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GENERALIZATION OF A THEOREM OF STEINHAUS

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Abstract. We present a multidimensional version of the Three Gap Theorem of Steinhaus, proving that the number of the so-called primitive arcs is bounded in any dimension.

1. Introduction. Let α be an irrational number and $d \geq 1$ an integer. The distribution of the local spacings between the members of the sequence $n^d\alpha \pmod 1$ in the case $d = 1$ is completely different from the distribution when $d > 1$ (see [4] and the references therein). If $d > 1$ one expects that for almost all α the distribution is Poissonian, and one knows for instance that the pair correlation is Poissonian indeed (see [1] and [4]). If $d = 1$ one knows that the distribution is not Poissonian, and this is a consequence of the following Three Gap Theorem of Steinhaus:

Let α be a positive number and N a non-negative integer. Then the fractional parts $\{n\alpha\}$, $0 \leq n \leq N$, partition the unit interval into $N + 1$ intervals which have at most 3 different lengths.

Proofs were subsequently provided by several authors (see [3], [5], [6]). In this paper we are interested in seeing whether such a phenomenon also holds in a multidimensional setting. A two-dimensional Steinhaus theorem was proved in [2]. Thus, if α_1, α_2 are real numbers and N_1, N_2 are non-negative integers, then the set $\{\{n_1\alpha_1 + n_2\alpha_2\} : 0 \leq n_i \leq N_i, 1 \leq i \leq 2\}$ partitions the unit interval into subintervals having at most $N_1 + 4$ different lengths and this bound can be attained. Hence, in the two-dimensional case the number of different lengths is unbounded as $N_1 \to \infty$ and therefore the same thing will happen in higher dimensions. We say that a gap is *primitive* if its length is not a sum of shorter lengths (not necessarily distinct) of other intervals in the partition. Returning to the one-dimensional Three Gap Theorem, we mention that it has a second part, which states that in case there are 3 different lengths, one of them equals the sum of the other two. Thus in dimension one there are at most two primitive gaps. This is

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the statement which will be generalized in the present paper. In Section 2 we prove that the number of primitive gaps is bounded in any dimension k by 2^k. It might be that this bound is best possible for any $k \geq 1$, but this remains an open problem. In the last section we give examples for $k = 2$ and $k = 3$ with respectively 4 and 8 primitive gaps.

2. Primitive gaps. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a vector with positive entries and for any positive integers $N_1, \ldots, N_k$ we consider the set of k-vectors

$$\mathcal{N} = \{\mathbf{n} = (n_1, \ldots, n_k) : 0 \leq n_i \leq N_i \text{ for } 1 \leq i \leq k\}.$$

We investigate the spacings between consecutive members of the sequence

$$\mathcal{M} = \big\{\{\mathbf{n} \cdot \boldsymbol{\alpha}\} : \mathbf{n} \in \mathcal{N}\big\} \subset [0, 1),$$

where the dot product is defined as usual by $\mathbf{n} \cdot \boldsymbol{\alpha} = n_1\alpha_1 + \ldots + n_k\alpha_k$. It is convenient to view $\mathcal{M}$ on the circle $\mathcal{C} = \mathcal{C}(0, 1/(2\pi))$, with center at the origin and circumference equal to 1. Thus any vector $\mathbf{n} \in \mathcal{N}$ corresponds to a point $\frac{1}{2\pi}e(\{\mathbf{n} \cdot \boldsymbol{\alpha}\}) = \frac{1}{2\pi}e^{2\pi i\{\mathbf{n}\cdot\boldsymbol{\alpha}\}} \in \mathcal{C}$. Then $\mathcal{M}$ might be seen as a set of "footprints" of a walk on the circle $\mathcal{C}$. The *gap* (or *spacing*) between two footprints $\frac{1}{2\pi}e(\{\mathbf{m} \cdot \boldsymbol{\alpha}\}), \frac{1}{2\pi}e(\{\mathbf{n} \cdot \boldsymbol{\alpha}\}) \in \mathcal{M}$ is the length $l(\mathbf{m}, \mathbf{n})$ of the arc $\big(\frac{1}{2\pi}e(\{\mathbf{m} \cdot \boldsymbol{\alpha}\}), \frac{1}{2\pi}e(\{\mathbf{n} \cdot \boldsymbol{\alpha}\})\big)$. Let

$$\mathcal{A} = \left\{\left(\frac{1}{2\pi}e(\{\mathbf{m} \cdot \boldsymbol{\alpha}\}), \frac{1}{2\pi}e(\{\mathbf{n} \cdot \boldsymbol{\alpha}\})\right) : \mathbf{m}, \mathbf{n} \in \mathcal{N}\right\} \tag{1}$$

be the set of all arcs determined by $\mathcal{M}$. With no danger of confusion, we will briefly write any element of $\mathcal{A}$ as $(\mathbf{m}, \mathbf{n})$. In what follows all the arcs are oriented in the counterclockwise direction. Given an arc $(\mathbf{m}, \mathbf{n})$, we call $\mathbf{m}$ the *initial point* of $(\mathbf{m}, \mathbf{n})$, and $\mathbf{n}$ the *endpoint* of $(\mathbf{m}, \mathbf{n})$. Note that

$$l(\mathbf{m}, \mathbf{n}) = 1 - l(\mathbf{n}, \mathbf{m}).$$

For any $\mathbf{m} \in \mathcal{N}$ and $i \in \{1, \ldots, k\}$ we write

$$\mathbf{m}(m_i \to u) := (m_1, \ldots, u, \ldots, m_k)$$

for the point obtained from $\mathbf{m}$ by replacing the component m_i by u.

A set $\mathcal{B} \subset \mathcal{A}$ of arcs is called *invariant under translations* if for every arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$, for any $1 \leq i \leq k$, and for any integers u, v with $0 \leq u \leq \min\{m_i, n_i\}$ and $\max(m_i, n_i) \leq \max(m_i, n_i) + v \leq N_i$, both arcs $(\mathbf{m}(m_i - u), \mathbf{n}(n_i - u))$ and $(\mathbf{m}(m_i + v), \mathbf{n}(n_i + v))$ are in $\mathcal{B}$. Note that all those arcs have the same length, equal to $l(\mathbf{m}, \mathbf{n})$.

Let $\mathcal{B} \subset \mathcal{A}$ be a set of arcs. We say that an arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$ is *indecomposable in* $\mathcal{B}$ if $l(\mathbf{m}, \mathbf{n}) > 0$ and there is no point $\mathbf{s} \in (\mathbf{m}, \mathbf{n})$ such that $l(\mathbf{m}, \mathbf{s}) > 0$, $l(\mathbf{s}, \mathbf{n}) > 0$ and both arcs $(\mathbf{m}, \mathbf{s})$ and $(\mathbf{s}, \mathbf{n})$ are in $\mathcal{B}$.

We call a set $\mathcal{B} \subset \mathcal{A}$ of arcs *complete* if for any arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$ the arc $(\mathbf{n}, \mathbf{m})$ belongs to $\mathcal{B}$ and for any pair of arcs $(\mathbf{m}, \mathbf{n}), (\mathbf{n}, \mathbf{s}) \in \mathcal{B}$ we also have $(\mathbf{m}, \mathbf{s}) \in \mathcal{B}$.

We call an arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$ *primitive in* $\mathcal{B}$ if $l(\mathbf{m}, \mathbf{n}) > 0$ and $l(\mathbf{m}, \mathbf{n})$ cannot be written as a sum of smaller lengths $l(\mathbf{r}, \mathbf{s})$ (not necessarily distinct) of non-overlapping arcs $(\mathbf{r}, \mathbf{s})$ from $\mathcal{B}$. Clearly any arc which is primitive in $\mathcal{A}$ is also indecomposable in $\mathcal{A}$, but the converse is not true.

Note that for any $\mathbf{m} \in \mathcal{N}$ there is a unique point $\mathbf{s} \in \mathcal{C}$ such that the arc $(\mathbf{m}, \mathbf{s})$ belongs to $\mathcal{A}$ and is indecomposable. Indeed, among all the arcs in $\mathcal{A}$ which have $\mathbf{m}$ as their initial point, there is one which has the smallest length; call it $(\mathbf{m}, \mathbf{s})$. Then $(\mathbf{m}, \mathbf{s})$ is indecomposable, while any other arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{A}$ decomposes into two arcs $(\mathbf{m}, \mathbf{s})$ and $(\mathbf{s}, \mathbf{n})$, both of which belong to $\mathcal{A}$. Similarly, there is a unique indecomposable arc in $\mathcal{A}$ that has $\mathbf{m}$ as its endpoint. The number of primitive arcs, however, is much smaller than the cardinality of $\mathcal{N}$.

LEMMA 1. *Let $\mathcal{B} \subset \mathcal{A}$ be a complete set of arcs. Suppose $(\mathbf{m}, \mathbf{n})$ and $(\mathbf{m}, \mathbf{r})$ are arcs in $\mathcal{B}$, $l(\mathbf{m}, \mathbf{n}) > 0$ and $(\mathbf{m}, \mathbf{r})$ is indecomposable in $\mathcal{B}$. Then*

$$l(\mathbf{m}, \mathbf{r}) \leq l(\mathbf{m}, \mathbf{n}).$$

Proof. Suppose $l(\mathbf{m}, \mathbf{r}) > l(\mathbf{m}, \mathbf{n})$. Then $\mathbf{n} \in (\mathbf{m}, \mathbf{r})$ and $l(\mathbf{n}, \mathbf{r}) > 0$. Moreover, since $\mathcal{B}$ is complete it follows that $(\mathbf{n}, \mathbf{r}) \in \mathcal{B}$. This implies that $(\mathbf{m}, \mathbf{r})$ is not indecomposable in $\mathcal{B}$, contradicting the hypothesis. ∎

LEMMA 2. *Let $\mathcal{B}$ be a complete subset of $\mathcal{A}$ invariant under translations. Let*

$$\Omega = \{(\mathbf{m}, \mathbf{n}) \in \mathcal{B} : (\mathbf{m}, \mathbf{n}) \textit{ indecomposable in } \mathcal{B} \textit{ and } m_i \in \{0, N_i\} \textit{ for } i = 1, \ldots, k\}.$$

Then for every arc $(\mathbf{r}, \mathbf{s}) \in \mathcal{B}$ with $l(\mathbf{r}, \mathbf{s}) > 0$ there exists an arc $(\mathbf{m}, \mathbf{n}) \in \Omega$ such that either

$$l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{n}),$$

or there exists an arc $(\mathbf{n}, \mathbf{t}) \in \mathcal{B}$ with $l(\mathbf{n}, \mathbf{t}) > 0$ such that

$$l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{n}) + l(\mathbf{n}, \mathbf{t}).$$

Proof. Let

$$\Omega_0 = \{(\mathbf{m}, \mathbf{n}) \in \mathcal{B} : l(\mathbf{m}, \mathbf{n}) > 0,\ m_i \in \{0, N_i\} \text{ for } i = 1, \ldots, k\}.$$

Let $(\mathbf{r}, \mathbf{s}) \in \mathcal{B}$. Since the length of an arc is translation invariant and by hypothesis $\mathcal{B}$ is invariant under translations, it follows that Ω_0 contains an arc $(\mathbf{m}, \mathbf{t})$ for which $l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{t})$. Precisely, one defines $\mathbf{m} = (m_1, \ldots, m_k)$ and $\mathbf{t} = (t_1, \ldots, t_k)$ by

$$m_i = \begin{cases} 0 & \text{if } r_i \le s_i, \\ N_i & \text{if } r_i > s_i, \end{cases} \qquad t_i = \begin{cases} s_i - r_i & \text{if } r_i \le s_i, \\ N_i + s_i - r_i & \text{if } r_i > s_i, \end{cases}$$

for any $1 \le i \le k$. We distinguish two cases.

1. The arc $(\mathbf{m}, \mathbf{t})$ is indecomposable in $\mathcal{B}$. Then $(\mathbf{m}, \mathbf{t}) \in \Omega$ and $(\mathbf{m}, \mathbf{n}) = (\mathbf{m}, \mathbf{t})$ is the required arc.

2. The arc $(\mathbf{m}, \mathbf{t})$ is not indecomposable in $\mathcal{B}$. Then there is a unique point $\mathbf{n} \in (\mathbf{m}, \mathbf{t})$ with $l(\mathbf{m}, \mathbf{n}) > 0$, $l(\mathbf{n}, \mathbf{t}) > 0$ such that $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$ and $(\mathbf{m}, \mathbf{n})$ is indecomposable in $\mathcal{B}$. Then $(\mathbf{m}, \mathbf{n}) \in \Omega$. Since $\mathcal{B}$ is complete one has $(\mathbf{n}, \mathbf{t}) \in \mathcal{B}$. Also

$$l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{t}) = l(\mathbf{m}, \mathbf{n}) + l(\mathbf{n}, \mathbf{t}). \ \blacksquare$$

LEMMA 3. *Let $\mathcal{B} \subset \mathcal{A}$ be complete and invariant under translations, $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$, $(\mathbf{r}, \mathbf{s}) \in \mathcal{B}$ and assume that $(\mathbf{m}, \mathbf{n})$ is a translation of $(\mathbf{r}, \mathbf{s})$. Then $(\mathbf{m}, \mathbf{n})$ is primitive if and only if $(\mathbf{r}, \mathbf{s})$ is primitive.*

This follows immediately from the definition of primitive arcs and the fact that translations leave the length of an arc unchanged.

THEOREM 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a vector with positive entries and let $N_1, \ldots, N_k$ be positive integers. Let $\mathcal{A}$ be the set of arcs given by* (1) *and let $\mathcal{B}$ be a subset of $\mathcal{A}$ complete and invariant under translations. Then the number of lengths of primitive arcs in $\mathcal{B}$ is at most 2^k.*

Proof. Let $(\mathbf{r}, \mathbf{s}) \in \mathcal{B}$ be a primitive arc. By Lemma 2 there exists an arc $(\mathbf{m}, \mathbf{n}) \in \Omega$ such that either

$$l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{n}),$$

or there exists an arc $(\mathbf{n}, \mathbf{t}) \in \mathcal{B}$ with $l(\mathbf{n}, \mathbf{t}) > 0$ such that

$$l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{n}) + l(\mathbf{n}, \mathbf{t}).$$

Since the arc $(\mathbf{r}, \mathbf{s})$ is primitive in $\mathcal{B}$, the second case cannot occur. It follows that $l(\mathbf{r}, \mathbf{s}) = l(\mathbf{m}, \mathbf{n})$. Now each arc $(\mathbf{m}, \mathbf{n}) \in \Omega$ is uniquely determined by its initial point $\mathbf{m}$. Hence Ω contains at most 2^k elements. $\blacksquare$

THEOREM 2. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a vector with positive entries, let $N_1, \ldots, N_k$ be positive integers and let r, $1 \le r \le k$, be an integer. Assume that for $1 \le i \le r$ one has $\alpha_i = a_i/q_i$ for some positive integers a_i, q_i with $q_i \le N_i$. Let $\mathcal{A}$ be the set of arcs given by* (1) *and $\mathcal{B}$ a subset of $\mathcal{A}$ complete and invariant under translations. Then the number of lengths of primitive arcs in $\mathcal{B}$ is at most 2^{k-r}.*

Proof. Note first that we may assume $q_i = N_i$ for $1 \le i \le r$, since the fractional parts $\{\mathbf{m} \cdot \boldsymbol{\alpha}\}$ give all the elements of $\mathcal{M}$ when $\mathbf{m}$ runs over $\mathcal{N}$, but restricted by the conditions $m_i \le q_i$ for $1 \le i \le r$. Let $(\mathbf{m}, \mathbf{n}) \in \mathcal{B}$ be a primitive arc. Using Lemmas 2 and 3 we know that there is a translation which sends $(\mathbf{m},\mathbf{n})$ to an arc $(\mathbf{s},\mathbf{t})\in\Omega$. On the other hand since $\{\alpha_i N_i\}=\{\alpha_i q_i\}=0$ for $1\le i\le r$, it follows that for any $\mathbf{m}\in\mathcal{N}$ with $m_i\in\{0,N_i\}, 1\le i\le r$, the vector $\mathbf{m}'=(m_1',\dots,m_k')$ given by

$$m_i'=\begin{cases}0 & \text{if } 1\le i\le r,\\ m_i & \text{if } r<i\le k,\end{cases}$$

produces the same point as $\mathbf{m}$ on the circle $\mathcal{C}$. This shows that the number of lengths of primitive arcs from $\mathcal{B}$ is not larger than the number of indecomposable arcs of the form $((0,\dots,0,x_{r+1},\dots,x_k),\mathbf{y})$, where $x_i\in\{0,N_i\}$ for $r+1\le i\le k$. In each such arc the endpoint is uniquely determined by its initial point. Thus the number of lengths of primitive arcs is at most 2^{k-r}. ∎

Theorem 1 can be generalized as follows. By an *interval of integers* J we mean a finite set of consecutive integers $J=\{a,a+1,\dots,a+b\}$. We write $\partial J=\{a,a+b\}$. Suppose that we have positive integers $d_1,\dots,d_k$ and for each $1\le i\le k$, we are given a set $\mathcal{N}_i$ which is a union of d_i intervals of integers, say $\mathcal{N}_i=J_{i1}\cup\ldots\cup J_{id_i}$. The intervals $J_{i1},\dots,J_{id_i}$ do not need to be disjoint, but we will assume that they have the same number of elements, say $|J_{i1}|=\ldots=|J_{id_i}|=N_i$. Set $\partial\mathcal{N}_i=\partial J_{i1}\cup\ldots\cup\partial J_{id_i}$. Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ be a vector with positive components. We consider the set of k-vectors

$$\mathcal{N}=\{\mathbf{n}=(n_1,\dots,n_k): n_i\in\mathcal{N}_i \text{ for } 1\le i\le k\}$$

and define the sets $\mathcal{M}$ and $\mathcal{A}$ as at the beginning of this section. Let $\mathcal{B}$ be a complete subset of $\mathcal{A}$ invariant under translations and

$$\Omega=\{(\mathbf{m},\mathbf{n})\in\mathcal{B}:(\mathbf{m},\mathbf{n}) \text{ indecomposable in } \mathcal{B} \text{ and } m_i\in\partial\mathcal{N}_i \text{ for } i=1,\dots,k\}.$$

Note that Ω contains at most $2^k d_1\ldots d_k$ elements. Arguing as before one obtains the following theorem, which shows that the number of lengths of primitive arcs produced by the set $\mathcal{B}$ is bounded in terms of $d_1,\dots,d_k$, regardless of the size of $N_1,\dots,N_k$.

THEOREM 3. *Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ be a vector with positive entries and let $d_1,\dots,d_k$ be positive integers. For $1\le i\le k$ let $\mathcal{N}_i$ be a union of d_i intervals of integers $J_{i1},\dots,J_{id_i}$ such that $|J_{i1}|=\ldots=|J_{id_i}|$. Let $\mathcal{A}$ be the set of arcs given by* (1) *and let $\mathcal{B}$ be a subset of $\mathcal{A}$ complete and invariant under translations. Then the number of lengths of primitive arcs in $\mathcal{B}$ is at most $2^k d_1\ldots d_k$.*

REMARK 1. (1) In particular, when $d_1=\ldots=d_k=1$ this gives the bound from Theorem 1.

(2) If we drop the condition that $|J_{i1}|=\ldots=|J_{id_i}|$, $1\le i\le k$, then the problem becomes more complicated, but there are instances when we obtain reasonably good upper bounds for the number of primitive arcs by

the same method. For example, if the intervals $J_{i1}, \dots, J_{id_i}$ have about the same length, in the sense that $|J_{i1}|, \dots, |J_{id_i}| \in [N_i, 2N_i]$ for some N_i, then we again have an upper bound for the number of lengths of primitive arcs which does not depend on N_i.

Corollary 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ be a vector with positive entries and let $d_1, \dots, d_k$ be positive integers. Let $\mathcal{N}_i$ be a set which is a union of d_i intervals of integers $J_{i1}, \dots, J_{id_i}$ such that $|J_{i1}|, \dots, |J_{id_i}| \in [N_i, 2N_i]$ for some positive integer N_i, $1 \le i \le k$. Let $\mathcal{A}$ be the set of arcs given by* (1) *and let $\mathcal{B}$ be a subset of $\mathcal{A}$ complete and invariant under translations. Then the number of lengths of primitive arcs in $\mathcal{B}$ is at most $4^k d_1 \dots d_k$.*

Proof. This follows from Theorem 3 and the fact that for any $1 \le i \le k$ and any $1 \le j \le d_i$, the interval J_{ij} can be written as a union of two intervals of integers $J_{ij} = J'_{ij} \cup J''_{ij}$ with $|J'_{ij}| = |J''_{ij}| = N_i$. Then one applies Theorem 3 with d_i replaced by $2d_i$ for $1 \le i \le k$ to get the result. ∎

If $\mathcal{B} = \mathcal{A}$, the gap between two elements $\{\mathbf{m} \cdot \boldsymbol{\alpha}\}$ and $\{\mathbf{n} \cdot \boldsymbol{\alpha}\}$ that corresponds to a primitive arc $(\mathbf{m}, \mathbf{n}) \in \mathcal{A}$ is called a *primitive gap*. Then by Theorem 1 it follows that the number of lengths of primitive gaps is at most 2^k.

3. Examples for $k = 2$ and $k = 3$. The bound 2^k from Theorem 1 is clearly the best possible when $k = 1$. In the first part of this section we construct a class of examples to show that for $k = 2$ the bound 2^k is attained for every $N_1 > 1$. To simplify the notations, we will briefly write N, α, β instead of N_1, α_1, α_2 respectively. We take $k = 2$, $N > 1$, $N_2 = 1$, $\varepsilon = 1/(2N^2 + 3N)$ and we consider the interval

$$\mathcal{I} = \left(\frac{1}{N+1}, \frac{(1-\varepsilon)N}{N^2 + N - 1} \right).$$

For any α restricted for the moment only to belong to $\mathcal{I}$, we take $\beta = ((2N-1)/N)\alpha + \varepsilon$. This selection allows us to calculate precisely the fractional part of the numbers $i\alpha + j\beta$ when $0 \le i \le N$ and $0 \le j \le 1$. To do this, let us notice that by definition it follows that

$$\alpha < \beta < 2\alpha \quad \text{and} \quad (N-1)\alpha + \beta = \left(N - 1 + \frac{2N-1}{N} \right)\alpha + \varepsilon < 1.$$

Consequently, we have the following sequence of inequalities (notice that they sort sub-unitary numbers, which therefore coincide with their own fractional parts):

$$(2) \qquad \begin{aligned} &0 < \alpha < \beta < 2\alpha < \alpha + \beta < 3\alpha < 2\alpha + \beta < 4\alpha < 3\alpha + \beta < \dots \\ &< (N-1)\alpha < (N-2)\alpha + \beta < N\alpha < (N-1)\alpha + \beta < 1. \end{aligned}$$

It remains to find the position of $\{N\alpha+\beta\}$ in this sequence. Again, by definition we have

$$N\alpha+\beta=\frac{N^2+2N-1}{N}\alpha+\varepsilon>\frac{N^2+2N-1}{N(N-1)}+\varepsilon>1 \tag{3}$$

and

$$0<N\alpha+\beta-1=\frac{N^2+2N-1}{N}\alpha+\varepsilon-1<\alpha. \tag{4}$$

Then by (2)–(4) we obtain

$$\begin{aligned}0&<\{N\alpha+\beta\}<\{\alpha\}<\{\beta\}<\{2\alpha\}<\{\alpha+\beta\}\\ &<\{3\alpha\}<\{2\alpha+\beta\}<\{4\alpha\}<\{3\alpha+\beta\}<\dots\\ &<\{(N-1)\alpha\}<\{(N-2)\alpha+\beta\}<\{N\alpha\}<\{(N-1)\alpha+\beta\}<1.\end{aligned} \tag{5}$$

From (2) and (5) we see that there are only four distinct gaps between consecutive points on the circle $\mathcal{C}(0,1/(2\pi))$ represented by $\{i\alpha+j\beta\}$, where $0\le i\le N$ and $0\le j\le 1$. Then the distinct lengths of the gaps are:

$$\begin{aligned}d_1&=\{N\alpha+\beta\}=\frac{N^2+2N-1}{N}\alpha-\frac{2N^2+3N-1}{2N^2+3N},\\ d_2&=\alpha-\{N\alpha+\beta\}=-\frac{N^2+N-1}{N}\alpha+\frac{2N^2+3N-1}{2N^2+3N},\\ d_3&=\beta-\alpha=\frac{N-1}{N}\alpha+\frac{1}{2N^2+3N},\\ d_4&=2\alpha-\beta=\frac{1}{N}\alpha-\frac{1}{2N^2+3N}.\end{aligned}$$

Also, we find that there is one gap with length d_1, two gaps with length d_2, N gaps with length d_3 and $N-1$ gaps with length d_4. Then it is easy to show that

$$d_2<d_4<d_3<d_1. \tag{6}$$

Now we are ready to prove that d_1, d_2, d_3 and d_4 are all lengths of primitive gaps as announced at the beginning of this section. For this we shall see that an additional restriction on α is needed. Thus we will choose $\alpha\in\mathcal{I}$ in such a way that a finite number of elements of $\mathcal{I}$ are excepted. Since there are only two gaps with length d_2 and $d_4\ne 2d_2$, by (6) and the definition it follows that both d_2 and d_4 are lengths of primitive gaps.

The gap d_3 is primitive unless

$$d_3=id_2+jd_4 \tag{7}$$

for some $i\in\{0,1,2\}$ and $j\in\{0,\dots,N-1\}$. Condition (7) is equivalent to

$$\alpha((N^2+N-1)i-j+N-1)=\frac{(2N^2+3N-1)i-j-1}{2N+3}$$

and we see that there are finitely many (at most $3N$) values of $\alpha \in \mathcal{I}$ for which this is true with $i \in \{0,1,2\}$ and $j \in \{0,\dots,N-1\}$. Let $\mathcal{U}$ be the set of those bad values of α, that is,

$$\mathcal{U} = \{\alpha \in \mathcal{I} : d_3 = id_2 + jd_4 \text{ for some } i \in \{0,1,2\},\ j \in \{0,\dots,N-1\}\}.$$

Similarly, d_1 is primitive unless

$$d_1 = id_2 + jd_4 + hd_3 \tag{8}$$

for some $i \in \{0,1,2\}$, $j \in \{0,\dots,N-1\}$, $h \in \{0,\dots,N\}$. Condition (8) is equivalent to

$$\begin{aligned}\alpha((N^2+N-1)i - j - (N-1)h + N^2 + 2N - 1)\\ = \frac{(2N^2+3N-1)i - j + h + 2N^2 + 3N - 1}{2N+3}.\end{aligned} \tag{9}$$

Since the coefficient of α on the left-hand side of (9) is

$$\geq -(N-1) - N(N-1) + N^2 + 2N - 1 = 2N > 0,$$

it follows that there are only finitely many (at most $3N(N+1)$) values of $\alpha \in \mathcal{I}$ for which (8) is satisfied with $i \in \{0,1,2\}$, $j \in \{0,\dots,N-1\}$, $h \in \{0,\dots,N\}$. The set of those bad values is

$$\begin{aligned}\mathcal{V} = \{\alpha \in \mathcal{I} : d_1 = id_2 + jd_4 + hd_3 \text{ for some } i \in \{0,1,2\},\\ j \in \{0,\dots,N-1\},\ h \in \{0,\dots,N\}\}.\end{aligned}$$

The set $\mathcal{U} \cup \mathcal{V}$ being finite, there are only a finite number of values of $\alpha \in \mathcal{I}$ for which d_3 or d_1 is not primitive.

In conclusion, for $\alpha \in \mathcal{I} \setminus (\mathcal{U} \cup \mathcal{V})$ all the gaps are primitive and there exist exactly four primitive arcs, as claimed.

We conclude the case $k = 2$ with the remark that in the above example, the lengths of the primitive arcs are not linearly independent over $\mathbb{Z}$.

In the second part of this section we move in 3 dimensions and we present a numerical example where we find eight primitive gaps. Trying to get such an example, we found that for small values of N_1, N_2, N_3 and $\alpha_1, \alpha_2, \alpha_3$ chosen at random, it is quite unlikely to encounter an example with eight primitive gaps. There is an interesting behavior though which combines *continuity* with *jumps* in the number of distinct lengths of gaps and the number of primitive gaps, if one changes slightly the values of α's and the N's. We also observed that the number of primitive gaps is near the bound 2^k when most of the lengths of the gaps are close to one another, and very often there are few primitive gaps when some gaps are small, other large and the remaining have lengths well distributed in-between. Starting with three real numbers in $(0,1)$ and three integers N_1, N_2, N_3, moving back and forth with slight changes in the chosen values, we arrived at the following example. (This example is by no means singular; one can find similar examples in its *neighborhood* or with completely different values of α's and N's.) Choose $N_1 = 2$, $N_2 = 12$, $N_3 = 13$ and $\alpha_1 = 0.601189$, $\alpha_2 = 0.131320$, $\alpha_3 = 0.580500$. Then there are $3 \cdot 13 \cdot 14 = 546$ arcs between the points generated by the set

$$\{\{n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3\} : n_1 = 0, 1, 2,\ n_2 = 0, 1, \dots, 12,\ n_3 = 0, 1, \dots, 13\}.$$

The arcs have only 13 distinct lengths, $d_1, \dots, d_{13}$. We present the results in Table 1.

Table 1

Arc	Length	No. of apparitions	
d_1	0.000191	104	primitive
d_2	0.000711	176	primitive
d_3	0.000722	8	primitive
d_4	0.000902	10	
d_5	0.001242	12	primitive
d_6	0.002718	5	primitive
d_7	0.002898	12	primitive
d_8	0.003238	77	primitive
d_9	0.003418	40	primitive
d_{10}	0.003429	30	
d_{11}	0.003609	6	
d_{12}	0.003949	36	
d_{13}	0.004140	30	

The gaps $d_4, d_{10}, d_{11}, d_{12}, d_{13}$ are not primitive because

$$d_4 = d_1 + d_2, \quad d_{10} = d_2 + d_6, \quad d_{11} = d_2 + d_7, \quad d_{12} = d_2 + d_8, \quad d_{13} = d_5 + d_7.$$

The other eight gaps are primitive.

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