac{\partial f}{\partial x} - \frac{F}{W}\frac{\partial f}{\partial y}.$$

This implies that $\Delta_\mu f = 0$ and $\Delta_\mu g = 0$ on $\mathbb{C}$.

Forty years ago J. Moser ([11], [12]) proved the following theorem:

Let Ω be a domain in $\mathbb{R}^n$, $\Omega_1 \subset\subset \Omega$, and suppose the differential operator P has the form $P = \sum_{i,j=1}^n (\partial/\partial x_j)(a_{ij}(x)\partial/\partial x_i)$ where the $a_{ij}(x)$ are bounded measurable functions on Ω and there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\|\xi\|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le C_2\|\xi\|^2$ for each $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then there exist $1 > \alpha > 0$ and $M > 0$ depending only on C_1, C_2, and $\operatorname{dist}(\overline{\Omega}_1, \partial\Omega)$ such that if u satisfies the equation $Pu = 0$, $u \in W_2^1(\Omega)$ and $\|u\|_{L^2(\Omega)} \le 1$, then $u \in \Lambda_\alpha(\overline{\Omega}_1)$ and $\|u\|_{\Lambda_\alpha(\Omega)} \le M$ where Λ_α denotes the usual Hölder (Lipschitz) space. (For the proof of Moser's theorem see [11], [12] and also [8], Ch. 9, §5.)

The operator Δ_μ satisfies the assumptions of Moser's theorem and the solution Φ of the Beltrami equation constructed above belongs to $\Lambda_\alpha(B(0,R))$ for some $\alpha > 0$ for each $R > 0$; the α and the Hölder norm of Φ depend only on k, R and $\|\Phi\|_{L^2(B(0,2R))}$ (see also Remark 2 below).

Now we can proceed in a standard way. Let μ be a function from $L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \le k < 1$. Assume that $\operatorname{supp}\mu \subset\subset B(0,R)$. Then there exists a sequence $\mu_n \in C_0^\infty(\mathbb{C})$, $\operatorname{supp}\mu \subset\subset B(0,R)$, $\|\mu_n\|_\infty \le \|\mu\|$, such that $\mu_n \to \mu$ on $\mathbb{C}$ a.e.

Let Φ_n be a sequence of homeomorphic solutions of the Beltrami equation with μ_n, normalized so that $\Phi_n(\infty) = \infty$, $\Phi_n(0) = 0$ and $\Phi_n(1) = 1$. Note that all Φ_n are holomorphic outside $B(0,R)$.

By Moser's theorem the sequence Φ_n is equicontinuous on $\mathbb{C}$ and therefore we can take a subsequence with $\Phi_{n_k} \to \Phi$ almost uniformly on $\mathbb{C}$. It is easy to show that Φ is a homeomorphic solution of the equation $\partial\Phi/\partial\overline{z} = \mu\partial\Phi/\partial z$. (It suffices to choose an almost uniformly convergent subsequence of $\Phi_{n_k}^{-1}$.) Finally let μ be any function from $L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \le k < 1$. Put $\mu_1 = \mu\chi_{B(0,1)}$, $\mu_2 = \mu - \mu_1$ and $\mu_2^*(z) = 1/\overline{\mu_2(1/\overline{z})}$. Let Ψ^* be a homeomorphic solution of the μ_2^*-Beltrami equation. Put $\Psi = 1/\overline{\Psi^*(1/\overline{z})}$. The complex dilatation of Ψ is equal to μ_2. Take now the homeomorphism F with complex dilatation

$$(\mu_1 \circ \Psi^{-1}) \exp\left(2i \arg \frac{\partial\Psi}{\partial z} \circ \Psi^{-1}\right).$$

The mapping $\Phi = F \circ \Psi$ has the complex dilatation $\mu_1 + \mu_2 = \mu$, since $\mu_1 \cdot \mu_2 = 0$.

This ends the whole proof.

REMARK 1. The proof of Moser's theorem is not very complicated and uses only the imbedding $W_2^1(\Omega) \hookrightarrow L^p(\Omega)$ where $p = n/(n-2)$ if $n > 2$ and p is any number from $(1,\infty)$ for $n = 2$. (This is a special case of the Sobolev imbedding theorem.)

REMARK 2. In the case $P = \Delta_\mu$ the Hölder exponent α can be taken equal to (or sometimes greater than) $1/K$, where $K = (1+\|\mu\|_\infty)/(1-\|\mu\|_\alpha)$.

The simple proof of this estimate due to Buff can be found in [5]. This proof uses only the Grötsch inequalities (see [1]). However we prefer to use Moser's theorem for the following ideological reason: We want to point out that the theory of the Beltrami equation and quasiregular functions on the plane can be viewed as a part of the classical theory of strongly elliptic differential operators of second order.

3. Further results. In this section we study non-homeomorphic solutions of the Beltrami equation. In order to avoid pathologies, we shall always *assume* that the solutions of the Beltrami equation, and of the other differential equations we consider, are *always* in $W_2^1(\cdot, \mathrm{loc})$.

Let us start with the following.

PROPOSITION 1. (a) *Let U be a domain in $\mathbb{C}$ and let $\mu \in L^\infty(U)$ with $\|\mu\|_\infty \le k < 1$. If Ψ satisfies the Beltrami equation $\partial\Psi/\partial\overline{z} = \mu\partial\Psi/\partial z$, $\Psi \in W_2^1(U, \mathrm{loc})$, then $\Delta_\mu u = \Delta_\mu v = 0$ for $u = \operatorname{Re}\Psi$ and $v = \operatorname{Im}\Psi$.*

(b) *If U is a simply connected domain, μ is as above, $u \in W_2^1(U, \mathrm{loc})$ and $\Delta_\mu u = 0$, then there exists $\Psi \in W_2^1(U, \mathrm{loc})$ such that $\partial\Psi/\partial\overline{z} = \mu\partial\Psi/\partial z$ and $u = \operatorname{Re}\Psi$.*

Proof. (a) Let $U_1 \subset\subset U_2 \subset\subset U$. There exists a sequence $\mu_n \in C_0^\infty(\mathbb{C})$, $\|\mu_n\|_\infty \le \|\mu\|$, $\operatorname{supp}\mu_n \subset U_2$ such that $\mu_n(z) \to \mu(z)$ on U_1 a.e.

Let $\Phi, \Phi_1, \Phi_2, \ldots, \Phi_n, \ldots$ denote the normalized homeomorphic solutions of the Beltrami equation for $\mu, \mu_1, \mu_2, \ldots, \mu_n, \ldots$ respectively. We can assume that $\Phi_n \to \Phi$ almost umiformly on $\mathbb{C}$. (We put $\mu = 0$ outside U.) The function $h = \Psi \circ \Phi^{-1}$ is holomorphic on $\Phi(U)$. The sequence of functions $\Psi_n = h \circ \Phi_n$ tends to Ψ uniformly on U_1. Thus $u_n = \operatorname{Re}\Psi_n \to u$ and $v_n = \operatorname{Im}\Psi_n \to v$. Since u_n and v_n are smooth, $\Delta_{\mu_n} u_n = \Delta_{\mu_n} v_n = 0$. We have $\Delta_{\mu_n} u_n \to \Delta_\mu u$ and $\Delta_{\mu_n} v_n \to \Delta_\mu v$ in the distribution space $D'(U_1)$. Hence $\Delta_\mu u = \Delta_\mu v = 0$.

(b) Let $u \in W_2^1(U, \mathrm{loc})$ with $\Delta_\mu u = 0$. The differential form $\omega = \omega_1 dx + \omega_2 dy$ with $L^2(U, \mathrm{loc})$ coefficients

$$\omega_1 = \frac{F}{W}\frac{\partial u}{\partial x} - \frac{E}{W}\frac{\partial u}{\partial y}, \qquad \omega_2 = \frac{G}{W}\frac{\partial u}{\partial x} - \frac{F}{W}\frac{\partial u}{\partial y}$$

is d-closed. Let $U_1 \subset\subset U$ be a simply connected domain with smooth boundary. The distributions $\partial\omega_1/\partial x, \partial\omega_1/\partial y, \partial\omega_2/\partial x, \partial\omega_2/\partial y$ belong to $W_2^{-1}(U_2)$. The Laplace operator Δ is an isomorphism between $\mathring{W}_2^1(U_2)$ and $W_2^{-1}(U_2)$.

There exists $f \in \mathring{W}_2^1(U_2)$ for which $\Delta f = \partial\omega_1/\partial x + \partial\omega_2/\partial y$. The form

$$\omega' = \left(\omega_1 - \frac{\partial f}{\partial x}\right)dx + \left(\omega_2 - \frac{\partial f}{\partial y}\right)dy$$

is a d-closed form with harmonic coefficients. Since U_1 is simply connected, there exists $v_0 \in C^\infty(U_1)$ with $dv_0 = \omega'$.

Put $v = v_0 + f$. Then $\Psi = u + iv$ satisfies the Beltrami equation on U_2. Hence u is locally the real part of a solution of the Beltrami equation. Since U is simply connected, u is also globally the real part of such a solution. In what follows, we call functions u for which $\Delta_\mu u = 0$ *μ-harmonic functions*; solutions of the Beltrami equation which belong to $W_2^1(\cdot, \mathrm{loc})$ *μ-quasiregular functions*; and homeomorphic solutions to the Beltrami equation which are equal to ∞ at ∞, 0 at 0, and 1 at 1 *normalized μ-quasiconformal maps*.

Our proof of the existence of homeomorphic solutions together with Remark 2 and Proposition 1 yields immediately the following.

THEOREM 1. *If Ψ is a μ-quasiregular function on an open set U in $\mathbb{C}$, then Ψ belongs to the Hölder space $\Lambda_\alpha(U, \mathrm{loc})$ where $\alpha \geq 1/K$ with $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. If u is a μ-harmonic function on U, then u belongs to $\Lambda_\alpha(U, \mathrm{loc}), \alpha \geq 1/K$.*

Let us now consider the Dirichlet problem for μ-harmonic functions.

THEOREM 2. *Let D be a domain in $\mathbb{C}$ bounded by a finite number of Jordan curves and let φ be a continuous function on ∂D. For each $\mu \in L^\infty(D)$ with $\|\mu\|_\infty \leq k < 1$, there exists $u \in C(\overline{D})$ such that $\Delta_\mu u = 0$ on D and $u = \varphi$ on ∂D.*

Proof. By putting $\mu = 0$ on $\mathbb{C} \setminus D$ we can assume that $\mu \in L^\infty(\mathbb{C})$. Let Φ_μ be a normalized μ-quasiconformal map. By the Koebe theorem [6] there exists a conformal map h from $\Phi_\mu(D)$ onto a domain D_1 whose boundary consists of a finite number of circles. Put $\Phi = h \circ \Phi_\mu$. By the Carathéodory theorem Φ extends to a homeomorphism between $\overline{D}$ and $\overline{D}_1$. Let $\varphi_1 = \varphi \circ \Phi^{-1}$ on ∂D_1. We now solve the usual Dirichlet problem and obtain u_1 on $\overline{D}_1$ with $\Delta u_1 = 0$ on D_1 and $u_1 = \varphi_1$ on ∂D_1.

It is obvious that $u_1 \circ \Phi$ is continuous on $\overline{D}$ and $u = u_1 \circ \varphi = \varphi$ on ∂D. The harmonic function u_1 is locally the real part of a holomorphic function. Let $V \subset \Phi(D)$, $u_1 = \operatorname{Re} g$ on V. Thus $u = \operatorname{Re} g \circ \Phi^{-1}$. Since $g \circ \Phi$ is μ-quasiregular, u is μ-harmonic on $\Phi^{-1}(V)$. Such sets cover all of D and u is μ-harmonic on the whole D.

REMARK 3. In Theorem 2 it suffices to assume that D is a bounded domain whose boundary consists of a finite number of locally connected components with non-zero capacities.

THEOREM 3. *Let U be an open subset of $\mathbb{C}$. Suppose that $\mu \in L^\infty(U) \cap \Lambda_{m+\alpha}(U, \mathrm{loc})$ with $\|\mu\|_\infty \le k < 1$, $m \ge 1$. Then every μ-harmonic function u and every μ-quasiregular function Ψ belong to $\Lambda_{1+m+\alpha}(U, \mathrm{loc})$.*

Proof. Let $B(a, R) \subset\subset U$. Let $\varphi = \varphi_1 + i\varphi_2$ be a C^∞ diffeomorphism of the circle $C(a, R)$ onto itself. Let $G = u_1 + iv_1$ be a solution of the Dirichlet problem $\Delta_\mu u_1 = \Delta_\mu v_1 = 0$ on $B(a, R), u_1 = \varphi_1$ and $v_1 = \varphi_2$ on $\partial B(a, R) = C(a, R)$. We have $G \in \Lambda_{1+m+\alpha}(\overline{B(a, R)})$ by Schauder's estimates ([8], Ch. III, §2).

Put $\mu = 0$ on $\mathbb{C} \setminus U$ and let Φ_μ be a normalized μ-quasiconformal map. Denote by h the Riemann map from $\Phi_\mu(B(a, R))$ onto $B(0, 1)$ and put $\Phi = h \circ \Phi_\mu$. As above Φ extends to a homeomorphism from $\overline{B(a, R)}$ onto $\overline{B(0, 1)}$. The map $\varphi \circ \Phi^{-1}$ maps homeomorphically $C(0, 1)$ onto $C(a, R)$. By the Rado theorem (see e.g. [13]) the harmonic extension F of $\varphi \circ \Phi^{-1}$ is a diffeomorphism of $B(0, 1)$ onto $B(a, R)$. We have $G = F \circ \Phi$. Since $G \in \Lambda_{1+m+\alpha}(\overline{B(a, R)})$ the mappings Φ and Φ_μ belong to $\Lambda_{1+m+\alpha}(B(a, R), \mathrm{loc})$. If u is μ-harmonic on $B(a, R)$, then $u = w \circ \Phi_\mu$ where w is harmonic on $\Phi(B(a, R))$, hence $u \in \Lambda_{1+m+\alpha}(B(a, R), \mathrm{loc})$. Since (a, R) can be chosen arbitrarily, we have $u \in \Lambda_{1+m+\alpha}(U, \mathrm{loc})$. Then Proposition 1 shows that each μ-quasiregular map Ψ also belongs to $\Lambda_{1+m+\alpha}(U, \mathrm{loc})$.

Finally let us return to the inhomogeneous equation $\Delta_\mu u = -(\partial J/\partial y + \partial I/\partial x)$ considered in Section 2.

If we consider the Dirichlet problem for this equation, we get the following

THEOREM 4. *Let D be a simply connected domain bounded by a Jordan curve of class $\Lambda_{2+\alpha}$. Suppose that $\mu \in \Lambda_{2+\alpha}(\overline{D})$ and $\|\mu\|_\infty \le k < 1$. Let φ be a function of class $\Lambda_{2+\alpha}$ on ∂D. Then there exists a μ-quasiregular function $\Psi \in \Lambda_{2+\alpha}(\overline{D})$ which is a local homeomorphism such that $|\partial\Psi/\partial z| = e^\varphi$ on ∂D.*

Proof. Repeat the first part of Dittmar's proof using Schauder estimates [8].

REMARK 4. The simple example of $f_n(z) = z^n/n$ on $B(0, 1)$ shows that the condition $|\partial f/\partial z| = e^\varphi$ on ∂D does not determine a μ-quasiregular function uniquely.

However if we assume additionally that f is a local homeomorphism on D, then the correspondence $f \leftrightarrow |\partial f/\partial z|\,|_{\partial D}$ is one-to-one.

REMARK 5. Let U be an open set in $\mathbb{C}$ and $\mu \in L^\infty(U)$, $\|\mu\|_\infty \le k < 1$. Assume that U_1 is an open subset of U such that $\mu \in C^\infty(U_1)$. Then each μ-harmonic or μ-quasiregular function is of class C^∞ on U_1. If μ is real-analytic on U, then each μ-harmonic or μ-quasiregular function must be real-analytic on U_1. This is a direct consequence of the ellipticity of the operators Δ_μ and $\partial/\partial\overline{z} - \mu\partial/\partial z$ and Theorems 8.6.1 and 8.3.1 from Hörmander's book [7].

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