SOME REMARKS ON DUALITY OF
STATIONARY SEQUENCES

BY

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Abstract. The paper clarifies the connection between Urbanik’s and Miamee and
Pourahmadi’s concepts of duality for univariate weakly stationary random sequences.
Some of Urbanik’s results are proved in an alternative way and at the same time gen-
eralized to the multivariate case.

1. Introduction. In [8] K. Urbanik introduced a concept of duality for
univariate weakly stationary random sequences over the group of integers
and studied relations between linear prediction problems of the sequence
in question and its dual. His main tool was the one-sided moving average
representation of a completely non-deterministic sequence and his proofs
were essentially “spectral free”.

In [5] A. G. Miamee and M. Pourahmadi introduced a notion of du-
alsequences of univariate weakly stationary random sequences (in fact, even of
certain harmonizable stable sequences) which have spectral densities whose
inverses are integrable. Their method uses the spectral domain of a sequence
and is also applicable if the group of integers is replaced by an arbitrary
discrete abelian group. Miamee and Pourahmadi’s method was generaliz-
to the multivariate case in [2].

It turns out that the duality concept of Urbanik reduces to that of Mi-
amee and Pourahmadi if the spectral measure of the sequence has the above
mentioned properties. The aim of the present short note is to point out
this connection and this way obtain alternative proofs of some of Urbanik’s
results. At the same time we will study multivariate processes where some
results of Makagon and Weron (see [3], [4]) will be helpful.

2. Preliminaries. Let $G$ be an (additively written) discrete abelian
group, $\Gamma$ its dual group, and $\sigma$ the normalized Haar measure of $\Gamma$. For a
subset $Q$ of $G$ denote by $Q^c$ the complement $Q^c := G \setminus Q$. Set $G_g := \{g\}^c,
g \in G$. If $Q \subseteq G_0$, let $Q^*$ be the complement $Q^* := G_0 \setminus Q$.

2000 Mathematics Subject Classification: Primary 60G10; Secondary 60G25.
For \( q \in \mathbb{N} \), let \( \mathcal{M}_q \) be the algebra of \( q \times q \)-matrices with entries from the field of complex numbers \( \mathbb{C} \). If \( A \in \mathcal{M}_q \), the symbol \( A^+ \) stands for the Moore–Penrose inverse of \( A \) (see [6, p. 406]).

The class of all \( q \)-variate weakly stationary random processes over \( G \) will be denoted by \( \mathcal{S}_q(G) \). For some basic facts concerning such processes we refer to [7].

Let \( X := \{X_g\}_{g \in G} \in \mathcal{S}_q(G) \). If \( Q \subseteq G \), denote by \( [X,Q] \) the left Hilbert-\( \mathcal{M}_q \)-module spanned by \( X_g, g \in Q \). Set \( [X,G] =: [X] \). If \( L \) is a submodule of \( [X] \), let \( L^\perp \) be the (Gramian) orthogonal complement of \( L \) with respect to \( [X] \). For a subset \( Q \) of \( G \), it is an important problem of linear prediction theory to calculate the orthogonal projection \( \hat{X}_0(Q) \) of \( X_0 \) onto \( [X,Q] \) as well as the prediction error matrix

\[
\Delta(X,Q) := \langle X_0 - \hat{X}_0(Q), X_0 - \hat{X}_0(Q) \rangle.
\]

Here \( \langle \cdot, \cdot \rangle \) stands for the Gramian matrix.

We recall that a process \( X \) of \( \mathcal{S}_q(G) \) is said to be \( J_0 \)-regular if

\[
\bigcap_{g \in G} [X,G_g] = \{0\}
\]

and \( J_0 \)-singular if

\[
[X,G_g] = [X] \quad \text{for all } g \in G
\]

(cf. [7, Definition 2.10] and [4, p. 133]).

**Lemma 2.1** (cf. [7, Theorem 2.13]). For \( X \in \mathcal{S}_q(G) \), there exists a unique Wold decomposition \( X = U + V \) into a \( J_0 \)-regular process \( U \) and a \( J_0 \)-singular process \( V \).

Let \( F \) be the (non-stochastic) spectral measure of \( X \) and \( F' \) the derivative of its absolutely continuous part with respect to \( \sigma \). We will say that \( X \) has a spectral density if \( F \) is absolutely continuous and in that case \( F' \) is said to be the spectral density of \( X \).

**Lemma 2.2** (cf. [4, Theorem 5.3]). A process \( X \) of \( \mathcal{S}_q(G) \) is \( J_0 \)-regular if and only if it has a spectral density \( F' \) with the following two properties:

(i) The range of \( F' \) is constant \( \sigma \)-a.e.

(ii) The integral \( \int \Gamma F' \, d\sigma \) exists.

**3. Dual processes.** Following Urbanik ([8, p. 155]) we introduce a concept of a dual process. Note that our definitions of the dual process as well as of the sets \( K \) and \( K_0 \) below slightly differ from Urbanik’s. Let \( P \) denote the orthogonal projector in \( \mathbb{C}^q \) onto the range of \( \Delta(X,G_0) \).

**Definition 3.1.** Let \( X \in \mathcal{S}_q(G) \). A process \( X^* \) of \( \mathcal{S}_q(G) \) is called the dual process of \( X \) if \( [X^*] \subseteq [X], \langle X^*_g, X_h \rangle = 0 \) for \( g \neq h \) and \( \langle X^*_g, X_g \rangle = P, g, h \in G \).
The dual process exists and is unique. It can be described more explicitly. In fact, if $Y_g$ denotes the orthogonal projection of $X_g$ onto $[X, G^\perp_g]$ and $Z_g := (Y_g, Y_g)^+ Y_g$, $g \in G$, then the process $Z := \{Z_g\}_{g \in G}$ coincides with $X^*$. From results of Makagon and Weron ([3, p. 49 and formula (4.12)]) we now obtain the following assertion.

**Lemma 3.2.** Let $F$ be the spectral measure of the process $X$ of $\mathcal{S}_q(G)$. Then $X^*$ has the spectral density $PF^{r+}P$.

**Definition 3.3.** Set

$$K := \{X \in \mathcal{S}_q(G) : [X] = \{0\} \text{ or } [X^*] \neq \{0\}\},$$

$$K_0 := \{X \in K : [X^*] = [X]\}.$$

**Lemma 3.4.** A process $X$ of $\mathcal{S}_q(G)$ belongs to $K_0$ if and only if it is $\mathcal{J}_0$-regular.

**Proof.** First note that $Z_g$ spans $[X, G^\perp_g]$, $g \in G$. Hence, $[X^*] = [Z] = \{0\}$ if and only if $\bigcap_{g \in G} [X, G_g] = \{0\}$.

**Proposition 3.5.** Let $X \in \mathcal{S}_q(G)$. If $X^*$ is its dual, then $X^* \in K_0$.

**Proof.** Let $U$ be the $\mathcal{J}_0$-regular part of the Wold decomposition of $X$. By Lemma 3.4, $U \in K_0$. Hence, $U^* \in K_0$ (cf. [8, Proposition 2.1]). On the other hand, it is obvious that $X^* = U^*$.

Let $X \in K_0$, $[X] \neq \{0\}$. Using Lemmas 3.4 and 2.2 and considering instead of $X$ the process $TXT^*$ for a suitable unitary $T$ of $\mathcal{M}_q$, without loss of generality, we can assume that $X$ has a spectral density of the form

$$\begin{pmatrix}
F_{11}' & 0 \\
0 & 0
\end{pmatrix}$$

where $F_{11}'$ is invertible $\sigma$-a.e. and $F_{11}^{r-1}$ is integrable with respect to $\sigma$. Then by Lemma 3.2 we conclude that the spectral density of $X^*$ is equal to

$$\begin{pmatrix}
F_{11}^{r-1} & 0 \\
0 & 0
\end{pmatrix}$$

$\sigma$-a.e. Of course, we can confine ourselves to the upper left corner and thus apply results of [2].

Let $X \in \mathcal{S}_q(G)$ and $Q \subseteq G$. It is obvious that $[X, Q] \subseteq [X^*, Q^c]^\perp$. We denote by $\Lambda(X)$ the set of all subsets $Q$ of $G$ such that $[X, Q] = [X^*, Q^c]^\perp$.

**Theorem 3.6.** Let $X \in \mathcal{S}_q(G)$ and $Q \subseteq G_0$. Then

$$\Delta(X, Q) \geq \Delta(X^*, Q^*)^+.$$  \hspace{1cm} (3.1)

If, moreover, $X \in K_0$ and $Q \in \Lambda(X)$, then

$$\Delta(X, Q) = \Delta(X^*, Q^*)^+.$$  \hspace{1cm} (3.2)
Proof. Let $U$ be the $J_0$-regular part of the Wold decomposition of $X$. Then $\Delta(X, Q) \geq \Delta(U, Q)$. Since $U \in K_0$, we have $\Delta(U, Q) \geq \Delta(U^*, Q^*)^+$ by Theorem 2.2 of [2]. Using these two inequalities and the fact that $X^* = U^*$, we immediately get (3.1). Equality (3.2) is a consequence of Theorem 2.2 of [2].

Theorem 3.6 shows that conditions for $Q$ belonging to $\Lambda(X)$ are of interest. Some results can be found in [5] and [8] if $q = 1$ and in [2] if $q$ is an arbitrary positive integer. Moreover, in [1], [2] and [5] duality is used to conclude some explicit linear prediction results.

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Received 1 October 2001 (4130)