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ON THE ORBIT OF THE CENTRALIZER OF A MATRIX

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Abstract. Let A be a complex $n\times n$ matrix. Let $\{A\}'$ be its commutant in $M_n(\mathbb{C})$, and $C(A)$ be its centralizer in $\mathrm{GL}(n,\mathbb{C})$. Consider the standard $C(A)$-action on $\mathbb{C}^n$. We describe the $C(A)$-orbits via invariant subspaces of $\{A\}'$. For example, we count the number of $C(A)$-orbits as well as that of invariant subspaces of $\{A\}'$.

1. Introduction. Let $M_{n\times m}(\mathbb{C})$ be the $n\times m$ complex matrices, and $M_n(\mathbb{C})=M_{n\times n}(\mathbb{C})$. Let $\mathrm{GL}(n,\mathbb{C})$ be the invertible matrices in $M_n(\mathbb{C})$. Fix $A\in M_n(\mathbb{C})$, and let

$$C(A)=\{T\in \mathrm{GL}(n,\mathbb{C}) : TA=AT\}$$

denote the centralizer of A. For any vector $x\in\mathbb{C}^n$,

$$\mathcal{O}_x=\{Tx : T\in C(A)\}\subset\mathbb{C}^n$$

is the orbit of the standard action of $C(A)$ on $\mathbb{C}^n$. The purpose of this paper is to study the number of $C(A)$-orbits in $\mathbb{C}^n$.

Consider the orbit space

$$\mathcal{O}_{C(A)}=\{\mathcal{O}_x : x\in\mathbb{C}^n\}.$$

It is clear that $\mathcal{O}_{C(A)}$ is a partition of the whole vector space $\mathbb{C}^n$. As mentioned above, this article studies the number of elements in $\mathcal{O}_{C(A)}$.

We show that the problem of studying the number of $C(A)$-orbits for an arbitrary square matrix A reduces to the case where A is nilpotent. First, we study the properties of $\mathcal{O}_x$. Let

$$\{A\}'=\{T\in M_n(\mathbb{C}) : TA=AT\}.$$

For a vector x, let

$$\mathcal{M}_x=\{Tx : T\in\{A\}'\}.$$

The linear space $\mathcal{M}_x$ is closed in $\mathbb{C}^n$. Also, let $\overline{\mathcal{O}}_x\subset\mathbb{C}^n$ denote the closure of $\mathcal{O}_x$ in $\mathbb{C}^n$.

LEMMA 1.1. *For any vector x, $\overline{\mathcal{O}}_x=\mathcal{M}_x$.*

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Proof. Since $\mathcal{O}_x \subset \mathcal{M}_x$ and $\mathcal{M}_x$ is closed, it follows that $\overline{\mathcal{O}}_x \subset \mathcal{M}_x$. On the other hand, for any $T \in \{A\}'$, there exists a sequence $\{T_n\}_{n=1}^{\infty} \subset C(A)$ with $T_n \to T$. Hence $\mathcal{M}_x \subset \overline{\mathcal{O}}_x$, which completes the proof. ■

PROPOSITION 1.2. *If* $\overline{\mathcal{O}}_x = \overline{\mathcal{O}}_y$, *then* $\mathcal{O}_x = \mathcal{O}_y$.

Proof. Observe that $\overline{\mathcal{O}}_y$ is a disjoint union of $C(A)$-orbits. So we may write $\overline{\mathcal{O}}_y = \mathcal{O}_y \cup \bigcup_\alpha \mathcal{O}_{y_\alpha}$, where $\mathcal{O}_y$ and all $\mathcal{O}_{y_\alpha}$ are distinct. Further, the dimension of each manifold $\mathcal{O}_{y_\alpha}$ is less than the dimension of $\mathcal{O}_y$.

Suppose that $\mathcal{O}_x \neq \mathcal{O}_y$. Then from $\mathcal{O}_x \subset \overline{\mathcal{O}}_y$, it follows that $\mathcal{O}_x = \mathcal{O}_{y_\alpha}$ for some α. Since $\dim \mathcal{O}_x < \dim \mathcal{O}_y$, the closure $\overline{\mathcal{O}}_x$ cannot contain $\mathcal{O}_y$, which contradicts $\overline{\mathcal{O}}_x = \overline{\mathcal{O}}_y$, and the proposition follows. ■

In view of the previous lemma and proposition, we obtain a partial order $\preceq$ in the following definition.

DEFINITION 1.3. (a) Let $\preceq$ be the partial order on $\mathcal{O}_{C(A)}$ defined by $\mathcal{O}_x \preceq \mathcal{O}_y$ if and only if $\overline{\mathcal{O}}_x \subset \overline{\mathcal{O}}_y$, where $\mathcal{O}_x, \mathcal{O}_y \in \mathcal{O}_{C(A)}$.

(b) Two orbit spaces $\mathcal{O}_{C(A)}$ and $\mathcal{O}_{C(B)}$ are said to be *isomorphic* if there exists an order-preserving bijection between them.

The following two propositions are easy to prove, yet important.

PROPOSITION 1.4. *Let* $A, B \in M_n(\mathbb{C})$. *If* $A = S^{-1}BS$ *for some* $S \in \mathrm{GL}(n, \mathbb{C})$, *then* $C(A) = \{S^{-1}RS : R \in C(B)\}$. *Hence* $\mathcal{O}_{C(A)}$ *and* $\mathcal{O}_{C(B)}$ *are isomorphic.*

For any $A \in M_n(\mathbb{C})$, we use $\sigma(A)$ to denote the spectrum of A. Also, we write $\sharp B$ for the number of elements in a set B.

PROPOSITION 1.5. *Let* $A \in M_n(\mathbb{C})$ *and* $B \in M_m(\mathbb{C})$. *If* $\sigma(A)$ *and* $\sigma(B)$ *are disjoint*, *then* $C(A \oplus B) = \{X \oplus Y : X \in C(A) \text{ and } Y \in C(B)\}$. *Moreover*, $\mathcal{O}_{C(A\oplus B)} = \{\mathcal{O}_x \oplus \mathcal{O}_y : \mathcal{O}_x \in \mathcal{O}_{C(A)} \text{ and } \mathcal{O}_y \in \mathcal{O}_{C(B)}\}$. *Hence*, $\sharp\mathcal{O}_{C(A\oplus B)} = \sharp\mathcal{O}_{C(A)} \times \sharp\mathcal{O}_{C(B)}$.

For any $A \in M_n(\mathbb{C})$, we use $J(A)$ to denote the Jordan form of A, which is similar to A, that is, there exists $S \in \mathrm{GL}(n, \mathbb{C})$ such that $S^{-1}AS = J(A)$. By Proposition 1.4, $\mathcal{O}_{C(J(A))}$ and $\mathcal{O}_{C(A)}$ are isomorphic. Consequently, by Proposition 1.5, it suffices to consider the case of a one-eigenvalue matrix. In fact, it may be reduced to the case of nilpotent matrices.

If we view $C(A)$ as a Lie group, then $\{A\}'$ is its Lie algebra. In Section 2, we will prove the following Main Theorem by transforming our setting from the Lie group $C(A)$ to the Lie algebra $\{A\}'$.

MAIN THEOREM. *Let* J_n *be the* $n \times n$ *Jordan block and let*

$$J = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} J_{n_i} \in M_n(\mathbb{C}),$$

where $n_1 > \ldots > n_m \geq 1$. *Set* $n_{m+1} = 0$. *Then*

$$\sharp\mathcal{O}_{C(J)} = \prod_{i=1}^{m}(n_i - n_{i+1} + 1).$$

We consider the problem of determining $\sharp\mathcal{O}_{C(A)}$ for general matrices A in Corollary 2.14. Finally, we consider the same problem for real matrices in Section 3.

2. Nilpotent matrices. Let J_n be the standard $n \times n$ Jordan block. That is,

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in M_n(\mathbb{C}). \tag{1}$$

Let $n_1 > \ldots > n_m \geq 1$, $k_1, \ldots, k_m \in \mathbb{N}$, and

$$J = \bigoplus_{i=1}^{m} \bigoplus_{s=1}^{k_i} J_{n_i}. \tag{2}$$

In this section, we want to study $\mathcal{O}_{C(J)}$. For a given matrix J, we may compute $C(J)$ directly, but this may not help to understand $\mathcal{O}_{C(J)}$ because the structure of $C(J)$ is too complex. We provide an example as follows.

Example 2.1. Let $J = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. By direct computation, every element in $C(J)$ is of the form

$$\begin{bmatrix} a & e & b & f \\ 0 & a & 0 & b \\ c & g & d & h \\ 0 & c & 0 & d \end{bmatrix}$$

for some complex numbers $a, b, \ldots, h$.

Let J be defined as in (2). The treatment of $C(J)$ can be difficult when elements in $C(J)$ are not block-upper-triangular. To remedy this, we will transform J to $\widetilde{J}$ via a unitary equivalence such that every element in $C(\widetilde{J})$ is block-upper-triangular. We note that $\mathcal{O}_{C(J)}$ and $\mathcal{O}_{C(\widetilde{J})}$ are isomorphic by Proposition 1.4.

Lemma 2.2. *Let* J *as in* (2) *be a matrix with respect to the basis*

$$\begin{aligned} F = \{&f(n_1,1,1), f(n_1,1,2), \ldots, f(n_1,1,n_1), \\ &f(n_1,2,1), f(n_1,2,2), \ldots, f(n_1,2,n_1), \ldots, f(n_1,k_1,n_1), \\ &f(n_2,1,1), f(n_2,1,2), \ldots, f(n_2,1,n_2), \\ &f(n_2,2,1), \ldots, f(n_2,k_2,n_2), \\ &f(n_3,1,1), \ldots, f(n_m,k_m,1), \ldots, f(n_m,k_m,n_m)\}. \end{aligned}$$

Consider the following reordering of F:

$$\begin{aligned}E = \{&f(n_1,1,1), f(n_1,2,1), \ldots, f(n_1,k_1,1),\\ &f(n_2,1,1), \ldots, f(n_2,k_2,1), \ldots,\\ &f(n_m,1,1), \ldots, f(n_m,k_m,1),\\ &f(n_1,1,2), f(n_1,2,2), \ldots, f(n_m,k_m,2), \ldots,\\ &f(n_1,1,n_1), \ldots, f(n_1,k_1,n_1)\}.\end{aligned}$$

Then the matrix J written in the basis E is

$$\widetilde{J} = \begin{bmatrix} 0_{u_1} & T_1 & & \\ & 0_{u_2} & \ddots & \\ & & \ddots & T_{n_1-1} \\ & & & 0_{u_{n_1}} \end{bmatrix} \quad on \bigoplus_{j=1}^{n_1} \mathbb{C}^{u_j}, \tag{3}$$

where

$$T_j = \begin{bmatrix} I_{u_{j+1}} \\ 0_{(u_j-u_{j+1})\times u_{j+1}} \end{bmatrix} \in M_{u_j\times u_{j+1}}(\mathbb{C}), \tag{4}$$

and

$$\begin{aligned} u_1 = \ldots = u_{n_m} &= k_1 + \ldots + k_m,\\ u_{n_m+1} = \ldots = u_{n_{m-1}} &= k_1 + \ldots + k_{m-1},\\ &\vdots\\ u_{n_2+1} = \ldots = u_{n_1} &= k_1. \end{aligned} \tag{5}$$

Instead of considering J as in (2), we consider $\widetilde{J}$ as in (3), because every element in $C(\widetilde{J})$ is block-upper-triangular. We explain it more clearly as follows.

THEOREM 2.3. *Let $\widetilde{J}$ be as in (3). Then for any $X \in C(\widetilde{J})$ we have $X = [X_{i,j}]_{i,j=1}^{n_1}$ on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{u_i}$ with $X_{i,j} = 0$ for $i > j$, and for $i \le j$,*

$$X_{i,j} = \begin{bmatrix} X_{i+1,j+1} & A_{i,j} \\ 0 & B_{i,j} \end{bmatrix} \in M_{u_i\times u_j}(\mathbb{C}) \tag{6}$$

for some $A_{i,j} \in M_{u_{i+1}\times(u_j-u_{j+1})}(\mathbb{C})$ and $B_{i,j} \in M_{(u_i-u_{i+1})\times(u_j-u_{j+1})}(\mathbb{C})$.

Proof. For any $X \in C(\widetilde{J})$, we know that

$$X\widetilde{J} = \widetilde{J}X. \tag{7}$$

By examining the first column of (7), we see that $X_{2,1}, X_{3,1}, \ldots, X_{n_1,1}$ are all zero matrices. Then examining the second column of (7), we see that so are $X_{3,2}, X_{4,2}, \ldots, X_{n_1,2}$. By induction, $X_{j+1,j}, \ldots, X_{n_1,j}$ are all zero matrices.

Therefore, $X_{i,j} = 0$ whenever $i > j$ and so X is block-upper-triangular with respect to $\bigoplus_{j=1}^{n_1} \mathbb{C}^{u_j}$. Now, we only have to consider $X_{i,j}$ for $i \le j$. By (7) again, we have

$$X_{i,j}T_j = T_{i+1}X_{i+1,j+1}.$$

Therefore, for each $1 \le j \le n_1$, $X_{1,j}, X_{2,j+1}, \ldots, X_{n_1-j+1,n_1}$ are related and (6) follows by direct computation. ∎

However, $C(\widetilde{J})$ is still too complicated for us to find its orbit space. So we want to further simplify the setting. We first obtain two corollaries of Lemma 2.2 and Theorem 2.3.

COROLLARY 2.4. *Let $B = \bigoplus_{i=1}^{m} J_{n_i}$, where $n_1 > \ldots > n_m \ge 1$. The matrix B is with respect to the basis $F = \{f(n_1,1), f(n_1,2), \ldots, f(n_1,n_1), f(n_2,1), f(n_2,2), \ldots, f(n_2,n_2), \ldots, f(n_m,1), f(n_m,2), \ldots, f(n_m,n_m)\}$. Consider the following reordering E of F : $\{f(n_1,1), f(n_2,1), \ldots, f(n_m,1), f(n_1,2), f(n_2,2), \ldots, f(n_m,2), \ldots, f(n_1,n_1)\}$. Then the matrix B written in the basis E is*

$$\widetilde{B} = \begin{bmatrix} 0_{t_1} & B_1 & & \\ & 0_{t_2} & \ddots & \\ & & \ddots & B_{n_1-1} \\ & & & 0_{t_{n_1}} \end{bmatrix} \quad \text{on } \bigoplus_{j=1}^{n_1} \mathbb{C}^{t_j}, \tag{8}$$

where

$$B_j = \begin{bmatrix} I_{t_{j+1}} \\ 0_{(t_j - t_{j+1}) \times t_{j+1}} \end{bmatrix} \in M_{t_j \times t_{j+1}}(\mathbb{C}), \tag{9}$$

and

$$\begin{aligned} t_1 = \ldots = t_{n_m} &= m, \\ t_{n_m+1} = \ldots = t_{n_{m-1}} &= m-1, \\ &\vdots \\ t_{n_2+1} = \ldots = t_{n_1} &= 1. \end{aligned} \tag{10}$$

COROLLARY 2.5. *If $Y \in C(\widetilde{B})$, then $Y = [Y_{i,j}]_{i,j=1}^{n_1}$ on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{t_i}$ with $Y_{i,j} = 0$ for $i > j$, and for $i \le j$,*

$$Y_{i,j} = \begin{bmatrix} Y_{i+1,j+1} & C_{i,j} \\ 0 & D_{i,j} \end{bmatrix} \in M_{t_i \times t_j}(\mathbb{C}) \tag{11}$$

for some $C_{i,j} \in M_{t_{i+1} \times (t_j - t_{j+1})}(\mathbb{C})$ and $D_{i,j} \in M_{(t_i - t_{i+1})) \times (t_j - t_{j+1})}(\mathbb{C})$.

We now want to prove that $\mathcal{O}_{C(\widetilde{J})}$ and $\mathcal{O}_{C(\widetilde{B})}$ are isomorphic. Consider $\widetilde{J}$ in (3) as $[J_{i,j}]_{i,j=1}^{n_1}$ on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{u_i}$, and $\widetilde{B}$ in (8) as $[B_{i,j}]_{i,j=1}^{n_1}$ on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{t_i}$.

It is easy to see that $[J_{i,j}]_{i,j=1}^{n_1}$ and $[B_{i,j}]_{i,j=1}^{n_1}$ have the following similar structures:

(1) Only for $1 \leq i \leq n_1 - 1$, $J_{i,i}$, $J_{i,i+1}$, $B_{i,i}$, $B_{i,i+1}$, J_{n_1,n_1} and B_{n_1,n_1} could be nonzero matrices. The other entries of $[J_{i,j}]_{i,j=1}^{n_1}$ and $[B_{i,j}]_{i,j=1}^{n_1}$ are all zero matrices.

(2) For each $1 \leq i \leq n_1$, both $J_{i,i}$ and $B_{i,i}$ are scalar matrices.

(3) For each $1 \leq i \leq n_1 - 1$, both $J_{i,i+1}$ and $B_{i,i+1}$ are of the form $\begin{bmatrix} I \\ 0 \end{bmatrix}$.

By Theorem 2.3, we have a clear picture of elements in $C(\widetilde{J})$. For $X \in C(\widetilde{J})$, X is block-upper-triangular with respect to $\bigoplus_{j=1}^{n_1} \mathbb{C}^{u_j}$. In addition, if $\mathcal{P} \in \mathcal{O}_{C(\widetilde{J})}$, then $\mathcal{P} = \bigoplus_{j=1}^{n_1} \mathcal{P}_j$, where each $\overline{\mathcal{P}}_j$ is either 0 or $\mathbb{C}^{k_j}$. Similarly, by Corollary 2.5, for $Y \in C(\widetilde{B})$, Y is upper-triangular with respect to $\bigoplus_{j=1}^{n_1} \mathbb{C}^{t_j}$. By the structure of $C(\widetilde{B})$, if $\mathcal{Q} \in \mathcal{O}_{C(\widetilde{B})}$, then $\mathcal{Q} = \bigoplus_{i=1}^{n_1} \mathcal{Q}_i$, where each $\overline{\mathcal{Q}}_i$ is either 0 or $\mathbb{C}$. These observations help us to prove the following lemma. Recall from Definition 1.3 that two orbit spaces are isomorphic if and only if there exists an order-preserving bijection between them.

LEMMA 2.6. *The orbit spaces $\mathcal{O}_{C(\widetilde{J})}$ and $\mathcal{O}_{C(\widetilde{B})}$ are isomorphic.*

Proof. Define $\phi : \mathcal{O}_{C(\widetilde{J})} \to \mathcal{O}_{C(\widetilde{B})}$ by $\phi(\mathcal{P}) = \mathcal{Q}$, where $\mathcal{P} = \bigoplus_{i=1}^{n_1} \mathcal{P}_i$, $\mathcal{Q} = \bigoplus_{i=1}^{n_1} \mathcal{Q}_i$ with $\overline{\mathcal{Q}}_i = \mathbb{C}$ if $\overline{\mathcal{P}}_i = \mathbb{C}^{k_i}$ and $\mathcal{Q}_i = 0$ if $\mathcal{P}_i = 0$. By Proposition 1.2, ϕ is well defined. In addition, it is clear that ϕ is an order-preserving bijection. Therefore, $\mathcal{O}_{C(\widetilde{J})}$ and $\mathcal{O}_{C(\widetilde{B})}$ are isomorphic. ∎

By Lemma 2.6, from now on it suffices to consider $\mathcal{O}_{C(\widetilde{B})}$. Recall that $\{\widetilde{B}\}'$ consists of all matrices which commute with $\widetilde{B}$. As is often the case in Lie group theory, we transfer the problem about the Lie group $C(\widetilde{B})$ to one about its Lie algebra $\{\widetilde{B}\}'$. As usual, a linear space $\mathcal{M}$ is said to be invariant for an algebra $\{\widetilde{B}\}'$ if $T\mathcal{M} \subset \mathcal{M}$ for any $T \in \{\widetilde{B}\}'$. Consider the lattice of invariant subspaces of $\{\widetilde{B}\}'$:

$$\operatorname{Lat}\{\widetilde{B}\}' = \{\mathcal{M} \subseteq \mathbb{C}^n : \mathcal{M} \text{ is an invariant subspace of } \{\widetilde{B}\}'\}.$$

As a side remark, for any matrix A, the study of Lat $\{A\}$ and Lat $\{A\}'$ is of interest. Brickman and Fillmore [1] study Lat $\{A\}$ under many different conditions. [3] considers more general topics in invariant subspaces. In this paper we focus on Lat $\{\widetilde{B}\}'$.

For a vector x, recall that $\mathcal{M}_x = \{Tx : T \in \{\widetilde{B}\}'\}$, which is the smallest invariant subspace of $\{\widetilde{B}\}'$ containing x. Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be subsets of a given vector space. As usual, we use $\bigvee_{i=1}^{m} \mathcal{A}_i$ to denote the linear span of

$\mathcal{A}_1, \ldots, \mathcal{A}_m$. In general, it is easy to see that for a square matrix T, and $\mathcal{M} \in \operatorname{Lat}\{T\}'$, $\mathcal{M}$ is $\bigvee_{i=1}^{m} \mathcal{M}_{x_i}$ for some vectors $x_1, \ldots, x_m$.

Recall that $\widetilde{B}$ is taken with respect to the basis E which is defined in Corollary 2.4. For the convenience of computing, we rename the element $f(n_k, l) \in E$ as $e(l, k)$ from now on.

By Corollary 2.5, each matrix in $\{\widetilde{B}\}'$ is upper-triangular. For each $\mathcal{M} \in \operatorname{Lat}\{\widetilde{B}\}'$, $\mathcal{M} = \bigvee_{l=1}^{k} \mathcal{M}_{e(i_l, j_l)}$ for some $e(i_1, j_1), \ldots, e(i_k, j_k) \in E$. Since $\bigvee_{l=1}^{k} \mathcal{M}_{e(i_l,j_l)} = \mathcal{M}_{\sum_{l=1}^{k} e(i_l,j_l)}$, we conclude that $\mathcal{M} = \mathcal{M}_{\sum_{l=1}^{k} e(i_l,j_l)}$. Note that $\sharp \operatorname{Lat}\{\widetilde{B}\}'$ is not equal to the number of subsets of E, because different subsets of E may lead to the same invariant subspace. For example, $\mathcal{M}_{e(1,2)} = \mathcal{M}_{e(1,1)+e(1,2)}$.

LEMMA 2.7. *Let* $\{t_l\}_{l=1}^{n_1}$ *be as in* (10). *For each* $\mathcal{M} \in \operatorname{Lat}\{\widetilde{B}\}'$, *there exist* $e(i_1, j_1), e(i_2, j_2), \ldots, e(i_k, j_k) \in E$, *with* $1 \le i_1 < \ldots < i_k \le n_1$, *and* $e(i_{l-1}, j_{l-1}) \notin \mathcal{N}(i_l, j_l)$ *for each* $l = 2, \ldots, k$, *such that* $\mathcal{M} = \mathcal{M}_{\sum_{l=1}^{k} e(i_l,j_l)}$. *In addition,*

$$\begin{aligned}\mathcal{M} = {} & (\mathbb{C}^{t_{n_{j_1}-i_1+1}} \oplus 0) \oplus (\mathbb{C}^{t_{n_{j_1}-i_1+2}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_1}}} \oplus 0) \\ & \oplus (\mathbb{C}^{t_{n_{j_2}-i_2+i_1+1}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_2}}} \oplus 0) \oplus \ldots \\ & \oplus (\mathbb{C}^{t_{n_{j_k}-i_k+i_{k-1}+1}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_k}}} \oplus 0)\end{aligned}$$

on $\bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}$, *and* $(t_{n_{j_1}-i_1+1}, t_{n_{j_1}-i_1+2}, \ldots, t_{n_{j_k}})$ *is a subsequence of* $\{t_l\}_{l=1}^{n_1}$.

Proof. For $X \in \{\widetilde{B}\}'$, let $X(i, j)$ be the (i, j)th column of X. By Theorem 2.3, for each $1 \le k \le m$,

$$X(n_k, t_{n_k}) = X(n_k, k) = \bigoplus_{j=1}^{n_k} X(n_k, k, j) \oplus \bigoplus_{j=n_k+1}^{n_1} 0 \tag{12}$$

on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{t_i}$, where each $X(n_k, k, j)$ is a $t_j \times 1$ complex matrix:

$$X(n_k, k, j) = \begin{bmatrix} a_{(n_k,k)}^{(j,1)} \\ \vdots \\ a_{(n_k,k)}^{(j,t_j)} \end{bmatrix} \in M_{t_j \times 1}(\mathbb{C}).$$

In addition, by Theorem 2.3 again, for each $1 \le i \le n_1$, $1 \le j \le t_{n_i}$,

$$X(i, j) = \Big(\bigoplus_{l=1}^{i} X(n_j, j, n_j - i + l) \oplus 0 \Big) \oplus \bigoplus_{l=i+1}^{n_1} 0 \tag{13}$$

on $\bigoplus_{i=1}^{n_1} \mathbb{C}^{t_i}$.

Let $\mathcal{N}(i,j)$ be the column space of $X(i,j)$. That is, by (13),

$$\mathcal{N}(i,j) = \Big(\bigoplus_{l=1}^{i} \mathbb{C}^{t_{n_j-i+l}} \oplus 0\Big) \oplus \bigoplus_{l=i+1}^{n_1} 0 \subset \bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}. \tag{14}$$

Since $\{\widetilde{B}\}'$ is an algebra, $\mathcal{N}(i,j) \in \operatorname{Lat}\{\widetilde{B}\}'$. In fact, $\mathcal{N}(i,j) = \mathcal{M}_{e(i,j)}$ is the smallest invariant subspace of $\{\widetilde{B}\}'$ which contains $e(i,j)$. Moreover, by (14), for each i with $1 \le i \le n_1$, $\mathcal{N}(i,j) \subset \mathcal{N}(i,k)$ if $1 \le j \le k \le t_{n_i}$. Similarly, for each j with $1 \le j \le n_1$, $\mathcal{N}(i,j) \subset \mathcal{N}(k,j)$ if $i \le k$. Therefore, for each $\mathcal{M} \in \operatorname{Lat}\{\widetilde{B}\}'$, there exist $e(i_1,j_1)$, $e(i_2,j_2)$, …, $e(i_k,j_k) \in E$, with $1 \le i_1 < \ldots < i_k \le n_1$, and $e(i_{l-1},j_{l-1}) \notin \mathcal{N}(i_l,j_l)$ for each $l = 2,\ldots,k$, such that

$$\mathcal{M} = \bigvee_{l=1}^{k} \mathcal{N}(i_l,j_l) = \bigvee_{l=1}^{k} \mathcal{M}_{e(i_l,j_l)} = \mathcal{M}_{\sum_{l=1}^{k} e(i_l,j_l)}.$$

Moreover, by (14),

$$\begin{aligned}\mathcal{M}_{\sum_{l=1}^{k} e(i_l,j_l)} = {} & (\mathbb{C}^{t_{n_{j_1-i_1+1}}} \oplus 0) \oplus (\mathbb{C}^{t_{n_{j_1-i_1+2}}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_1}}} \oplus 0)\\ & \oplus (\mathbb{C}^{t_{n_{j_2-i_2+i_1+1}}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_2}}} \oplus 0) \oplus \ldots\\ & \oplus (\mathbb{C}^{t_{n_{j_k-i_k+i_{k-1}+1}}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{n_{j_k}}} \oplus 0)\end{aligned}$$

on $\bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}$. Now, it remains to show that $(t_{n_{j_1-i_1+1}}, t_{n_{j_1-i_1+2}}, \ldots, t_{n_k})$ is a subsequence of $\{t_l\}_{l=1}^{n_1}$. That is to say, we should prove that

$$\begin{aligned} n_{j_1-i_1+1} < n_{j_1-i_1+2} < \ldots < n_{j_1} < n_{j_2-i_2+i_1+1} < \ldots < n_{j_2} &\\ < n_{j_3-i_3+i_2+1} < \ldots < n_{j_k}&.\end{aligned}$$

Since $e(i_1,j_1) \notin \mathcal{N}(i_2,j_2)$, we have $t_{n_{j_1}} > t_{n_{j_2-i_2+i_1}} \ge t_{n_{j_2-i_2+i_1+1}}$. It follows that $n_{j_1} < n_{j_2-i_2+i_1} < n_{j_2-i_2+i_1+1}$ because $\{t_i\}_{i=1}^{n_1}$ is a decreasing sequence. Similarly, $n_{j_p} < n_{j_{p+1}-i_{p+1}+i_p+1}$ for $p = 1,\ldots,k-1$. This completes the proof. ■

The following lemma deals with the converse of Lemma 2.7.

LEMMA 2.8. *Each subsequence* $(t_{l_1},\ldots,t_{l_k})$ *of* $\{t_l\}_{l=1}^{n_1}$ *produces an invariant space* $(\mathbb{C}^{t_{l_1}} \oplus 0) \oplus (\mathbb{C}^{t_{l_2}} \oplus 0) \oplus \ldots \oplus (\mathbb{C}^{t_{l_k}} \oplus 0) \oplus 0 \oplus \ldots \oplus 0$ *for* $\{\widetilde{B}\}'$ *on* $\bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}$.

Proof. Let $\mathcal{A}$ be the collection of all subsequences of $\{t_j\}_{j=1}^{n_1}$ and

$$\mathcal{A}(i,j) = \{(a_1,\ldots,a_i) \in \mathcal{A} : a_i = j\}.$$

Let

$$\mathcal{B}(i,j) = \{\mathcal{M} \in \operatorname{Lat}\{\widetilde{B}\}' \colon e(i,j) \in \mathcal{M},\ e(k,l) \notin \mathcal{M} \text{ if } k > i \text{ or } k = i,\ l > j\}.$$

It is clear that $\{\mathcal{A}(i,j) : i = 1,\dots,n_1,\ j = 1,\dots,t_i\}$ and $\{\mathcal{B}(i,j) : i = 1,\dots,n_1,\ j = 1,\dots,t_i\}$ are partitions of $\mathcal{A}$ and $\operatorname{Lat}\{\widetilde{B}\}'$ respectively. It suffices to show that for each $(a_1,\dots a_j) \in \mathcal{A}(i,j)$, there exists $\mathcal{M} \in \operatorname{Lat}\{\widetilde{B}\}'$ such that

$$\mathcal{M} = (\mathbb{C}^{a_1} \oplus 0) \oplus (\mathbb{C}^{a_2} \oplus 0) \oplus \dots \oplus (\mathbb{C}^{a_j} \oplus 0) \oplus 0 \oplus \dots \oplus 0$$

on $\bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}$. We do this by induction.

We first consider the case of $i = 1$. For each $1 \le j \le t_1 = m$, $\mathcal{A}(1,j) = (j)$, and $\mathcal{M}_{e(1,j)} = (\mathbb{C}^j \oplus 0) \oplus 0 \oplus \dots \oplus 0$. So we have proved the statement for $i = 1$. Assume that it is true for $i = 1,\dots,p-1$ and $j = 1,\dots,t_i$. We now consider the case of $i = p$ and $j = q$, where $1 \le q \le t_p$. Let $a_p = q$ and $(a_1,\dots,a_p) \in \mathcal{A}(p,q)$. If $\mathcal{M}_{e(p,q)} = (\mathbb{C}^{a_1} \oplus 0) \oplus (\mathbb{C}^{a_2} \oplus 0) \oplus \dots \oplus (\mathbb{C}^{a_j} \oplus 0) \oplus 0 \oplus \dots \oplus 0$, then we are done. Otherwise, by (14),

$$\mathcal{M}_{e(p,q)} = \Big(\bigoplus_{l=1}^{q} \mathbb{C}^{t_{n_q-p+l}} \oplus 0 \Big) \oplus \bigoplus_{l=i+1}^{n_1} 0 \subset \bigoplus_{l=1}^{n_1} \mathbb{C}^{t_l}.$$

Since $(t_{n_q-p+1}, t_{n_q-p+2}, \dots, t_{n_q})$ is the smallest subsequence of $\{t_i\}_{i=1}^{n_1}$ with length p and last term q, there exists l with $1 \le l < p$ such that the last l terms of $(a_1,\dots,a_p)$ and $(t_{n_q-p+1}, t_{n_q-p+2}, \dots, t_{n_q})$ are equal, but $a_{p-q} > t_{n_q-p+l}$. By assumption, there exists $\mathcal{M}_0 \in \mathcal{B}(p-l, a_{p-l})$ such that $\mathcal{M}_0 = (\mathbb{C}^{a_1} \oplus 0) \oplus (\mathbb{C}^{a_2} \oplus 0) \oplus \dots \oplus (\mathbb{C}^{a_{p-l}} \oplus 0) \oplus 0 \oplus \dots \oplus 0$. Let $\mathcal{M}$ be the linear span of $\mathcal{M}_0$ and $\mathcal{M}_{e(p,q)}$. It follows that $\mathcal{M} = (\mathbb{C}^{a_1} \oplus 0) \oplus (\mathbb{C}^{a_2} \oplus 0) \oplus \dots \oplus (\mathbb{C}^{a_q} \oplus 0) \oplus 0 \oplus \dots \oplus 0$, which completes the proof. ∎

Let $\{a_j\}_{j=1}^n$ be a sequence of numbers, and let $\mathcal{A}$ be the collection of all subsequences of $\{a_j\}_{j=1}^n$. Note that the empty set is also an element of $\mathcal{A}$. For any $b = \{b_j\}_{j=1}^k$ and $c = \{c_j\}_{j=1}^l$ in $\mathcal{A}$, we define $b \preceq c$ if $k < l$ and $b_j \le c_j$ for all j. It is obvious that $(\mathcal{A}, \preceq)$ is a partially ordered set.

Definition 2.9. Let $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}\{A\}'$, where $A \in M_n(\mathbb{C})$. We define $\mathcal{M} \preceq \mathcal{N}$ if and only if $\mathcal{M} \subset \mathcal{N}$.

Thus $\operatorname{Lat}\{A\}'$ is also a partially ordered set. Let $\{t_l\}_{l=1}^{n_1}$ be defined as in (10), and let $\mathcal{A}$ be the collection of all subsequences of $\{t_l\}_{l=1}^{n_1}$. Let $\widetilde{B}$ be as in (8). As in Definition 1.3, two partially ordered sets are said to be *isomorphic* if there exists an order-preserving bijection between them. Combining Lemmas 2.7, 2.8, we get the following corollary.

Corollary 2.10. $\operatorname{Lat}\{\widetilde{B}\}'$ *and* $\mathcal{A}$ *are isomorphic.*

Next, we want to prove that $\operatorname{Lat}\{\widetilde{B}\}'$ and $\mathcal{O}_{C(\widetilde{B})}$ are isomorphic. Recall that an element in $\mathcal{O}_{C(\widetilde{B})}$ is of the form $\mathcal{O}_x = \{Tx : T \in C(\widetilde{B})\}$ for some $x \in \mathbb{C}^n$.

Proposition 2.11. *The partially ordered sets* $\mathcal{O}_{C(\widetilde{B})}$ *and* Lat $\{\widetilde{B}\}'$ *are isomorphic and so* $\sharp\mathcal{O}_{C(\widetilde{B})} = \sharp\,\mathrm{Lat}\,\{\widetilde{B}\}'$.

Proof. Define $\phi : \mathcal{O}_{C(\widetilde{B})} \to \mathrm{Lat}\,\{\widetilde{B}\}'$ by $\phi(\mathcal{O}_x) = \mathcal{M}_x$ for any $x \in \mathbb{C}^n$. If x and y are distinct vectors with $\mathcal{O}_x = \mathcal{O}_y$, then by Lemma 1.1, $\mathcal{M}_x = \overline{\mathcal{O}}_x = \overline{\mathcal{O}}_y = \mathcal{M}_y$. Thus, ϕ is well defined. Next, for each $\mathcal{M} \in \mathrm{Lat}\,\{\widetilde{B}\}'$, by Lemma 2.7, $\mathcal{M} = \mathcal{M}_x$ for some vector x. Therefore, $\phi(\mathcal{O}_x) = \mathcal{M}_x = \mathcal{M}$ and so ϕ is onto. If $\mathcal{M}_x = \mathcal{M}_y$, then Lemma 1.1 and Proposition 1.2 show that $\mathcal{O}_x = \mathcal{O}_y$. Finally, it is clear that ϕ is order-preserving. ∎

Let $n_{m+1} = 0$. Since $\sharp\mathcal{A} = \prod_{i=1}^m (n_i - n_{i+1} + 1)$, it follows that $\sharp\mathcal{O}_{C(\widetilde{B})} = \prod_{i=1}^m (n_i - n_{i+1} + 1)$. By Corollary 2.10 and Proposition 2.11, we obtain the following theorem.

Theorem 2.12. *The three partially ordered sets* $\mathcal{O}_{C(\widetilde{B})}$, Lat $\{\widetilde{B}\}'$ *and* $\mathcal{A}$ *are isomorphic. Hence* $\sharp\mathcal{O}_{C(\widetilde{B})} = \prod_{i=1}^m (n_i - n_{i+1} + 1)$.

Proof of the Main Theorem. Recall that J is defined as in (2). By Lemma 2.2, Proposition 1.4, and Lemma 2.6, we know that $\mathcal{O}_{C(J)}$ is isomorphic to $\mathcal{O}_{C(\widetilde{B})}$. By the previous theorem, $\mathcal{O}_{C(\widetilde{B})}$, Lat $\{\widetilde{B}\}'$ and $\mathcal{A}$ are isomorphic. Hence $\sharp\mathcal{O}_{C(J)} = \prod_{i=1}^m (n_i - n_{i+1} + 1)$ and so the Main Theorem is proved. ∎

Corollary 2.13. *Let N be a normal matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then* $\sharp\,\mathrm{Lat}\,\{N\}' = 2^m$.

Proof. Via Jordan form, we may assume that $N = \bigoplus_{i=1}^m N_i$, where each N_i is normal with the only eigenvalue λ_i. We let n and n_i denote the cardinality of Lat $\{N\}'$ and Lat $\{N_i\}'$ for each i respectively. By Proposition 1.5, $n = n_1 \ldots n_m$. By the Main Theorem, $n_i = 2$ for each i. Hence $n = 2^m$. ∎

We now consider $\sharp\mathcal{O}_{C(A)}$ for an arbitrary matrix $A \in M_n(\mathbb{C})$. By Proposition 1.4, we know that $\sharp\mathcal{O}_{C(A)} = \sharp\mathcal{O}_{C(J(A))}$, where $J(A)$ is the Jordan form of A. That is, there exist distinct complex numbers $\lambda_1, \ldots, \lambda_k$ such that $J(A) = \bigoplus_{j=1}^k J(\lambda_j)$, and each $J(\lambda_j) = \bigoplus_{i=1}^{m_j} \bigoplus_{s=1}^{k_{j,i}} J_{n_{j,i}}(\lambda_j)$, where

$$J_{n_{j,i}}(\lambda_j) = \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix} \in M_{n_{j,i}}(\mathbb{C}),$$

with $n_{j,1} > n_{j,2} > \ldots > n_{j,m_j} \geq 1$ and each $k_{j,i} \in \mathbb{N}$. Set $n_{j,m_j+1} = 0$. By Propositions 1.4, 1.5, and the Main Theorem, we obtain the following corollary.

COROLLARY 2.14. *For an arbitrary matrix $A \in M_n(\mathbb{C})$,*

$$\sharp\mathcal{O}_{C(A)} = \prod_{j=1}^{k}\prod_{i=1}^{m_j}(n_{j,i} - n_{j,i+1} + 1).$$

Finally, we provide an example to illustrate the computation of $\mathcal{O}_{C(A)}$ for a given matrix A.

EXAMPLE 2.15. Let

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{bmatrix} \in M_n(\mathbb{C})$$

and $A = J_4 \oplus J_2 \oplus J_1$. We will list all the elements in $\mathcal{O}_{C(A)}$.

First, we change the order of the basis from $\{e_i\}_{i=1}^{7}$ to $\{e_1, e_5, e_7, e_2, e_6, e_3, e_4\}$. Also, define $t_1 = 3$, $t_2 = 2$, and $t_3 = t_4 = 1$. Then the matrix A written in the new basis is

$$\widetilde{A} = \left[\begin{array}{c|c|c|c} 0_3 & \begin{matrix} I_2 \\ 0_{2\times 1} \end{matrix} & & \\ \hline & 0_2 & \begin{matrix} 1 \\ 0 \end{matrix} & \\ \hline & & 0 & 1 \\ \hline & & & 0 \end{array}\right]$$

on $\mathbb{C}^{t_1} \oplus \mathbb{C}^{t_2} \oplus \mathbb{C}^{t_3} \oplus \mathbb{C}^{t_4}$.

It is easy to see that $\{t_i\}_{i=1}^{4}$ has 12 subsequences: $\{3,2,1,1\}$, $\{3,2,1\}$, $\{3,1,1\}$, $\{2,1,1\}$, $\{3,2\}$, $\{3,1\}$, $\{2,1\}$, $\{1,1\}$, $\{3\}$, $\{2\}$, $\{1\}$ and the empty set. By Lemmas 2.7 and 2.8, $\operatorname{Lat}\{\widetilde{A}\}'$ has 12 elements. Moreover, every subsequence $\{m_1, m_2, m_3, m_4\}$ of $\{t_i\}_{i=1}^{4}$ represents an element $(\mathbb{C}^{m_1} \oplus 0) \oplus (\mathbb{C}^{m_2} \oplus 0) \oplus (\mathbb{C}^{m_3} \oplus 0) \oplus (\mathbb{C}^{m_4} \oplus 0) \in \operatorname{Lat}\{\widetilde{A}\}'$ with respect to $\mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1$. By Lemma 1.1, each $(\mathbb{C}^{m_1} \oplus 0) \oplus (\mathbb{C}^{m_2} \oplus 0) \oplus (\mathbb{C}^{m_3} \oplus 0) \oplus (\mathbb{C}^{m_4} \oplus 0)$ is the closure of some orbit $\mathcal{O}_x$. We get the 12 orbits of $\mathcal{O}_{C(A)}$ as follows:

1. $\mathcal{O}_{e_4} = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) : x_4 \neq 0\}$,
2. $\mathcal{O}_{e_3+e_6} = \{(x_1, x_2, x_3, 0, x_5, x_6, x_7) : x_3, x_6 \neq 0\}$,
3. $\mathcal{O}_{e_3+e_7} = \{(x_1, x_2, x_3, 0, x_1, 0, x_7) : x_3, x_7 \neq 0\}$,
4. $\mathcal{O}_{e_3} = \{(x_1, x_2, x_3, 0, x_5, 0, 0) : x_3 \neq 0\}$,
5. $\mathcal{O}_{e_6} = \{(x_1, x_2, 0, 0, x_5, x_6, x_7) : x_6 \neq 0\}$,
6. $\mathcal{O}_{e_2+e_7} = \{(x_1, x_2, 0, 0, x_5, 0, x_7) : x_2, x_7 \neq 0\}$,
7. $\mathcal{O}_{e_2+e_5} = \{(x_1, x_2, 0, 0, x_5, 0, 0) : x_2, x_5 \neq 0\}$,
8. $\mathcal{O}_{e_2} = \{(x_1, x_2, 0, 0, 0, 0, 0) : x_2 \neq 0\}$,
9. $\mathcal{O}_{e_7} = \{(x_1, 0, 0, 0, x_5, 0, x_7) : x_7 \neq 0\}$,
10. $\mathcal{O}_{e_5} = \{(x_1, 0, 0, 0, x_5, 0, 0) : x_5 \neq 0\}$,

11. $\mathcal{O}_{e_1} = \{(x_1, 0, 0, 0, 0, 0, 0) : x_1 \neq 0\}$,
12. $\mathcal{O}_0 = \{(0, 0, 0, 0, 0, 0, 0)\}$.

3. General real square matrix. In this section, the whole setting is restricted to $M_n(\mathbb{R})$, the set of all real $n \times n$ matrices. Let $\mathrm{GL}(n, \mathbb{R})$ be the Lie group of invertible matrices in $M_n(\mathbb{R})$. Given $A \in M_n(\mathbb{R})$, we use $C(A) = \{T \in \mathrm{GL}(n, \mathbb{R}) : TA = AT\}$ to denote the centralizer of A. For any $x \in \mathbb{R}^n$, $\mathcal{O}_x = \{Tx : T \in C(A)\} \subset \mathbb{R}^n$ is the orbit of the standard action of $C(A)$ on $\mathbb{R}^n$. Consider the collection of orbits, $\mathcal{O}_{C(A)} = \{\mathcal{O}_x : x \in \mathbb{R}^n\}$. We aim to compute $\sharp\mathcal{O}_{C(A)}$. As in Section 2, it suffices to consider $\mathcal{O}_{C(J(A))}$, where $J(A)$ is the real Jordan form of A which we now recall.

Let $A \in M_n(\mathbb{R})$ and $J(A)$ be its real Jordan form. Since A is a real matrix, we may assume that the distinct eigenvalues of A are $\alpha_1 \pm i\beta_1, \ldots, \alpha_k \pm i\beta_k$, $\nu_1, \ldots, \nu_l$ where each $\alpha_j, \beta_j, \nu_j \in \mathbb{R}$ and $\beta_j > 0$. Let $J(\nu_i)$ be the direct sum of all Jordan blocks associated with the eigenvalues ν_i as explained in Section 2. Also, let $J(\alpha_i, \beta_i)$ be the direct sum of all Jordan blocks associated with the eigenvalues $\alpha_i \pm i\beta_i$. That is, there exist $m_i \in \mathbb{N}$, $n_{i,1}, \ldots, n_{i,m_i}$, $k_{i,1}, \ldots, k_{i,m_i} \in \mathbb{N}$ with $n_{i,1} > \ldots > n_{i,m_i} \geq 1$, such that

$$K = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \in M_2(\mathbb{R}),$$

and

$$J(\alpha_i, \beta_i) = \bigoplus_{l=1}^{m_i} \bigoplus_{j=1}^{k_{i,m_j}} \begin{bmatrix} K & I_2 & & \\ & K & \ddots & \\ & & \ddots & I_2 \\ & & & K \end{bmatrix}_{2n_{i,l}} \in M_{\sum_{l=1}^{m_i} 2n_{i,l} \times k_{i,l}}(\mathbb{R}).$$

$J(A)$ is the direct sum of all $J(\nu_i)$ and $J(\alpha_i, \beta_i)$:

$$J(A) = \bigoplus_{i=1}^{k} J(\alpha_i, \beta_i) \oplus \bigoplus_{j=1}^{l} J(\nu_j).$$

Propositions 1.4 and 1.5 still hold in $M_n(\mathbb{R})$. So it suffices to consider each $\mathcal{O}_{C(J(\nu_i))}$ and each $\mathcal{O}_{C(J(\alpha_i, \beta_i))}$ in order to understand $\mathcal{O}_{C(A)}$. Moreover, we note that the Main Theorem holds if we consider the whole setting in $M_n(\mathbb{R})$ in place of $M_n(\mathbb{C})$. So each $\mathrm{Lat}\,\{J(\nu_i)\}'$ is well understood. We now only have to consider each $\mathrm{Lat}\,\{J(\alpha_i, \beta_i)\}'$. Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. For simplicity, we may consider $\mathrm{Lat}\,\{J(\alpha, \beta)\}'$ directly.

Recall that $J(\alpha, \beta)$ is the direct sum of all Jordan blocks associated with $\alpha \pm i\beta$. That is, there exist $m \in \mathbb{N}$, $n_1, \ldots, n_m$, $k_1, \ldots, k_m \in \mathbb{N}$ with

$n_1 > \ldots > n_m \geq 1$ such that

$$K = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$

$$J(\alpha,\beta) = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \begin{bmatrix} K & I_2 & & \\ & K & \ddots & \\ & & \ddots & I_2 \\ & & & K \end{bmatrix}_{2n_i} \in M_{\sum_{i=1}^m 2n_i \times k_i}(\mathbb{R}). \tag{15}$$

Using the same approach as in Section 2, we get the following analogue of the Main Theorem.

THEOREM 3.1. *Let $J(\alpha,\beta) \in M_n(\mathbb{R})$ be as in* (15) *and $n_{m+1} = 0$. Then*

$$\sharp \operatorname{Lat} \{J(\alpha,\beta)\}' = \prod_{i=1}^{m} (n_i - n_{i+1} + 1).$$

Thus, we can also completely solve the lattice structure for the commutant of real matrices.

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