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KOSZUL AND QUASI-KOSZUL ALGEBRAS OBTAINED BY TILTING

BY

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Abstract. Given a finite-dimensional algebra, we present sufficient conditions on the projective presentation of the algebra modulo its radical for a tilted algebra to be a Koszul algebra and for the endomorphism ring of a tilting module to be a quasi-Koszul algebra. One condition we impose is that the algebra has global dimension no greater than 2. One of the main techniques is studying maps between the direct summands of the tilting module. Some applications are given. We also show that a Brenner–Butler tilted algebra is simply connected if and only if the original algebra is simply connected.

Koszul algebras and tilting theory have played important role in several areas of mathematics, in particular, in the representation theory of algebras. These theories were shown to be part of a larger theory in [10]. Here we consider the question of when a tilted algebra is Koszul or quasi-Koszul.

Let Λ be a finite-dimensional algebra over an algebraically closed field k. We denote the radical of the algebra Λ by $\mathbf{r}_\Lambda$ or simply by $\mathbf{r}$. It is well known that Λ is Morita equivalent to a quotient of a finite quiver algebra by an ideal I which is contained in the square of the ideal generated by the arrows (see [5, III.1.9]). We assume throughout this work that the algebra Λ is of the form kQ/I where kQ is a quiver algebra and I is a *graded* ideal I in the sense that I is homogeneous with respect to the length grading on paths.

We briefly describe the paper. Section 1 contains well known facts about both Koszul and tilting theories. In Section 2 we study the case where Γ is a tilting of kQ/I and in Section 3 we assume $I = 0$. Assuming Γ is of global dimension 2, we characterize when Γ is a Koszul algebra in terms of the minimal projective resolution of $\Gamma/\mathbf{r}$ (see Theorem 2.7). We then obtain some applications of that result. We introduce the concept of T-sink maps, for T a tilting module, which is a concept related to irreducible maps. Our

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main theorem in Section 2 shows that a T-sink map determines a linear resolution for $\Gamma/\mathbf{r}$. As an application, we consider the Brenner–Butler tilted algebras, or BB-tilted algebras for short. Recall that a BB-tilted algebra is the endomorphism ring of a Brenner–Butler tilting module over a hereditary algebra. The full definition is given in Section 3.1. We show in Theorem 3.1 that BB-tilted algebras are Koszul algebras. We prove that BB-tilted algebras of a hereditary algebra are simply connected if and only if the hereditary algebra is also simply connected (Proposition 3.4). In Section 3.2 we present a class of finite representation type algebras which are also Koszul algebras. This class includes the class of iterated tilted algebras of type A_n (see [2]).

In Section 4, we present a new characterization of quasi-Koszul modules over non-graded algebras using the notion of an essentially linear resolution. We introduce the definition of essentially T-irreducible maps which extends the concept of T-sink maps. These maps occur in the description of quasi-Koszul modules over endomorphism rings of tilting modules and appear in the same way as the T-sink maps appear in Section 2. Proposition 4.8 uses essential linearity of the projective resolution of a module over the endomorphism ring of a tilting module to give a criterion for a module to be quasi-Koszul. Finally, starting with a quadratic monomial algebra, we describe a class of tilting modules whose endomorphism rings are always quasi-Koszul algebras (see Theorem 4.16).

1. Preliminaries. In this section we recall some definitions and basic facts of the theory of Koszul algebras and the theory of tilting algebras. The reader can find more details in [9] and [1] respectively. We also fix the notation and terminology which we will use throughout this work.

1.1. *Tilted algebras.* Let Λ be an algebra. A Λ-module T is called a *tilting module* when the following conditions are satisfied:

(i) $\mathrm{pd}_\Lambda T \leq 1$.
(ii) $\mathrm{Ext}^1_\Lambda(T,T) = 0$.
(iii) There is a short exact sequence $0 \to \Lambda \to T' \to T'' \to 0$ with $T', T'' \in \mathrm{add}(T)$.

The opposite endomorphism ring $\Gamma = \mathrm{End}^{\mathrm{op}}_\Lambda(T)$ of a tilting module T is called an *algebra tilted from* Λ. If Λ is hereditary, we just say that Γ is a *tilted algebra*. The Γ-module homomorphism $\mathrm{Hom}_\Lambda(T,f) : \mathrm{Hom}_\Lambda(T,M) \to \mathrm{Hom}_\Lambda(T,N)$ induced from a Λ-module homomorphism $f : M \to N$ will be denoted by f_*. If T_l is an indecomposable direct summand of T, we denote the indecomposable projective Γ-module $\mathrm{Hom}_\Lambda(T,T_l)$ by P_l. We denote the category of finitely generated left Λ-modules by Λ-mod. For any two Λ-modules A and B, we denote the group $\mathrm{Hom}_\Lambda(A,B)$ by (A,B). By the *top* of a Λ-module M, we mean $M/\mathbf{r}M$.

There is a close connection between the representation theory of the algebras Λ and Γ. Given a tilting Λ-module T we consider the following two full subcategories of the category of Λ-modules: the category $\mathcal{T}(T)$ of all modules generated by T and the category $\mathcal{F}(T)$ of modules M satisfying $\mathrm{Hom}_\Lambda(T, M) = 0$. The pair $(\mathcal{T}(T), \mathcal{F}(T))$ defines a torsion theory for Λ-mod. There are two corresponding full subcategories of Γ-mod, namely $\mathcal{X}(T)$ of all modules N such that $T \otimes_\Gamma N = 0$ and $\mathcal{Y}(T)$ of all modules N such that $\mathrm{Tor}_1^\Gamma(T, N) = 0$. We have the following.

Theorem of Brenner–Butler [13]. *Let T be a tilting Λ-module with $\mathrm{End}_\Lambda^{\mathrm{op}}(T) = \Gamma$. Then T is also a tilting Γ-module, and $\Lambda = \mathrm{End}_\Gamma^{\mathrm{op}}(T)$, canonically. Moreover, we have equivalences of the categories $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ under the restrictions of the functors $\mathrm{Hom}_\Lambda(T, -)$ and $- \otimes_\Gamma T$, and of $\mathcal{F}(T)$ and $\mathcal{X}(T)$ under the restrictions of the functors $\mathrm{Ext}_\Lambda^1(T, -)$ and $\mathrm{Tor}_1^\Gamma(T, -)$.* ∎

It is known that if Λ is hereditary then the torsion theory defined by T in the category Γ-mod splits; that is, any indecomposable module $M \in \Gamma$-mod is either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$.

1.2. *Linear resolutions and Koszul algebras.* A graded algebra Γ is called a *Koszul algebra* when the Yoneda algebra $E(\Gamma) = \coprod_{n \geq 0} \mathrm{Ext}_\Gamma^n(\Gamma/\mathbf{r}, \Gamma/\mathbf{r})$ is 1-*generated*; that is, the elements in $\mathrm{Ext}_\Gamma^1(\Gamma/r, \Gamma/r)$ generate all higher extension groups under the Yoneda product. In [9], Green and Martinez give the following necessary and sufficient condition for an algebra to be a Koszul algebra. A graded algebra Γ is a Koszul algebra if and only if $\Gamma/\mathbf{r}$ has a linear resolution, that is, there exists a graded projective resolution

$$\ldots \to P_{(n)} \to P_{(n-1)} \to \ldots \to P_{(2)} \to P_{(1)} \to P_{(0)} \to \Gamma/\mathbf{r} \to 0$$

such that $P_{(j)}$ is generated in degree j, for all $j \geq 0$.

As examples of Koszul algebras we have hereditary algebras, quadratic algebras with global dimension 2 (see [9], for instance), quadratic monomial algebras ([11]) and, as we shall show, the Brenner–Butler tilted algebras (see 4.1).

We recall that tilted algebras have global dimension two. It follows from [9] that tilted algebras are Koszul if and only if they are quadratic algebras.

2. Tilted algebras.

In this section we consider a finite-dimensional algebra Λ over the algebraically closed field k. We fix a tilting Λ-module T and a decomposition $T = \bigoplus_{j=1}^n T_j$ into indecomposable direct summands. Furthermore, we assume that the $\{T_j\}$ is multiplicity free. The next result is restricted to the tilted algebras and does not hold in general.

Proposition 2.1. *Let Λ be a hereditary algebra, T a tilting Λ-module and Γ the tilted algebra $\mathrm{End}_\Lambda^{\mathrm{op}}(T)$. Let $1 \leq j \leq n$ and $f : T' \to T_j$ be a*

Λ-module homomorphism between modules in $\operatorname{add}(T)$, *such that f induces a minimal projective presentation* $\operatorname{Hom}_\Lambda(T,T') \xrightarrow{f_*} \operatorname{Hom}_\Lambda(T,T_j) \to S_j \to 0$ *of the simple Γ-module S_j. Then f is either a monomorphism or an epimorphism. Moreover*, $S_j \in \mathcal{Y}(T)$ *if and only if f is a monomorphism. Otherwise*, $S_j \cong \operatorname{Ext}^1_\Lambda(T,\ker f)$.

Proof. Let $P_j = \operatorname{Hom}_\Lambda(T,T_j)$. Suppose that $f : T' \to T_j$ induces a minimal projective presentation of the simple Γ-module S_j. Note that $f = i \circ l$ where $l : T' \to \operatorname{Im} f$ and $i : \operatorname{Im} f \to T_j$. Now $\operatorname{Ext}^1_\Lambda(T,\operatorname{Im} f) = 0$, since $\operatorname{Im} f \in \mathcal{T}(T)$. We also have the following exact sequences in Γ-mod:

$$0 \to (T,\ker f) \to (T,T') \xrightarrow{l_*} (T,\operatorname{Im} f) \to \operatorname{Ext}^1_\Lambda(T,\ker f) \to 0,$$
$$0 \to (T,\operatorname{Im} f) \xrightarrow{i_*} P_j \to (T,\operatorname{coker} f) \to 0.$$

Suppose first that f is not an epimorphism. Then $\operatorname{coker} f \neq 0$ and so also $(T,\operatorname{coker} f) \neq 0$. Since $f_* = (il)_*$ is a projective presentation of S_j, it follows that $(T,\operatorname{coker} f) \cong S_j$. Now $\operatorname{coker} f \in \mathcal{T}(T)$ and so $S_j \in \mathcal{Y}(T)$ and hence, since Λ is hereditary, $\operatorname{pd}_\Gamma S_j = 1$. Thus $(T,\operatorname{Im} f)$ is projective and it follows from the minimality of f_* that l_* is an isomorphism. Since T' and $\operatorname{Im} f$ are both in $\mathcal{T}(T)$, it follows from the Brenner–Butler Theorem that they are isomorphic and hence $\ker f = 0$.

If f is an epimorphism, then i_* is an isomorphism and so $f_* = l_*$ and it follows that $S_j \cong \operatorname{Ext}^1_\Lambda(T,\ker f)$. We have seen that $S_j \in \mathcal{Y}(T)$ implies $\ker f = 0$ and so, in this case $S_j \in \mathcal{X}(T)$. ∎

Let $M, N \in \mathcal{T}(T)$ with M indecomposable. We say that the non-zero Λ-morphism $\alpha : N \to M$ is a *sink-torsion map* if it is a sink map in the category $\mathcal{T}(T)$. In other words, α is a minimal non-split homomorphism and every non-zero non-split homomorphism $\beta : L \to M$ with $L \in \mathcal{T}(T)$ factors through α.

We observe that if $f : E \to M$ is a right minimal almost split map and $M \in \mathcal{T}(T)$, then the restriction of f to $\operatorname{tr}_T(E)$ given by $f' : \operatorname{tr}_T(E) \to M$ is a sink-torsion map, where $\operatorname{tr}_T(E)$ is the trace of T in E, since the class $\mathcal{T}(T)$ of modules is closed under homomorphic images. The next result shows that the radical of an algebra Γ which is tilted from Λ may be defined by the Γ-module $\operatorname{Hom}_\Lambda(T,\operatorname{tr}_T(E))$ where $E = \bigoplus_{j=1}^n E_j$ for each $f_j : E_j \to T_j$ a right minimal almost split map.

PROPOSITION 2.2. *Let Λ be a finite-dimensional k-algebra and T a tilting Λ-module. Let T_l be an indecomposable direct summand of T such that $P_l = \operatorname{Hom}_\Lambda(T,T_l)$ is not a simple Γ-module. Then there exists exactly one sink-torsion map $\alpha : E \to T_l$, up to isomorphism.*

Proof. We know that $P_l \in \mathcal{Y}(T)$. It follows that $\mathbf{r}P_l \in \mathcal{Y}(T)$. Therefore there exists a Λ-module E such that $\mathbf{r}P_l \cong (T,E)$ and a map $\alpha : E \to T_l$

inducing the natural inclusion $\mathbf{r}P_l \hookrightarrow P_l$. We may assume that α is minimal and we claim that α is a sink-torsion map. Given a non-zero non-split homomorphism $\beta : N \to T_l$ with $N \in \mathcal{T}(T)$, we find that β_* is a non-split homomorphism. Therefore, β_* factors through α_* and so also β through α. The uniqueness is immediate. We observe that if a subcategory of an abelian category has sink maps then they are unique, up to isomorphism. ∎

Corollary 2.3. *A sink-torsion map $\alpha : E \to T_l$ as defined in Proposition* 2.2 *is either a monomorphism or an epimorphism. Furthermore let S_l be the top of P_l. If α is a monomorphism then $S_l \cong \mathrm{Hom}_\Lambda(T, \operatorname{coker}\alpha)$. Otherwise*, $S_l \cong \mathrm{Ext}^1_\Lambda(T, \ker\alpha)$.

Proof. Assume that a sink-torsion map α is not an epimorphism. Hence, $\operatorname{Im}\alpha$ is properly contained in T_l. By hypothesis, the inclusion $j : \operatorname{Im}\alpha \to T_l$ must factor through α, and it follows that $E \cong \ker\alpha \oplus \operatorname{Im}\alpha$. Since α is minimal, we see that $\ker\alpha = 0$.

Now if α is a monomorphism then the short exact sequence of Λ-modules $0 \to E \xrightarrow{\alpha} T_l \to \operatorname{coker}\alpha \to 0$ induces the short exact sequence of Γ-modules $0 \to (T, E) \xrightarrow{\alpha_*} P_l \to (T, \operatorname{coker}\alpha) \to 0$. Since we know that $\mathbf{r}P_l \cong (T, E)$, it follows that $S_l \cong \mathrm{Hom}_\Lambda(T, \operatorname{coker}\alpha)$.

If α is an epimorphism, the short exact sequence of Λ-modules $0 \to \ker\alpha \to E \xrightarrow{\alpha} T_l \to 0$ induces the exact sequence of Γ-modules

$$0 \to (T, \ker\alpha) \to (T, E) \xrightarrow{\alpha_*} P_l \to \mathrm{Ext}^1_\Lambda(T, \ker\alpha) \to 0,$$

where $(T, E) \cong \mathbf{r}P_l$. But α_* is a monomorphism and hence $(T, \ker\alpha) = 0$. It follows that $\ker\alpha \in \mathcal{F}(T)$ and $S_l \cong \mathrm{Ext}^1_\Lambda(T, \ker\alpha)$. ∎

The next definition introduces modules in Λ which play an important role in the study of the radical of Γ.

Definition 2.4. The *torsion-predecessor* of an indecomposable direct summand T_l of T is a Λ-module $E_l \in \mathcal{T}(T)$ such that $\mathbf{r}P_l \cong \mathrm{Hom}_\Lambda(T, E_l)$ with $P_l = \mathrm{Hom}_\Lambda(T, T_l)$. Let $M = \bigoplus_{i=1}^n M_i$ be a module in $\mathrm{add}(T)$, with M_i indecomposable. Then a module $E \in \mathcal{T}(T)$ will be called the *torsion-predecessor* of M if E is the direct sum of the torsion-predecessors of M_i for each i.

Let M be a non-zero Λ-module in $\mathrm{add}(\mathcal{T}(T))$. We denote the $\mathrm{add}(T)$-*approximation* of M by (T_M, π) (see [5] for a definition), or by T_M for short. This means that $T_M \in \mathrm{add}(T)$, π is a left minimal map and every morphism $\psi : T' \to M$ with $T' \in \mathrm{add}(T)$ factors through π.

Since $\mathrm{add}(T)$ is functorially finite we see that every module has a minimal $\mathrm{add}(T)$-approximation which is unique up to isomorphism. We observe that the map π above is an epimorphism, since $M \in \mathcal{T}(T)$. For, consider a projective presentation $p_* : \mathrm{Hom}_\Lambda(T, T') \to \mathrm{Hom}_\Lambda(T, M) \to 0$. We see that p is an epimorphism since the functor $T \otimes -$ is right exact and $p = T \otimes p_*$. Since p must factor through π, we see that π is also an epimorphism. Furthermore, $\mathrm{Hom}_\Lambda(T, T_M)$ is the projective cover of $\mathrm{Hom}_\Lambda(T, M)$.

The next result is valid for a graded algebra tilted from any finite-dimensional algebra Λ. We observe that there are examples of tilted algebras which are not graded; that is, the ideal of a presentation of a tilted algebra is not always a homogeneous ideal. The reader can find examples in [14].

PROPOSITION 2.5. *Let T be a tilting Λ-module. Suppose that $\Gamma = kQ/I = \mathrm{End}(T)^{\mathrm{op}}$ and that I is a homogeneous ideal in KQ. Let $\ldots \to P_{(3)} \to P_{(2)} \to P_{(1)} \to \Gamma \to \Gamma/\mathbf{r} \to 0$ be the minimal projective resolution of $\mathrm{top}(\Gamma)$ where $P_{(j)} = \mathrm{Hom}_\Lambda(T, T'_j)$ with $T'_j \in \mathrm{add}(T)$. Let E_j be the torsion-predecessor of T'_j. Then Γ is a Koszul algebra if and only if, for each j, the canonical morphism from T'_{j+1} to the minimal left $\mathrm{add}(T)$-approximation of E_j is a split monomorphism.*

Proof. If Γ is a Koszul algebra then $P_{(j)}$ is generated in degree j, for each $j \geq 0$. Moreover $P_{(j)}$ is a graded direct summand of the projective cover of $\mathbf{r}P_{(j-1)}$, for $j \geq 0$. Let $E_{j-1} \in \mathcal{T}(T)$ be such that $\mathrm{Hom}_\Lambda(T, E_{j-1}) \cong \mathbf{r}P_{(j-1)}$ and $T_{E_{j-1}}$ be the minimal left $\mathrm{add}(T)$-approximation of E_{j-1}. Hence, $\mathrm{Hom}_\Lambda(T, T_{E_{j-1}})$ is the projective cover of $\mathbf{r}P_{(j-1)}$. Therefore, T'_j is a graded direct summand of $T_{E_{j-1}}$.

Conversely, if the canonical map is a split monomorphism then $P_{(j+1)}$ is a graded direct summand of the projective cover of $\mathbf{r}P_{(j)}$ for $j \geq 0$. Since I is homogeneous, it follows that Γ is Koszul. ■

The next result is the main theorem of this section. We first introduce the concept of *T-sink maps* and fix the notation which we will use to prove that result.

DEFINITION 2.6. Let $P_l = \mathrm{Hom}_\Lambda(T, T_l)$ be an indecomposable projective Γ-module, $\alpha_l : E_l \to T_l$ its sink-torsion map and (T_{E_l}, π_l) the minimal left $\mathrm{add}(T)$-approximation of E_l. The map $\alpha_l\pi_l$ will be called a *T-sink map* of T_l.

We fix a decomposition $T_{E_l} = \bigoplus_{s=1}^{r} T_{l_s}^{m_{l_s}}$ where $T_{l_1}, \ldots, T_{l_r}$ are indecomposable direct summands of T, pairwise non-isomorphic. We observe that $m_{l_s} = \dim_k \mathrm{Hom}_\Gamma(P_{l_s}, \mathbf{r}P_l/\mathbf{r}^2P_l)$ where $\mathbf{r}P_l = (T, E_l)$.

We now consider a finite-dimensional k-algebra Σ with quiver Q. We fix a ring surjection $\phi : kQ \to \Sigma$ and let $I = \ker\phi$. We assume that I is an admissible ideal; i.e., $J^N \subset I \subset J^2$ for some positive integer N where J is the ideal in KQ generated by the arrows of Q. Since Q is the quiver of Σ, such a ϕ exists. For each arrow $a \in Q_1$, say $a : i \to j$, we define a map $f_a : \phi(j)\Sigma \to \phi(i)\Sigma$, induced by the multiplication by $\phi(a)$ in Σ. We call f_a *multiplication by the arrow a*.

Theorem 2.7. *Let Λ be a finite-dimensional algebra over an algebraically closed field k and T a tilting Λ-module. Let $\Gamma = kQ/I = \mathrm{End}_\Lambda^{\mathrm{op}}(T)$ with Q a finite quiver and I an admissible ideal, such that Γ has global dimension 2. Consider the following minimal projective resolution of $\Gamma/\mathbf{r}$:*

$$0 \to \bigoplus_{t=1}^{r} P_{j_t}^{m_{j_t}} \xrightarrow{\varrho_*} \bigoplus_{v=1}^{s} P_{l_v}^{m_{l_v}} \xrightarrow{f_*} \Gamma \to \Gamma/\mathbf{r} \to 0.$$

Then Γ is a Koszul algebra if and only if the map ϱ may be defined by a matrix whose entries are maps with domain and codomain indecomposable which are components of T-sink maps.

Proof. Let T_l be an indecomposable direct summand of T. We recall that $((T, T_{E_l}), (\pi_l)_*)$ is the projective cover of $\mathbf{r}P_l$ and since $(\alpha_l)_* : \mathbf{r}P_l \to P_l$ is a sink map, one can conclude that S_l has a minimal presentation given by

$$\mathrm{Hom}_\Lambda(T, T_{E_l}) \xrightarrow{(\alpha_l \pi_l)_*} P_l \to S_l \to 0.$$

It follows that each non-zero map $f_* : P_{l_s} \to P_l$, with $l_s \neq l$, which is a multiplication by an arrow is also a component of the map $\alpha_l \pi_l$. Hence we may assume that $\{f_\alpha : T_{l_s} \to T_l \mid \alpha_* : l \to l_s \in Q_1(\Gamma)\} \cong \{\alpha_l (\pi_l)_s^{u_s} : T_{l_s} \to T_l \mid u_s = 1, \ldots, m_{l_s}\}$ for each $s = 1, \ldots, r$ fixed.

We now assume that Γ is a Koszul algebra. Then I is a quadratic ideal and by (1.1) of [6], $\bigoplus_{t=1}^{r} P_{j_t}^{m_{j_t}} \cong I/I^2$. It follows that the map ϱ is defined by components with domain and codomain indecomposable which are, up to isomorphism, multiplications by an arrow. We see from the projective presentation of the simple Γ-modules given above that each component of ϱ is a component of some T-sink map.

Conversely, suppose that each component of ϱ is defined by $\varrho_{j_t,l} : T_{j_t} \to T_l$, which are also components of T-sink maps. It follows that $\varrho_{j_t,l}$ is a component of a T-sink map of some T_l. Hence each component of ϱ_* with domain and codomain indecomposable is multiplication by an arrow. Therefore, Γ is quadratic. Since Γ has global dimension 2, it is a Koszul algebra. ∎

The following examples will illustrate our result.

Example 1 (*A finite type Brenner–Butler algebra*). Let Λ be a quiver algebra whose quiver is the following:

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

Let $T = \tau^- S_2 \oplus P_1 \oplus P_3 \oplus P_4$ be a tilting Λ-module. The Auslander–Reiten quiver of Λ, with the components of T parenthesized, is given by

$$
\begin{tikzcd}
(P_3) \arrow[dr] & & \tau^- P_3 \arrow[dr] & & I_3 \arrow[dr] & & \\
& P_2 \arrow[r] \arrow[ur] \arrow[dr] & (P_1) \arrow[r] & \tau^- P_2 \arrow[r] \arrow[ur] \arrow[dr] & S_2 \arrow[r] & I_2 \arrow[r] & (I_1) \\
(P_4) \arrow[ur] & & \tau^- P_4 \arrow[ur] & & I_4 \arrow[ur] & &
\end{tikzcd}
$$

The morphism $P_2 \to P_1$, in the graph above, is a sink map; moreover, $P_3 \oplus P_4$ is the T-generator of $\operatorname{tr}_T P_2$. Also, P_1 is the T-generator of I_2 and the morphism $I_2 \to \tau^- S_2 = I_1$, given by the graph above, is the map that induces the canonical sink map $r(T, \tau^- S_2) \hookrightarrow (T, \tau^- S_2)$. It follows that Γ has ordinary quiver given by

$$
\begin{tikzcd}
& & \widehat{3} \\
\widehat{2} \arrow[r, "\alpha"] & \widehat{1} \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\
& & \widehat{4}
\end{tikzcd}
$$

We observe that Γ is an algebra with radical square zero.

Example 2 (*A concealed algebra*). Let Λ be a quiver algebra whose quiver is the following:

$$
\begin{tikzcd}
1 & & & 5 \arrow[dl] \\
& 3 \arrow[ul] \arrow[dl] & 4 \arrow[l] & \\
2 & & & 6 \arrow[ul]
\end{tikzcd}
$$

A local sketch of the preprojective component of the Auslander–Reiten quiver of Λ is given by

$$
\begin{tikzcd}
P_1 \arrow[dr] & & \tau^- P_1 \arrow[dr] & & \tau^{-2} P_1 \arrow[dr] & & {} \\
P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau^- P_2 \arrow[r] & \tau^- P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau^{-2} P_2 \arrow[r] & \tau^{-2} P_3 \arrow[r] \arrow[ur] \arrow[dr] & {} \\
& & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_6 \arrow[r] & \tau^- P_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau^- P_6 \arrow[r] & {} \\
& & & P_5 \arrow[ur] & & \tau^- P_5 \arrow[uur] &
\end{tikzcd}
$$

We consider the tilting Λ-module $T = \tau^{-}P_5 \oplus \tau^{-}P_4 \oplus \tau^{-}P_3 \oplus \tau^{-}P_2 \oplus \tau^{-}P_1 \oplus P_6$ to define the endomorphism ring Γ. We see that the full subquiver of the Auslander–Reiten quiver of Λ, with vertices given by the indecomposable direct summands of T, forms a slice and it is the following quiver:

$$\begin{array}{ccccccc} \tau^{-}P_1 & \to & \tau^{-}P_3 & \to & \tau^{-}P_4 & \to & \tau^{-}P_5 \\ & \nearrow & & \nearrow & & & \\ \tau^{-}P_2 & & P_6 & & & & \end{array}$$

where the arrows represent irreducible maps, hence T-sink maps. It follows that Γ has projective radical.

3. Some applications

3.1. *Brenner–Butler tilted algebras.* Let $\Lambda = kQ$ be a finite-dimensional path algebra over a field k, with Q a finite connected quiver. Let $P_1, \ldots, P_n$ (respectively, $I_1, \ldots, I_n$) be a complete list of indecomposable projective non-isomorphic Λ-modules (respectively indecomposable non-isomorphic injective Λ-modules) corresponding to the vertices $1, \ldots, n$ in Q. We fix a simple Λ-module $S = S_i$ associated to the vertex i of Q and assume that $\tau^{-}S_i \neq 0$. The module $T = \tau^{-}S_i \oplus \bigoplus_{j\neq i} P_j$ is a tilting Λ-module. The endomorphism ring $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ will be called a *Brenner–Butler tilted algebra*, or *BB-tilted algebra* for short (see [1]). It is known that the class of torsion-free modules is given by $\mathcal{F}(T) = \mathrm{Cogen}(S)$. Since $\mathrm{Ext}^1_\Lambda(T, S) \cong \mathrm{DHom}_\Lambda(S, \tau T) = \mathrm{DHom}_\Lambda(S, S) \cong k$, we see that $\widehat{S} = \mathrm{Ext}^1_\Lambda(T, S)$ is a simple torsion $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$-module. We shall prove the following result.

THEOREM 3.1. *The endomorphism ring of a Brenner–Butler tilting module over a hereditary algebra is a Koszul algebra.*

Proof. We prove this theorem by showing that $\mathrm{top}(\Gamma)$ has a minimal projective resolution which satisfies the conditions of our main theorem.

We know that every simple module with projective dimension 1 has a linear presentation (see [9]). Moreover, if $\mathrm{pd}_\Gamma \widehat{S} = 1$ (where $\widehat{S} = \mathrm{Ext}^1_\Lambda(T, S)$) then Γ is hereditary. Hence we may assume that $\mathrm{pd}_\Gamma \widehat{S} = 2$. We observe that $\Omega^2(\Gamma/\mathbf{r}) = \Omega^2(\widehat{S})$, since S is the only simple module in $\mathcal{F}(T)$.

Let $0 \to S \to I_i \to I_1 \to 0$ be a minimal injective coresolution of the simple Λ-module $S = S_i$, where $I_1 = I_{l_1}^{m_1} \oplus \ldots \oplus I_{l_t}^{m_t}$ is such that l_s is an immediate predecessor of the vertex i in Q and m_s is the number of arrows from l_s to i for $s = 1, \ldots, t$. We have $\mathrm{top}(\tau^{-}S) = \mathrm{soc}\, I_1$ (see [5]) and since Λ is hereditary one concludes that

$$(*) \qquad 0 \to P_i \xrightarrow{f} \bigoplus_{s=1}^{t} P_{l_s}^{m_s} \xrightarrow{\pi} \tau^{-}S \to 0$$

is a Λ-projective minimal resolution of $\tau^- S$, where the matrix of f consists of multiplications by the arrows $\alpha_s^{u_s}$ from l_s to i in Q, with $s = 1, \dots, t$ and $1 \le u_s \le m_s$.

Applying the functor $\mathrm{Hom}_\Lambda(T, -)$ to the sequence $(*)$, we obtain the following exact sequence of Γ-modules:

$$0 \to (T, P_i) \xrightarrow{f_*} \bigoplus_{s=1}^{t} (T, P_{l_s})^{m_s} \xrightarrow{\pi_*} (T, \tau^- S) \to \mathrm{Ext}^1_\Lambda(T, P_i) \to 0.$$

Since π is a projective cover, we see that any map $\varphi_j : P_j \to \tau^- S$ must factor through π. Hence, given $\varphi \in \mathbf{r}_\Gamma(T, \tau^- S)$, we conclude that $\varphi \in \mathrm{Im}\, \pi_*$. Since π_* is not an epimorphism, we have $\mathrm{Im}\, \pi_* = \mathbf{r}_\Gamma(T, \tau^- S)$ and we see that $\mathrm{coker}\, \pi_* \cong \widehat{S}$.

We now consider the arrows $\beta_m^{v_m}$ from the vertex i to the vertex j_m, with $m = 1, \dots, r$ and $1 \le v_m \le v_{j_m}$, where v_{j_m} is the number of arrows between these vertices. We define $g = ((\beta_1^{v_1})_{1 \le v_1 \le v_{j_1}}, \dots, (\beta_r^{v_r})_{1 \le v_r \le v_{j_m}}) : \mathbf{r}_\Lambda P_i \to P_i$. It is easy to see that $\mathrm{coker}\, g = S_i$ and g_* is an isomorphism between the projective Γ-modules $(T, \mathbf{r}_\Lambda P_i)$ and (T, P_i). It follows that $\beta_m^{v_m} \alpha_s^{u_s} : P_{j_m} \to P_{l_s}$ is a component of the T-sink map of P_{l_s}. Moreover the minimal projective resolution of $\widehat{S}$ is the following exact sequence:

$$0 \to (T, \mathbf{r} P_i) \xrightarrow{f_*} \bigoplus_{s=1}^{t} (T, P_{l_s}^{m_s}) \xrightarrow{\pi_*} (T, \tau^- S) \to \widehat{S} \to 0,$$

where each component of f_* is defined by the map $(\beta_m^{v_m} \alpha_s^{u_s})_*$. It follows by Theorem 2.7 that Γ is a Koszul algebra. ∎

As an application of the result above, we describe a presentation of BB-tilted algebras.

Description of the quiver of BB-tilted algebras. The vertex $\widehat{i} \in Q_\Gamma$ corresponding to the simple Γ-module $\widehat{S}$ is a source with immediate successors given by the vertices associated with the simple Γ-modules $\widehat{S}_{\widehat{l}_s}$ for $s = 1, \dots, t$. The Γ-projective covers of the $\widehat{S}_{\widehat{l}_s}$ are given by $P_{\widehat{l}_s} = \mathrm{Hom}_\Lambda(T, P_{l_s})$. It follows from the fact that $\mathrm{Hom}_\Lambda(\tau^- S, P_j) = 0$ for $j \ne i$ that the projective cover of $\mathbf{r}(T, \tau^- S)$ is given by $\bigoplus_{s=1}^t P_{\widehat{l}_s}^{m_s}$.

The number of arrows from $\widehat{l}_s$ to some $\widehat{m}$ in the quiver of Γ, for $m \ne i$, is equal to

$$m_s \cdot \dim_k(P_m, \mathbf{r}_\Lambda P_i / \mathbf{r}_\Lambda^2 P_i) + \dim_k(P_m, P' / \mathbf{r}_\Lambda P'),$$

where $\mathbf{r}_\Lambda P_{l_s} = P_i^{m_s} \oplus P'$ for some projective Λ-module P'. Furthermore, at the remaining vertices, the quiver of Γ has the same description as the quiver Q. One can check these assertions by an easy computation.

Description of a set of relations for BB-tilted algebras. The ideal of relations for Γ has a minimal generating set given by the sum of paths which start at $\widehat{i}$ and end at $\widehat{j}_m$, for $m = 1, \ldots, r$, and is defined by $\sum_{s=1}^{t} \sum_{u_s=1}^{m_s} (\beta_m^{v_m} \alpha_s^{u_s}) \pi_s^{u_s}$, for each $v_m = 1, \ldots, v_{j_m}$.

As we said above, the quivers $Q(\Lambda)$ and $Q(\Gamma)$ of a hereditary algebra Λ and the associated BB-tilted algebra Γ have the same shape at the vertices $j \notin \{i, l_1, \ldots, l_r, j_1, \ldots, j_m\}$. In order to clarify the description above, we restrict ourselves to the case where $Q(\Lambda)$ is a tree, and present diagrams showing the connection between both quivers. We observe that this description will be useful in the next proposition. We first consider the following definition.

Definition 3.2. Let i be a vertex in a quiver Q. We define the *neighborhood* of i to be the full subquiver whose vertices are i, its immediate predecessors, and successors.

We now assume that the neighborhood of the vertex i in Q has the following diagram:

$$\begin{tikzcd}
l_1 \arrow[dr, "\alpha_1"] & & j_1 \\
\vdots & i \arrow[ur, "\beta_1"] \arrow[dr, "\beta_r"] & \vdots \\
l_t \arrow[ur, "\alpha_t"] & & j_r
\end{tikzcd}$$

As we have shown, the vertex $\widehat{i}$ is a source in the quiver $Q(\Gamma)$ and its immediate successors are given by the vertices $\widehat{l_1}, \ldots, \widehat{l_t}$ corresponding to the projective Γ-modules $\mathrm{Hom}_\Lambda(T, P_{l_s})$, $s = 1, \ldots, t$. We consider $\pi = (\pi_1, \ldots, \pi_t)$, a decomposition of π, and the arrow $\overline{\pi}_s$ corresponding to the homomorphism $(\pi_s)_* = \mathrm{Hom}(T, \pi_s)$. Each vertex $\widehat{l_s}$ is an immediate predecessor of each $\widehat{j}_m$. The arrows between $\widehat{l_s}$ and $\widehat{j}_m$ denoted by $(\beta_m \alpha_s)$ correspond to the homomorphism $(\beta_m \alpha_s)_*$, as defined in the proof of Theorem 2.7. For each $m = 1, \ldots, r$ we have the following local picture:

$$\begin{tikzcd}
 & \widehat{l_1} \arrow[dr, "(\beta_m \alpha_1)"] & \\
\widehat{i} \arrow[ur, "\overline{\pi}_1"] \arrow[dr, "\overline{\pi}_t"'] & \vdots & \widehat{j}_m \\
 & \widehat{l_t} \arrow[ur, "(\beta_m \alpha_t)"'] &
\end{tikzcd}$$

We present some explicit examples to further clarify the quiver with relations of a BB-tilted algebra.

EXAMPLE 3. If Λ is the quiver algebra whose quiver is

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ Q: \quad 1 \arrow[rr, "\beta"] & & 3 \end{tikzcd}$$

and $T = \tau^- S_1 \oplus \bigoplus_{j\neq 1} P_j$, we see that Γ has the following ordinary quiver:

$$\widehat{1} \xrightarrow{\overline{\pi}} \widehat{2} \overset{\beta\alpha}{\underset{\overline{\gamma}}{\rightrightarrows}} \widehat{3}$$

and relation $(\beta\alpha)\overline{\pi} = 0$.

EXAMPLE 4. Let Λ be the quiver algebra whose quiver is

$$1 \xleftarrow{\beta} 2 \overset{\alpha_1}{\underset{\alpha_2}{\leftleftarrows}} 3$$

and T the Brenner–Butler tilting module associated with the vertex 2, that is, $T = \tau^- S_2 \oplus P_1 \oplus P_3$. In this case Γ has a presentation whose ordinary quiver is given by

$$\widehat{2} \overset{\overline{\pi}_1}{\underset{\overline{\pi}_2}{\rightrightarrows}} \widehat{3} \overset{\beta\alpha_1}{\underset{\beta\alpha_2}{\rightrightarrows}} \widehat{1}$$

and only one relation $\overline{\pi}_1(\beta\alpha_1) + \overline{\pi}_2(\beta\alpha_2) = 0$.

It is conjectured that a tilted algebra is simply connected if and only if the original hereditary algebra is simply connected. We recall that a hereditary algebra is simply connected when its quiver is a tree. We know by [3] that this conjecture holds in the case of tame tilted algebras. Our objective now is to prove that this statement still holds for BB-tilted algebras irrespective of their representation type. We review some notation and definitions (see [4]).

Let (Q, I) be a presentation of a connected algebra Γ; we denote by $\Pi_1(Q, I)$ its fundamental group. Given any abelian group G, we denote by $Z^1(\Gamma, I, G)$ the set of all G-valued functions $f : Q_1 \to G$ such that $\sum_{i=1}^{u} f(\alpha_i) = \sum_{j=1}^{p} f(\beta_j)$ whenever there exists a minimal relation $\varrho = \sum_{i=1}^{q} \lambda_i w_i$, with $q \geq 2$, such that $w_1 = \alpha_1 \ldots \alpha_u$ and $w_2 = \beta_1 \ldots \beta_p$. In [4], Section 2.3, Assem and de la Peña show the existence of the following exact sequence of abelian groups:

$$0 \to G \to G^{|Q_0|} \to Z^1(\Gamma, I, G) \to \mathrm{Hom}(\Pi_1(Q, I), G) \to 0.$$

The next lemma is a consequence of the exact sequence above and will be used in our next proposition.

LEMMA 3.3. *Let (Q, I) be a presentation of a connected algebra, where I is generated by a set $\{\varrho_m = \sum_{i=1}^{u_m} \lambda_i w_i \mid m = 1, \ldots, r\} \cup \{\gamma_j\}$ where each γ_j is a monomial relation in I and each ϱ_m is a minimal relation with u_m*

terms. Then

$$\dim_{\mathbb{Q}}(\mathrm{Hom}(\varPi_1(Q,I),\mathbb{Q})) \geq |Q_1| - |Q_0| + 1 - \sum_{m=1}^{r} u_m + r.$$

Proof. This is a straightforward consequence of the sequence above with G being the additive group of the rational numbers. We just observe that $Z^1(\varGamma, I, \mathbb{Q})$ is a subspace of $\mathbb{Q}^{|Q_1|}$ which is determined by $\sum_{m=1}^{r} u_m - r$ linear equations. ∎

Proposition 3.4. *Let $\varGamma$ be a BB-tilted algebra from a hereditary algebra $\varLambda = kQ$. Then $\varGamma$ is simply connected if and only if Q is a tree.*

Proof. Assume first Q is a tree. Let $(Q(\varGamma), I)$ be the presentation of $\varGamma$ given above. Then it is easy to see that the fundamental group $\pi_1(Q(\varGamma), I)$ is trivial. Since $\dim o(\gamma)\varGamma t(\gamma) = 1$ for any arrow $\gamma \in Q(\varGamma)$, it follows by Theorem 3.5 of [8] that the fundamental group of any presentation is trivial. Since it is also known that $\varGamma$ is directed, it follows that $\varGamma$ is simply connected.

We assume now that the BB-tilted algebra $\varGamma$ is simply connected and we shall prove that $\varLambda$ is simply connected; that is, Q is a tree. We know that $H^1(\varGamma) = H^1(\varLambda)$ by Theorem 4.2 of [12]. Hence it is enough to prove that $H^1(\varGamma) = 0$.

We recall that outside the neighborhood of the vertex $i \in Q$ the quivers Q and $Q(\varGamma)$ have the same shape. Since $\varGamma$ is simply connected, we conclude that Q does not contain simple closed walks not involving arrows in the neighborhood of i. We also see that there is no path in Q which starts at some l_i and ends at some l_j. Otherwise, we would have the following picture of the neighborhood of i in Q:

$$\begin{array}{ccccc}
l_1 & & & & j_1 \\
\downarrow & \searrow & & \nearrow & \\
l_2 & \to & i & & \vdots \\
\vdots & \nearrow & & \searrow & \\
l_t & & & & j_r
\end{array}$$

Then $Q(\varGamma)$ would have the following picture of the neighborhood of $\widehat{i}$:

$$\begin{array}{ccccc}
 & & \widehat{l_1} & & \\
 & \nearrow & \downarrow & \searrow & \\
\widehat{i} & \to & \widehat{l_2} & \to & \widehat{j}_m \\
 & \searrow & \vdots & \nearrow & \\
 & & \widehat{l_t} & &
\end{array}$$

for each $m = 1, \ldots, r$. In this case, we claim that Γ is not simply connected. According to our description of BB-tilted algebras, if we compute the fundamental group of our presentation, the closed walk starting at $\widehat{i}$ and passing through $\widehat{l_1}$ and $\widehat{l_2}$ is not trivial in the fundamental group. The same kind of argument shows that there is no path in Q starting and ending at the vertices $j_1, \ldots, j_r$.

We consider the following set of vertices: $\mathcal{C} = \{\widehat{i}, \widehat{l_1}, \ldots, \widehat{l_t}, \widehat{j_1}, \ldots, \widehat{j_r}\}$. It follows from our presentation and the fact that Γ is simply connected that all simple closed walks in $Q(\Gamma)$ have vertices belonging to $\mathcal{C}$.

Let Γ' be a full subcategory of Γ whose vertices belong to $\mathcal{C}$. Hence Γ' is a full convex subcategory of Γ. If we identify all the vertices in the set $\mathcal{C}$, we obtain a quiver which is a tree. Using one-point extensions and coextensions, and Happel's long exact sequence, we see that $H^1(\Gamma) = H^1(\Gamma')$.

We claim that Γ' is the BB-tilted algebra from the hereditary algebra Λ' whose quiver is such that all the arrows start or end at i; that is, the quiver of Λ' has the following description:

$$\begin{array}{ccccc} l_1 & & & & j_1 \\ & \searrow & & \nearrow & \\ \vdots & & i & & \vdots \\ & \nearrow & & \searrow & \\ l_t & & & & j_r \end{array}$$

The result follows from the claim. Indeed, Λ' is hereditary and $Q(\Lambda')$ is a tree. Therefore $H^1(\Lambda') = 0$. We also know that $H^1(\Gamma') = H^1(\Lambda')$. Hence $H^1(\Lambda) = 0$, as we wished to prove.

We now prove the claim. We have shown that, in the quiver Q, there is no path between l's or between j's and no paths from some l to some j. Hence, in the quiver of Λ', there is no closed simple walk with origin at some l_s passing through some j_m or between themselves. Next, we prove that there is no multiple arrow starting at some l_s or ending at some j_m, for every $s = 1, \ldots, t$ and $m = 1, \ldots, r$.

We recall that Γ' is a connected algebra such that the relations are given by

$$\varrho_{v_m} = \sum_{s=1}^{t} \sum_{u_s=1}^{m_s} (\beta_m^{v_m} \alpha_s^{u_s}) \pi_s^{u_s},$$

for each pair (m, v_m). We shall prove that if $m_s > 1$ or $v_{j_m} > 1$ for some s or some m then Γ' is not simply connected.

We set $l = m_1 + \ldots + m_t$ and $v = v_{j_1} + \ldots + v_{j_r}$. Observe that the number of vertices of Γ' is $t + r + 1$, by the description for the quiver of the BB-tilted algebras.

If $l > 1$ then by the description of the minimal relations of Γ' and the lemma above we have

$$\dim_{\mathbb{Q}}(\mathrm{Hom}(\Pi_1(Q,I),\mathbb{Q})) \geq (l+v)-(t+r).$$

If $Q(\Lambda')$ has multiple arrows then $l > t$ or $v > r$; in this case

$$\dim_{\mathbb{Q}}(\mathrm{Hom}(\Pi_1(Q,I),\mathbb{Q})) > 0.$$

Therefore Γ' is not simply connected. Finally, if $l = 1$ then all relations are monomial. In this case, if $v_{j_m} > 1$ for some m, then Γ' is not simply connected. ∎

3.2. *A class of Koszul iterated tilted algebras.* A complete characterization of the generalized tilted algebra of type A_n is given by Assem and Happel in [2]. We recall that a quiver is called a *linear quiver* if for each vertex in the quiver there exists at most one arrow starting and one arrow ending at the vertex. In Section 2 of the cited paper the authors establish the following.

LEMMA ([2]). *Let A be a finite-dimensional algebra of finite representation type with the following properties*:

1. *There are at most two irreducible maps with prescribed domain or codomain.*
2. *If P_A is a projective module with indecomposable radical R then there is at most one irreducible map of codomain R. Dually if I_A is injective with $I/\operatorname{soc} I$ indecomposable then there is at most one irreducible map with domain $I/\operatorname{soc} I$.*

Then, *for every indecomposable M*, *the set of all* (*isomorphism classes of*) *indecomposable modules N such that* $\mathrm{Hom}(N,M) \neq 0$ *and* $\mathrm{Hom}(N,\tau M) = 0$ *is the union of two full linear subquivers of the AR-quiver intersecting at the vertex $[M]$. The dual conclusion also holds.*

Let Λ be a finite-dimensional algebra of finite representation type satisfying the hypothesis of the lemma above and Γ a tilted algebra from Λ. Using the Lemma above, Happel and Ringel have described the correspondence between indecomposable summands of the tilting module T and the vertices of the quiver $Q(\Gamma)$, in the following sense.

They consider an indecomposable direct summand T_l of T corresponding under the functor $\mathrm{Hom}_\Lambda(T,-)$ to the projective Γ-module associated with the vertex $l \in Q(\Gamma)$. Then, in the AR-quiver of Λ, there are at most two irreducible maps f and g of codomain T_l and at most two irreducible maps u, v of domain T_l. Furthermore, these irreducible maps determine at most four linear subquivers intersecting at T_l, such that any indecomposable module that does not lie in these subquivers is neither the domain of a non-zero map with codomain in T_l nor the codomain of a non-zero map with domain in T_l.

We denote by $L(f)$, $L(g)$, $R(u)$ and $R(v)$ the subquivers to the left and to the right of T_l, respectively, such that $L(f)$ together with $R(u)$ and $L(g)$ together with $R(v)$ are the full linear subquivers of the AR-quivers intersecting in T_l, as described in the lemma above. Let T_s be an indecomposable direct summand of T corresponding to the vertex $s \in Q(\Gamma)$. If $\mathrm{Hom}_\Lambda(T_l, T_s) \neq 0$ then we have $T_s \in R(u)$ or $T_s \in R(v)$. If $\mathrm{Hom}_\Lambda(T_s, T_l) \neq 0$ then $T_s \in L(f)$ or $T_s \in L(g)$.

Now we consider two neighbors s and j of the vertex l in the quiver of Γ such that there exist non-zero maps $T_s \to T_l$ and $T_l \to T_j$. If T_s, T_l belong to a linear subquiver determined by an irreducible map with codomain T_j then the composition $T_s \to T_l \to T_j$ is non-zero. Otherwise the composition $T_s \to T_l \to T_j$ is zero, since T_s is not in the linear subquiver determined by an irreducible map with codomain T_j in which T_l lies.

Happel and Ringel observed that the Lemma above is crucial for the proof of their main result since they have proved that every generalized tilted algebra of type A_n satisfies these hypotheses and hence its AR-quiver has the properties of that assertion. Using this lemma, Proposition 2.2 and the same kind of arguments presented in that paper, we show that the algebras tilted from an algebra Λ, where Λ satisfies the conditions of the lemma (which includes the generalized tilted algebras of type A_n), are monomial quadratic and by Green and Zacharia [11], they are Koszul algebras.

PROPOSITION 3.5. *Let Λ be a finite-dimensional algebra of finite representation type satisfying the hypotheses of the Lemma above and Γ a tilted algebra from Λ. Then all presentations of Γ are monomial quadratic. In particular Γ is a Koszul algebra.*

Proof. Let T_l be an indecomposable direct summand of T, $\alpha : E_l \to T_l$ the sink-torsion map and (T_{E_l}, π_l) the minimal left $\mathrm{add}(T)$-approximation of E_l. Since T_{E_l} belongs to $\mathrm{add}(T)$ and $\mathrm{Hom}(T_{E_l}, T_l) \neq 0$, it follows from the considerations above that each indecomposable direct summand of T_{E_l} (isomorphism class) lies on the subquiver $L(f)$ or on the subquiver $L(g)$. Since (T, T_{E_l}) is the projective cover of $\mathbf{r}(T, T_l)$ it follows that each indecomposable direct summand of T_{E_l} defines an immediate predecessor vertex in $Q(\Gamma)$ of the vertex l. Hence T_{E_l} has, at most, two indecomposable direct summands, say $T_{l,1}$ and $T_{l,2}$, belonging to the subquivers $L(f)$ and $L(g)$, respectively. Applying the same argument to the modules $T_{l,1}$ and $T_{l,2}$, we have the following picture, where the arrows represent components of T-sink maps:

$$\begin{tikzcd}
T_{l,1,1} \arrow[dr] & & \\
& T_{l,1} \arrow[dr] & \\
T_{l,1,2} \arrow[ur] & & T_l \\
T_{l,2,1} \arrow[dr] & & \\
& T_{l,2} \arrow[uur] & \\
T_{l,2,2} \arrow[ur] & &
\end{tikzcd}$$

We now describe the relations starting at $l \in Q(\Gamma)$. We first observe that there do not exist commutative relations in Γ. Suppose we have $T_{l,1,2} \cong T_{l,2,1}$. Since $T_{l,2,1}$ does not belong to $L(f)$ or to $L(g)$ we have $\mathrm{Hom}_\Lambda(T_{l,2,1}, T_l) = 0$. Hence both compositions $T_{l,1,2} \to T_{l,1} \to T_l$ and $T_{l,1,2} \to T_{l,2} \to T_l$ are zero maps. Therefore these composition maps define monomial relations. We also see that there is no non-zero homomorphism with domain an indecomposable module whose class is not in a full linear subquiver intersecting at T_l and with codomain T_l. It follows that the above presentation of Γ is monomial quadratic. Since we also have $\dim o(\gamma)\Gamma t(\gamma) = 1$ for all arrows γ, it follows by Proposition 2.5 of [8] that all presentations are monomial quadratic. ■

4. Quasi-Koszul algebras. The concept of quasi-Koszul algebras was introduced in [9], where Green and Martinez consider the case of non-graded algebras. They introduce the notion of linear resolutions in that case. They assume that Λ is a Noetherian semiperfect algebra over the field k and they prove that a Λ-module M is a quasi-Koszul module if and only if M has a *linear non-graded resolution* (see Thm. 4.4 in [9]). They show that this is equivalent to saying that $E(M) = \coprod_{n\geq 0} \mathrm{Ext}^n_\Lambda(M, \Lambda/\mathbf{r})$, as a graded $E(\Lambda)$-module, is generated in degree zero. We introduce a new method of dealing with the concept of quasi-Koszul modules.

Definition 4.1. Let Λ be a Noetherian semiperfect algebra over the field k. Let $\Theta : P \to Q$ be a non-zero Λ-map between projective modules. We say that Θ is an *essentially linear map* if $P \cong \bigoplus_i P_i$ and $Q \cong \bigoplus_j Q_j$ where the P_i and Q_j are indecomposable projective modules and, for each i and j, the map $\Theta_{i,j} : P_i \to Q_j$ induced by Θ induces a non-zero monomorphism $\overline{\Theta}_{i,j} : P_i/\mathbf{r}P_i \to \mathbf{r}Q_j/\mathbf{r}^2 Q_j$.

We say that a finitely generated Λ-module M has an *essentially linear resolution* if the minimal projective resolution of M

$$\ldots \to P_n \xrightarrow{f_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

is such that f_j is an essentially linear map for every $j > 0$. The next result shows that M has an essentially linear resolution if and only if it is a quasi-Koszul module.

PROPOSITION 4.2. *Let Λ be a Noetherian semiperfect k-algebra and M be a finitely generated Λ-module. Then M is a quasi-Koszul module if and only if M has an essentially linear resolution.*

Proof. We follow the argument in the proof of Theorem 4.4 of [9]. Let M be a finitely generated Λ-module with the following minimal projective resolution:

$$\ldots \to P_n \xrightarrow{f_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{f_1} P_0 \to M \to 0.$$

We consider $H = \ker f_{j-1}$ and we observe that if f_j is an essentially linear map for each $j \geq 1$ then the induced map $\bar{f}_j : P_j/\mathbf{r}P_j \to \mathbf{r}P_{j-1}/\mathbf{r}^2P_{j-1}$ is a non-zero monomorphism for each $j \geq 1$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{i} & \mathbf{r}P_{j-1} \\ \downarrow p & & q\downarrow \\ H/\mathbf{r}H & \xrightarrow{\bar{i}} & \mathbf{r}P_{j-1}/\mathbf{r}^2P_{j-1} \end{array}$$

where i is the inclusion map, $\bar{i}$ is induced by i and p, q are canonical epimorphisms. We observe that $\bar{i}$ is a non-zero monomorphism since f_j is an essentially linear map. It follows from the argument in the proof of Proposition 4.1 of [9] that $\mathrm{Ext}^j_\Lambda(M, \Lambda/\mathbf{r}) = \mathrm{Ext}^1_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cdot \mathrm{Ext}^{j-1}_\Lambda(M, \Lambda/\mathbf{r})$. It follows by induction that $E(M)$ is generated in degree zero. Hence M is a quasi-Koszul module.

Conversely, if we assume that M is a quasi-Koszul module, then it follows from the argument in the proof of Theorem 4.4 of [9] that the map $\ker f_0 \to \Lambda/\mathbf{r}$ factors through $\mathbf{r}P_0$. Hence f_1 is an essentially linear map. The proof follows by induction. Since M is a quasi-Koszul module we have $\mathrm{Ext}^j_\Lambda(M, \Lambda/\mathbf{r}) = \mathrm{Ext}^1_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cdot \mathrm{Ext}^{j-1}_\Lambda(M, \Lambda/\mathbf{r})$. We also have $\mathrm{Ext}^{j-1}_\Lambda(M, \Lambda/\mathbf{r}) \cong \mathrm{Hom}_\Lambda(\ker f_{j-2}, \Lambda/\mathbf{r})$. We now apply the same arguments in the case $j = 1$ to conclude that every map $\ker f_{j-1} \to \Lambda/\mathbf{r}$ factors through $\mathbf{r}P_{j-1}$. It follows that f_j is an essentially linear map for each $j > 1$. ∎

4.1. *Endomorphism rings of tilting modules.* In this subsection we shall introduce the concept of *essentially T-irreducible maps* and prove that they are strongly related to the essentially linear maps presented in the previous subsection. As a consequence, we shall give a condition for a module over an endomorphism ring of a tilting module to be a quasi-Koszul module. We fix a finite-dimensional algebra Λ over the algebraically closed field k and a tilting Λ-module T. We also choose a fixed decomposition $T = \bigoplus_{j=1}^{n} T_j$ into indecomposable direct summands and denote by Γ the opposite endomorphism ring of the tilting module T.

Definition 4.3. Let $f : T_v \to T_l$ be a non-zero Λ-morphism between indecomposable modules in $\mathrm{add}(T)$. We say that f is a *T-irreducible map* if f is not an isomorphism and for any factorization $f = gh$ through $\mathrm{add}(T)$, either h is a split monomorphism or g is a split epimorphism.

Lemma 4.4. *Let $f_* : P_v \to P_l$ be a Γ-morphism which is not an isomorphism and is not in $\mathbf{r}^2\Gamma$. Then f is a T-irreducible map.*

Proof. Assume that f is not a T-irreducible map and let $f = gh$ for some $g : T' \to T_l$ and $h : T_l \to T'$ with $T' \in \mathrm{add}(T)$. If h is not a split monomorphism then $h_*(P_v)$ is contained in $\mathbf{r}P'$, where $P' = (T, T')$. If g is not a split epimorphism then $g_*(h_*(P_v)) \subset g_*(\mathbf{r}P') \subset \mathbf{r}^2 P_l$. Hence f_* is in $\mathbf{r}^2\Gamma$. ■

Lemma 4.5. *Let $P_l = \mathrm{Hom}_\Lambda(T, T_l)$ and $P_j = \mathrm{Hom}_\Lambda(T, T_j)$ be indecomposable projective Γ-modules. Then there exists a T-irreducible map f : $T_j \to T_l$ if and only if P_j is a direct summand of the projective cover of $\mathbf{r}P_l$.*

Proof. Assume that f is a T-irreducible map. Let $\alpha_l\pi_l : T_{E_l} \to T_l$ be the T-sink map of T_l. Hence f factors through $\alpha_l\pi_l$. We consider a map $\beta : T_j \to T_{E_l}$ such that $f = \alpha_l\pi_l\beta$. Since $\alpha_l\pi_l$ is not a split epimorphism it follows by the hypothesis on f that β is a split monomorphism. Hence T_j is a direct summand of T_{E_l}.

Conversely, assume that $P_j = (T, T_j)$ is a direct summand of the projective cover of the radical of P_l. In this case, there exists a non-zero morphism $f_* : P_j \to P_l$ such that $\mathrm{Im}\, f_*$ is contained in $\mathbf{r}P_l$ and $\mathrm{Im}\, f_*$ is not contained in $\mathbf{r}^2 P_l$. We claim that f is a T-irreducible map. Suppose not; that is, there exist non-zero maps $g : T' \to T_l$ and $h : T_j \to T'$, with $T' \in \mathrm{add}(T)$, such that $f = gh$, where g is not a split epimorphism and h is not a split monomorphism. By the Brenner–Butler Theorem we have $f_* = g_*h_*$, $\mathrm{Im}\, g_* \subset \mathbf{r}P_l$ and $\mathrm{Im}\, h_* \subset \mathbf{r}(T, T')$. It follows that $\mathrm{Im}\, f_* \subset \mathbf{r}^2 P_l$, a contradiction with the hypothesis on f_*. ■

Remark. It follows from Lemma 4.5 that if $f : T_v \to T_l$ is T-irreducible then f induces a non-zero monomorphism $\bar{f}_* : P_v/\mathbf{r}P_v \to \mathbf{r}P_l/\mathbf{r}^2 P_l$; that is, if Γ is the endomorphism ring of a tilting module over an algebra Λ, then every T-irreducible map is also an essentially linear map. The next definition generalizes the concept of T-irreducible maps.

Definition 4.6. Let $\varrho_* : (T, T_{(1)}) \to (T, T_{(2)})$ be a non-zero Λ-module homomorphism. We say that ϱ is an *essentially T-irreducible map* if the following conditions hold:

(1) For every indecomposable direct summand $T'_{(1)}$ of $T_{(1)}$, there exists an indecomposable direct summand $T'_{(2)}$ of $T_{(2)}$ such that the composition $T'_{(1)} \xrightarrow{\varrho_{T'}} T_{(2)} \xrightarrow{p} T'_{(2)}$ is non-zero, where $\varrho_{T'_{(1)}}$ is the restriction of ϱ to $T'_{(1)}$ and p is the canonical projection.

(2) Whenever $\alpha = p\varrho_{T'_{(1)}}$ for p and $\varrho_{T'_{(1)}}$ as described above, then α is a T-irreducible map.

The next result relates the concepts of essentially linear maps and essentially T-irreducible maps over endomorphism rings of tilting modules.

LEMMA 4.7. *Let Λ be a finite-dimensional algebra over the field k, T be a tilting module and $\Gamma = \mathrm{End}^{\mathrm{op}}_{\Lambda}(T)$ the endomorphism ring of T over Λ. Let $\varrho_* : P_{(1)} = (T, T_{(1)}) \to P_{(2)} = (T, T_{(2)})$ be a non-zero Γ-module homomorphism. Then ϱ_* is an essentially linear map if and only if ϱ is an essentially T-irreducible map.*

Proof. Assume that ϱ is an essentially T-irreducible map. It follows from the remark above that ϱ_* induces a non-zero monomorphism $\overline{(\varrho_{T'})_*} : (T, T')/\mathbf{r}(T, T') \to \mathbf{r}P_{(2)}/\mathbf{r}^2 P_{(2)}$ for each indecomposable direct summand (T, T') of $P_{(1)}$. Therefore ϱ_* is an essentially linear map.

We now assume that ϱ_* is an essentially linear map and we shall prove that each component $\varrho' : T'_{(1)} \to T'_{(2)}$ of ϱ, for $T'_{(1)}$ and $T'_{(2)}$ indecomposable direct summands of $T_{(1)}$ and $T_{(2)}$ respectively, is a T-irreducible map. We consider a factorization of ϱ' through $\mathrm{add}(T)$, say $\varrho' = gh$ with $g : T'' \to T'_{(2)}$ and $h : T'_{(1)} \to T''$ for $T'' \in \mathrm{add}(T)$, and let $\overline{\varrho}'_*, \overline{h}_*$ and $\overline{g}_*$ be the induced maps on the tops of the modules (T, T''), $(T, T'_{(1)})$ and $(T, T'_{(2)})$, respectively. Since $\overline{\varrho}'_*$ is a split monomorphism, so is $\overline{h}_*$. Since (T, T'') is a projective Γ-module, h_* is a split monomorphism. Hence h is also a split monomorphism. Therefore ϱ' is a T-irreducible map and we conclude that ϱ is an essentially T-irreducible map. ∎

We now present the main result of this subsection.

PROPOSITION 4.8. *Let $\Gamma = \mathrm{End}^{\mathrm{op}}_{\Lambda}(T)$ be the endomorphism ring of the tilting module T over the finite-dimensional algebra Λ. Let M be a finitely generated Γ-module. Consider the minimal projective resolution of M given by the following long exact sequence*:

$$\ldots \to P_n \xrightarrow{(\varrho_n)_*} P_{n-1} \to \ldots \to P_1 \xrightarrow{(\varrho_1)_*} P_0 \to M \to 0.$$

Then M is a quasi-Koszul module if and only if ϱ_j is an essentially T-irreducible map for each $j > 0$.

Proof. It follows from Proposition 4.2 that M is a quasi-Koszul module if and only if M has an essentially linear resolution. By Lemma 4.7, the latter condition is equivalent to ϱ_j being an essentially T-irreducible map for each $j > 0$. ∎

We have the following characterization of quasi-Koszul modules in the context of the endomorphism rings of tilting modules, as a consequence of Definition 4.6 and Proposition 4.8. Let Λ be a finite-dimensional algebra over the field k and T be a tilting Λ-module. Let $\Gamma = \mathrm{End}_\Lambda^{\mathrm{op}}(T)$ be the endomorphism ring of T over Λ, and M be a finitely generated Γ-module. Observe that M is a quasi-Koszul module if and only if M has an essentially T-irreducible projective resolution.

Corollary 4.9. *Let $\Gamma = \mathrm{End}_\Lambda^{\mathrm{op}}(T)$ be the graded endomorphism ring of the tilting module T over the finite-dimensional algebra Λ. Let M be a finitely generated Γ-module generated in degree zero. Then M is a Koszul module if and only if M has an essentially T-irreducible projective resolution.*

Proof. Since Γ is graded the proof is straightforward from Proposition 4.8. ∎

4.2. *A class of quasi-Koszul endomorphism rings.* In this subsection we fix the following notation and assumptions. Recall that a finite-dimensional k-algebra is called *triangular* if its quiver has no oriented cycles. Recall that Λ is called a *quadratic monomial algebra* if $\Lambda = kQ/I$ where I is generated by a set of paths of length 2 in Q. In this subsection, Λ will denote a triangular, basic, finite-dimensional, quadratic, monomial k-algebra of global dimension smaller than or equal to two. Let $P_1, \ldots, P_n$ be a complete list of indecomposable projective non-isomorphic Λ-modules. Let X be an indecomposable Λ-module such that $T = X \oplus \bigoplus_{j\neq i} P_j$ is a tilting Λ-module. Assume moreover that $\Gamma = \mathrm{End}_\Lambda^{\mathrm{op}}(T)$ has global dimension at most 2. We shall prove that if X is a Koszul module then Γ is a quasi-Koszul algebra. This result will be obtained as a consequence of several lemmas.

If X is a projective module then we have $\Gamma \cong \Lambda$, which is a Koszul algebra. For the remainder of this subsection, we assume that X is not a projective Λ-module.

Lemma 4.10. *Let $0 \to P_{(1)} \xrightarrow{f} P_{(X)} \xrightarrow{\pi} X \to 0$ be a minimal projective resolution of X. Then $P_{(1)} \cong P_i$ and $P_{(X)} \in \mathrm{add}(T)$.*

Proof. We show first that $P_{(1)} \cong P_i^n$ and $P_X \in \mathrm{add}(T)$. Let $P_{(1)} \cong P_i^n \oplus P'$ with $P' \in \mathrm{add}(T)$ and $n \geq 0$ and

$$\begin{array}{ccccccccc}
0 & \to & P_{(1)} & \xrightarrow{f} & P_{(X)} & \to & X & \to & 0 \\
 & & \downarrow p & & \downarrow & & \| & & \\
0 & \to & P' & \to & E & \to & X & \to & 0
\end{array}$$

where $p : P_{(1)} \to P'$ is the projection and the commutative diagram is obtained by taking the pushout of p and f. Since $\mathrm{Ext}^1(X,T)=0$, the bottom row splits. Hence the map p can be factored through $P_{(X)}$. This contradicts the minimality of the resolution unless $P'=0$. Therefore $P_{(1)}=P_i^n$ for some $n\geq 0$. The triangularity of Λ and the fact that $P_{(1)}=P_i^n$ imply that $\mathrm{Hom}(P_{(X)},P_i)=0$. Hence, we conclude that $P_{(X)}$ is in $\mathrm{add}(T)$.

We now show that $n=1$. We apply the functor $\mathrm{Hom}_\Lambda(T,-)$ to the minimal projective resolution of X to obtain the exact sequence

$$\text{(I)}\qquad 0\to (T,P_i^n)\to (T,P_{(X)})\xrightarrow{\pi_*}(T,X)\to \mathrm{Ext}^1_\Lambda(T,P_i^n)$$
$$\to \mathrm{Ext}^1_\Lambda(T,P_{(X)})\to \mathrm{Ext}^1_\Lambda(T,X)\to 0.$$

Since $\mathrm{Ext}^1_\Lambda(T,P_{(X)})=0$ and (T,X) is an indecomposable projective Γ-module, it follows that $\mathrm{Ext}^1_\Lambda(T,P_i^n)$ is indecomposable and therefore $n=1$. ■

For each indecomposable projective module P_i, let S_i denote the top of P_i.

LEMMA 4.11. *Keeping the hypotheses and notation of this subsection, the following statements hold*:

1. *The canonical inclusion* $\beta : \mathbf{r}P_i \to P_i$ *defines an isomorphism* $\beta_* : (T,\mathbf{r}P_i)\to (T,P_i)$.
2. $\mathrm{Ext}^1_\Lambda(T,P_i)$ *is a simple* Γ*-module.*
3. $\mathrm{pd}_\Lambda S_i = 1$ *and hence* $\mathbf{r}P_i$ *is a projective* Λ*-module.*

Proof. We observe that $(T,S_i)=0$, since S_i is not a simple module in $\mathrm{top}(T)$. It follows by definition that $S_i\in\mathcal{F}(T)$. Using the projective resolution of X to compute the Ext groups we have $\mathrm{Ext}^1_\Lambda(T,S_i)\cong \mathrm{Ext}^1_\Lambda(X,S_i)\cong \mathrm{Hom}_\Lambda(P_i,S_i)\cong k$. It follows that $\mathrm{Ext}^1_\Lambda(T,S_i)$ is a Γ-simple module.

We apply the functor $\mathrm{Hom}_\Lambda(T,-)$ to the short exact sequence

$$0\to \mathbf{r}P_i\xrightarrow{\beta} P_i\to S_i\to 0$$

where β is the canonical inclusion and we obtain the exact sequence

$$\text{(II)}\qquad 0\to (T,\mathbf{r}P_i)\xrightarrow{\beta_*}(T,P_i)\to (T,S_i)\to \mathrm{Ext}^1_\Lambda(T,\mathbf{r}P_i)$$
$$\to \mathrm{Ext}^1_\Lambda(T,P_i)\to \mathrm{Ext}^1_\Lambda(T,S_i)\to 0.$$

Now the first statement follows from the fact that $(T,S_i)=0$.

Since Λ is triangular, the projective cover of $\mathbf{r}P_i$ does not have P_i as a direct summand. Now $\mathcal{T}(T)=\mathrm{Gen}(T)$ implies that $\mathbf{r}P_i\in\mathcal{T}(T)$. Hence $\mathrm{Ext}^1_\Lambda(T,\mathbf{r}P_i)=0$. It follows from the exact sequence above that $\mathrm{Ext}^1_\Lambda(T,P_i)$ $\cong \mathrm{Ext}^1_\Lambda(T,S_i)$, which is a simple Λ-module.

We now prove that $\mathrm{pd}\, S_i=1$. Since $\mathrm{gldim}\,\Gamma=2$ it follows from the sequence (I) that the following exact sequence is a projective resolution for the Γ-simple module $\mathrm{Ext}^1_\Lambda(T,P_i)$:

$$0 \to (T,\mathbf{r}P_i) \xrightarrow{(f\beta)_*} (T,P_{(X)}) \xrightarrow{\pi_*} (T,X) \to \mathrm{Ext}^1_\Lambda(T,P_i) \to 0$$

Hence $(T,\mathbf{r}P_i)$ is a projective Γ-module. Since $\mathbf{r}P_i \in \mathcal{T}(T)$ we have $\mathbf{r}P_i \in \mathrm{add}(T)$. From the triangularity of Γ it follows that (T,X) is not a direct summand of $(T,\mathbf{r}P_i)$. Therefore $\mathbf{r}P_i$ is a projective Λ-module. ∎

We now fix some more notation. We denote by $\widehat{S}_j$ the simple Γ-module $\mathrm{top}(T,P_j)$ for $j \neq i$ and by $\widehat{S}_X$ the simple Γ-module $\mathrm{top}(T,X)$. By the previous lemma, $\widehat{S}_X$ is isomorphic to $\mathrm{Ext}^1(T,S_i)$. Let $l_1,\dots,l_r,l_{r+1},\dots,l_t$ denote vertices which are the immediate predecessors of the vertex $i \in Q_\Lambda$, ordered in such a way that $P_{(X)} = \bigoplus_{s=1}^r P_{l_s}^{m_s}$ is the projective cover of X. We assume that $\mathbf{r}P_i = \bigoplus_{j=1}^m P_{v_j}$ is a decomposition of $\mathbf{r}P_i$ as direct sum of indecomposable projective modules and we consider the minimal almost split map $\beta : \mathbf{r}P_i \to P_i$ defined by $\beta_j : P_{v_j} \to P_i$ for $j = 1,\dots,m$, where β_j is multiplication by an arrow in Q_Λ.

Lemma 4.12. *With the hypotheses and notation of this subsection, let l_u be an immediate predecessor of the vertex i fixed above and $g : P_i \to P_{l_u}$ a linear map for some $u = 1,\dots,t$. Also, consider the map $\beta_j : P_{v_j} \to P_i$ for $j = 1,\dots,m$, defined above. For each j, if $g\beta_j$ is a non-zero map then it is a T-irreducible map.*

Proof. Suppose first that $g\beta_j$ factors through some projective module $P' \in \mathrm{add}(T)$, say $g\beta_j = fh$ for $f : P' \to P_{l_u}$ and $h : P_{v_j} \to P'$. We observe that $g\beta_j - fh$ defines a linear combination of paths in Λ which is a relation. Since Λ is monomial we see that each term of that linear combination is null. Hence, each component of the map $g\beta_j$ is a null function, a contradiction to our hypothesis.

If $g\beta_j$ factors through X, say $g\beta_j = f'h'$ for $f' : X \to P_{l_u}$ and $h' : P_{v_j} \to X$, then h' must factor through $P_{(X)}$ and we have the previous case. ∎

Corollary 4.13. (a) *If X is a Koszul module then the simple module $\widehat{S}_X$ is a quasi-Koszul module.*

(b) *Given any projective Λ-module P we have* $\mathrm{Ext}^1(T,P) \in \mathrm{add}(\widehat{S}_X)$.

Proof. If X is a Koszul module then, by Lemma 4.10, we have a linear resolution for X given by the short exact sequence

$$0 \to P_i \xrightarrow{f} P_{(X)} \xrightarrow{\pi} X \to 0,$$

with $P_{(X)} \in \mathrm{add}(T)$, and f a linear map. It follows from Lemma 4.12 that $f\beta$ is an essentially T-irreducible map. Since the exact sequence (II) is a projective resolution for the simple module $\mathrm{Ext}^1_\Lambda(T, S_i)$, Proposition 4.8 shows that $\widehat{S}_X$ is a quasi-Koszul module. Statement (b) follows from Lemma 4.11. ∎

REMARKS. (1) We have $(X, S_j) = 0$ for each $j \notin \{l_1, \ldots, l_r\}$. Hence the Γ-module (T, S_j) is a simple module.

(2) We have $\mathrm{Ext}^1_\Lambda(T, P_j) = \mathrm{Ext}^1_\Lambda(T, S_j) = 0$ since $P_j \in \mathrm{add}(T)$ and $S_j \in \mathcal{T}(T)$. Hence if we apply the functor $\mathrm{Hom}_\Lambda(T, -)$ to the short exact sequence $0 \to rP_j \to P_j \xrightarrow{\pi_j} S_j \to 0$ then we obtain the exact sequence

$$0 \to (T, \mathbf{r}P_j) \to (T, P_j) \xrightarrow{(\pi_j)_*} (T, S_j) \to \mathrm{Ext}^1_\Lambda(T, \mathbf{r}P_j) \to 0.$$

Since $\mathbf{r}(T, P_j) \cong (T, \mathbf{r}P_j)$ for each $j \neq i$ and (T, P_j) is an indecomposable projective module, the short sequence $0 \to (T, \mathbf{r}P_j) \to (T, P_j) \to \widehat{S}_j \to 0$ is exact.

(3) We recall from [9] and Remark (2) above that $\widehat{S}_j$ is a quasi-Koszul module if and only if $(T, \mathbf{r}P_j)$ is a quasi-Koszul module.

LEMMA 4.14. *Let $\Gamma = \mathrm{End}^{\mathrm{op}}_\Lambda(T)$ be the opposite endomorphism ring of the tilting module $T = X \oplus \bigoplus_{j\neq i} P_j$ over a quadratic, monomial, triangular algebra Λ. Let $f : Q \to P$ be a Λ-map with P and Q projective Λ-modules. If f is linear and $P \in \mathrm{add}(T)$ then f_* is an essentially linear map.*

Proof. It is enough to prove the statement for the case of P and Q indecomposable modules. We may also assume $P = P_j$ for $j \neq i$ since $P \in \mathrm{add}(T)$. We consider two cases:

CASE 1: Assume that $Q \in \mathrm{add}(T)$. Since f is a linear map and $P, Q \in \mathrm{add}(T)$ it follows from Lemma 4.4 that f is a T-irreducible map. Lemma 4.7 yields that the Γ-map $\mathrm{Hom}_\Lambda(T, f) : (T, Q) \to (T, P_j)$ is essentially linear.

CASE 2: Assume that $Q \notin \mathrm{add}(T)$. Since Q is an indecomposable projective Λ-module, we have $Q = P_i$. We consider the natural inclusion map $\beta : \mathbf{r}P_i \to P_i$. We observe that $\mathrm{Hom}_\Lambda(T, f\beta) : (T, \mathbf{r}P_i) \to (T, P_j)$ is a Γ-map given by $(f\beta)_* = \{(f\beta_j)_*\}_{j=1,\ldots,m}$ where $\beta_j : P_{v_j} \to P_i$ for $j = 1, \ldots, m$ are linear maps. It follows from Lemma 4.12 that $f\beta_j$ is an irreducible Λ-map for each $j = 1, \ldots, m$. Now Lemma 4.7 shows that $\mathrm{Hom}_\Lambda(T, f\beta)$ is an essentially linear map. We know by Lemma 4.11 that β_* is an isomorphism. Hence we may conclude that f_* is an essentially linear map. ∎

COROLLARY 4.15. *Let $\Gamma = \mathrm{End}^{\mathrm{op}}_\Lambda(T)$ be the opposite endomorphism ring of the tilting module $T = X \oplus \bigoplus_{j\neq i} P_j$ over a quadratic, monomial, triangular algebra Λ. Let $f : Q \to P$ be a Λ-map with P and Q projective Λ-modules. If f is linear then f_* is a direct sum of an essentially linear map and a split monomorphism.*

Proof. Using the previous lemma, it is enough to consider the case $P = P_i$; but then the result follows from Lemma 4.11. ■

We next present the main result of this section. We observe that the class of endomorphism rings considered in this statement includes the Brenner–Butler tilted algebras.

THEOREM 4.16. *Let Λ be a triangular finite-dimensional monomial quadratic algebra over the field k with* $\operatorname{gldim} \Lambda = 2$ *and $T = X \oplus \bigoplus_{j\neq i} P_j$ be a tilting Λ-module. Let $\Gamma = \operatorname{End}^{\mathrm{op}}_\Lambda(T)$ be the opposite endomorphism ring of T over Λ. Assume that Γ has global dimension 2. If X is a Koszul module then Γ is a quasi-Koszul algebra.*

Proof. It is enough to prove that $\mathbf{r}\Gamma$ is a quasi-Koszul module. Since $\mathbf{r}P_j$ is a Koszul module and has projective dimension at most one, we have a linear resolution of $\mathbf{r}P_j$ given by

$$(*)\qquad 0 \to P_{(1)} \xrightarrow{\varrho} P_{(0)} \xrightarrow{p_j} \mathbf{r}P_j \to 0,$$

with ϱ a linear map. Applying Corollary 4.13, we see that $\operatorname{Ext}^1_\Lambda(T, P_{(1)})$ is a semisimple module in $\operatorname{add}(\widehat{S}_X)$. Therefore all the submodules of $\operatorname{Ext}^1_\Lambda(T, P_{(1)})$ are also in $\operatorname{add}(\widehat{S}_X)$.

If we apply the functor $\operatorname{Hom}_\Lambda(T, -)$ to the short exact sequence $(*)$, we obtain the following exact sequence:

$$(\mathrm{III})\qquad 0 \to (T, P_{(1)}) \xrightarrow{\varrho_*} (T, P_{(0)}) \xrightarrow{(p_j)_*} (T, \mathbf{r}P_j) \to (\widehat{S}_X)^{n'} \to 0,$$

for some $n' \geq 0$.

We show first that $\operatorname{Im}(p_j)_*$ is a quasi-Koszul module. Since the sequence $(*)$ is a linear resolution, it follows from Corollary 4.15 that $\operatorname{Im}(p_j)_*$ is a quasi-Koszul module.

We now consider the following exact sequence:

$$0 \to \operatorname{Im}(p_j)_* \to (T, \mathbf{r}P_j) \to \widehat{S}_X^{n'} \to 0.$$

By Corollary 4.13, $\widehat{S}_X^{n'}$ a quasi-Koszul module. Lemma 4.14 tells us that $0 \to \operatorname{Im}(p_j)_* \to (T, \mathbf{r}P_j)$ is an essentially linear map hence it induces a monomorphism $\operatorname{top}(\operatorname{Im}(p_j)_*) \to \operatorname{top}(T, \mathbf{r}P_j)$. It follows from Proposition 5.3 of [9] that $(T, \mathbf{r}P_j)$ is a quasi-Koszul module. Hence $(T, \mathbf{r}P_j) = \mathbf{r}(T, P_j)$ is a quasi-Koszul module for every $j \neq i$. By Corollary 4.13, $\mathbf{r}(T, X)$ is also a quasi-Koszul module. Hence $\mathbf{r}\Gamma$ is a quasi-Koszul module. ■

We observe that the hypothesis that Λ is monomial is crucial in Theorem 4.16 above. It is easy to check that Lemma 4.12 does not hold if Λ is not a monomial algebra. It follows that $\widehat{S}_X$ is not a quasi-Koszul module in that case. We now give some examples of the class of endomorphism rings presented in Theorem 4.16. Examples 5 and 6 show that the quadratic property of Λ and the hypothesis that X be a Koszul module cannot be discarded in our main result. We also observe that we know of no example where Λ is a finite-dimensional k-algebra, T is a tilting module of the form $X \oplus \bigoplus_{j\neq i} P_j$ with X a Koszul module and $\Gamma = \operatorname{End}_\Lambda^{\mathrm{op}}(T)$ such that Γ is not actually Koszul (and not merely quasi-Koszul).

Example 5. Let Λ be the quiver algebra given by a quiver with relations as follows. The quiver Q is

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

with the relation $dcba = 0$. Let $T = \bigoplus_{j\neq 3} P_j \oplus \tau^- S_3$ be a tilting Λ-module and $\Gamma = \operatorname{End}_\Lambda^{\mathrm{op}}(T)$ the endomorphism ring of T over Λ.

Then Γ is given by the following quiver with relations:

$$\begin{tikzcd} & \widehat{3} \arrow[d, "\pi"] & & \\ 1 \arrow[r, "\bar{a}"] & 2 \arrow[r, "\bar{b}"] & 4 \arrow[r, "\bar{c}"] & 5 \end{tikzcd}$$

with the relations $\bar{c}\bar{b}\bar{a} = \bar{b}\pi = 0$. A minimal projective resolution for the Γ-simple module $\widehat{S}_1$ is given by the exact sequence

$$0 \to \widehat{P}_5 \xrightarrow{\bar{c}\bar{b}} \widehat{P}_2 \to \widehat{P}_1 \to \widehat{S}_1 \to 0.$$

Hence $\widehat{S}_1$ has no essentially linear resolution. Moreover Γ is not a quasi-Koszul algebra.

Example 6. Let Λ be the quiver algebra given by the quiver

$$\begin{tikzcd} & & 3 \arrow[dr, "c"] & & \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[rr, "d"] & & 4 \arrow[r, "e"] & 5 \end{tikzcd}$$

with the relations $ba = ed = 0$.

We consider a tilting module $T = (\bigoplus_{j\neq i} P_j) \oplus X$ with X given by the minimal projective resolution $0 \to P_4 \xrightarrow{cb} P_2 \to X \to 0$. Hence X is not a Koszul module. We see that the projective resolution of $\widehat{S}_X$ is given by

$$0 \to (T, P_5) \xrightarrow{(ec)_* b_*} (T, P_2) \to (T, X) \to \widehat{S}_X \to 0.$$

Therefore $\widehat{S}_X$ is not a quasi-Koszul module.

EXAMPLE 7. Let Λ be the quiver algebra of Example 6. We consider a tilting module $T = (\bigoplus_{j \neq i} P_j) \oplus X$ with X given by the minimal projective resolution $0 \to P_4 \xrightarrow{d} P_2 \to X \to 0$. Since X is a Koszul module, it follows that $\Gamma = \mathrm{End}_{\Lambda}^{\mathrm{op}}(T)$ is a quasi-Koszul algebra. The quiver with relations of Γ is given as follows:

$$\begin{array}{ccccccc} & & \widehat{4} & & & & \\ & & \downarrow \pi & & & & \\ 1 & \xrightarrow{\bar{a}} & 2 & \xrightarrow{\bar{b}} & 3 & \xrightarrow{\overline{ec}} & 5 \end{array}$$

with the relation $\overline{b}\overline{a} = 0$. We note that Γ is a Koszul algebra with global dimension 2.

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