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LOCAL DERIVATIONS IN POLYNOMIAL AND POWER SERIES RINGS

BY

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Abstract. We give a description of all local derivations (in the Kadison sense) in the polynomial ring in one variable in characteristic two. Moreover, we describe all local derivations in the power series ring in one variable in any characteristic.

1. Introduction. The notion of a local derivation appeared in the paper of Kadison [1]. Let k be a field and A a commutative k-algebra with unity. Recall that a k-linear map $d : A \to A$ is a *derivation of* A if $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$. Next, a k-linear map $\alpha : A \to A$ is called a *local derivation of* A if for each $a \in A$ there exists a derivation d_a of A such that $\alpha(a) = d_a(a)$.

Each derivation of A is a local derivation. It is known that every local derivation of a polynomial ring over an infinite field k is a derivation. In the case $k = \mathbb{C}$ this was proved by Kadison [1]. For any infinite field k the result was formulated by Yon [3], but his proof was incorrect. The correct proof was given by Nowicki [2]. Furthermore, in [2] Nowicki gave an example of a local derivation of the polynomial ring in n variables over a finite field which is not a derivation.

In Section 3 we describe all local derivations in any polynomial ring in one variable in characteristic two. In our proof we use, among other things, the lemma (from Section 2) which states, under some additional assumptions, the invariance of the quotient of the product by the least common multiple. These methods are unrelated to Nowicki's example, but at the end of Section 3 we present a certain natural generalization of that example.

The final Section 4 is devoted to power series rings. It contains a construction of an infinite family of local derivations in the power series ring in one variable over an arbitrary field. This family is a generalization of another example given by Nowicki [2]. Finally, we give a full description of local derivations in any power series ring in one variable.

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2. Preliminary lemmas. Let k be a finite field of a positive characteristic p. If l is a non-negative integer, then we denote by B_l the subset of $k[t]$ defined by

$$B_l := \{h \in k[t];\ \deg_t(h) < l\}.$$

Moreover, we use the notations

$$G_l(g) := \prod_{h \in B_l} (g + h), \qquad H_l(g) := \operatorname{lcm}\{g + h;\ h \in B_l\}$$

(the least common multiple is not uniquely defined, from now on we assume that it is a monic polynomial) for any $g \in k[t]$ such that $\deg_t(g) \geq l$ (then each element of the form $g+h$, for $h \in B_l$, is non-zero). It is clear that $G_l(g)$ is divisible by $H_l(g)$.

LEMMA 1. *Let $g_1, g_2 \in k[t]$ be monic polynomials and* $\deg_t(g_1) \geq l$, $\deg_t(g_2) \geq l$. *Then*

$$\frac{G_l(g_1)}{H_l(g_1)} = \frac{G_l(g_2)}{H_l(g_2)}.$$

Proof. Put $L_1 := \{g_1 + h;\ h \in B_l\}$ and $L_2 := \{g_2 + h;\ h \in B_l\}$. Define $v_g(f) := m$ for $f, g \in k[t] \setminus \{0\}$ such that $f = g^m \overline{f}$ and $\overline{f}$ is not divisible by g. Let $g \in k[t]$ be an irreducible polynomial. Assume that g is a factor of at least one polynomial from the set L_1, that is,

$$\{f \in L_1;\ v_g(f) \geq 1\} \neq \emptyset.$$

If $\#\{f \in L_1;\ v_g(f) \geq 1\} = 1$, then g is a factor (in some power s) of exactly one polynomial from L_1. Then $v_g(G_l(g_1)) = s = v_g(H_l(g_1))$ and

$$v_g\left(\frac{G_l(g_1)}{H_l(g_1)}\right) = 0.$$

Let s be a positive integer. We first prove that if

$$\#\{f \in L_1;\ v_g(f) \geq s\} \geq 2,$$

then

$$\#\{f \in L_1;\ v_g(f) \geq s\} = \#\{f \in L_2;\ v_g(f) \geq s\}.$$

Since $g^s \mid g_1 + h_1$ and $g^s \mid g_1 + h_2$ for some $h_1, h_2 \in B_l$, $h_1 \neq h_2$, we have $g^s \mid h_1 - h_2$ and thus $\deg_t(g^s) < l$. Let $g_1 - g_2 = g^s q + r$ for $q, r \in k[t]$ such that $\deg_t(r) < \deg_t(g^s)$. Then

$$g_1 \equiv g_2 + r \pmod{g^s}$$

and $r \in B_l$. Since B_l is an additive group, when we add consecutively to both sides of the congruence all the elements of B_l we get on the left-hand side all the polynomials from L_1 and on the right-hand side all the polynomials from L_2. Thus g^s divides the same number of polynomials from L_1 and of polynomials from L_2.

Suppose that $\#\{f \in L_1;\, v_g(f) \geq 1\} \geq 2$. Put

$$m_s := \#\{f \in L_1;\, v_g(f) \geq s\}$$

for any positive integer s. Let

$$j = \max\{s \geq 1;\, m_s \geq 2\},$$
$$i_1 = \max\{s \geq 1;\, m_s \geq 1\} = \max\{v_g(f);\, f \in L_1\}.$$

The numbers j and i_1 are well defined since L_1 is a finite set of non-zero elements. Then

$$v_g(G_l(g_1)) = m_1 + \ldots + m_j + (i_1 - j), \qquad v_g(H_l(g_1)) = i_1.$$

Hence

$$v_g\left(\frac{G_l(g_1)}{H_l(g_1)}\right) = m_1 + \ldots + m_j - j.$$

From what has already been proved, we conclude that m_s, for $s = 1, \ldots, j$, is also the number of polynomials from L_2 divisible by g^s, that is,

$$\#\{f \in L_2;\, v_g(f) \geq s\} = m_s.$$

Moreover, we easily deduce that

$$\#\{f \in L_2;\, v_g(f) \geq s\} \leq 1$$

for $s > j$. Let

$$i_2 = \max\{v_g(f);\, f \in L_2\}.$$

Then

$$v_g(G_l(g_2)) = m_1 + \ldots + m_j + (i_2 - j), \qquad v_g(H_l(g_2)) = i_2.$$

Hence

$$v_g\left(\frac{G_l(g_2)}{H_l(g_2)}\right) = m_1 + \ldots + m_j - j.$$

The same arguments are valid when $\{f \in L_2;\, v_g(f) \geq 1\} \neq \emptyset$. Thus the polynomials $G_l(g_1)/H_l(g_1)$ and $G_l(g_2)/H_l(g_2)$ have the same factorization. Since both are monic, they are equal. ∎

For example, for each monic $g \in \mathbb{Z}_2[t]$ such that $\deg_t(g) \geq 2$,

$$\frac{G_2(g)}{H_2(g)} = t(t+1),$$

and for each monic $g \in \mathbb{Z}_2[t]$ such that $\deg_t(g) \geq 3$,

$$\frac{G_3(g)}{H_3(g)} = t^4(t+1)^4(t^2+t+1).$$

Slight changes in the proof of Lemma 1 actually show that if g is a monic polynomial from $k[t]$ and $\deg_t(g) \geq l$, then

$$\frac{G_l(g)}{H_l(g)} = (-1)^l \prod_{h \in B_l \setminus \{0\}} h.$$

Moreover, the assumption of Lemma 1 that the polynomials are monic is not essential. If g_1, g_2 are arbitrary polynomials from $k[t]$ such that $\deg_t(g_1) \geq l$ and $\deg_t(g_2) \geq l$, then

$$\widetilde{g}_2 \frac{G_l(g_1)}{H_l(g_1)} = \widetilde{g}_1 \frac{G_l(g_2)}{H_l(g_2)},$$

where $\widetilde{g}_1$ and $\widetilde{g}_2$ are respectively the leading coefficients of g_1 and g_2. For our further purposes we require only Lemma 1.

LEMMA 2. *Let $f_1, \ldots, f_l \in k[t]$. Then the map $F_l : k[t] \to k[t]$ defined by*

$$F_l(a) = \prod_{(b_1, \ldots, b_l) \in k^l} (a + f_1 b_1 + \ldots + f_l b_l)$$

for $a \in k[t]$ is additive.

Proof. We prove by induction that for each $l \geq 1$ there exist $g_0, \ldots, g_s \in k[t]$ such that for every $a \in k[t]$,

$$F(a) = g_s a^{p^s} + \ldots + g_1 a^p + g_0 a.$$

This proves the lemma since in characteristic p raising to the power p^j is additive and a linear combination of additive maps is additive.

Assume $l = 1$ and let the field k consist of $\{c_1, \ldots, c_{p^r}\}$. Then

$$(a + f_1 c_1) \ldots (a + f_1 c_{p^r}) = a^{p^r} - f_1^{p^r - 1} a \tag{1}$$

since all the elements of the form $-f_1 c_i$ are roots of the polynomial $x^{p^r} - f_1^{p^r-1} x$.

Let $l > 1$. By (1),

$$\prod_{(b_1, \ldots, b_l) \in k^l} (a + f_1 b_1 + \ldots + f_l b_l)$$
$$= \prod_{(b_1, \ldots, b_{l-1}) \in k^{l-1}} ((a + f_1 b_1 + \ldots + f_{l-1} b_{l-1})^{p^r} - f_l^{p^r-1}(a + f_1 b_1 + \ldots + f_{l-1} b_{l-1})).$$

Hence $F(a)$ is equal to

$$\prod_{(b_1, \ldots, b_{l-1}) \in k^{l-1}} (a^{p^r} - f_l^{p^r-1} a + (f_1^{p^r} - f_l^{p^r-1} f_1) b_1 + \ldots + (f_{l-1}^{p^r} - f_l^{p^r-1} f_{l-1}) b_{l-1}).$$

We now apply the inductive assumption for $\widetilde{f}_i = f_i^{p^r} - f_l^{p^r-1} f_i$ and $i = 1, \ldots, l-1$. The proof is complete since $(a^{p^r} - f_l^{p^r-1} a)^{p^m}$ is for any natural

m again a polynomial expression in a, with coefficients in $k[t]$, such that the exponents of a are powers of p. ∎

COROLLARY 3. *The map* $F : k[t] \to k[t]$ *defined by* $F(a) = G_l(a)$, *for* $a \in k[t]$, *is* k*-linear.*

Proof. In Lemma 2 replace f_i by t^{i-1} for $i = 1, \ldots, l$. Then F is additive. Let $c \in k$ and $c \neq 0$. Then $cG_l(a) = G_l(ca)$ since for $c \neq 0$ we have $\{ch;\, h \in B_l\} = B_l$ and since $c^{p^{lr}} = c$ in the field k of cardinality p^r (then B_l contains p^{lr} elements). If $c = 0$, then $cG_l(a) = 0 = G_l(ca)$. ∎

3. Polynomial rings in characteristic two. Let k be a finite field of characteristic two and n be a fixed non-negative even integer. Consider the set

$$A_n := \Big\{h \in k[t];\, \deg_t(h) < n,\, h = \sum_{2|i} a_i t^i,\, a_i \in k\Big\}.$$

This is a finite set (and $A_n = \{0\}$ in the case $n = 0$). Moreover, it forms an additive group. Define a k-linear map $\alpha_n : k[t] \to k[t]$ by

$$\alpha_n(t^s) = \begin{cases} 0, & s \leq n, \\ \operatorname{lcm}\{t^{s-1} + h;\, h \in A_n\}, & s > n,\, 2 \,|\, (s-1), \\ 0, & s > n,\, 2 \,|\, s. \end{cases}$$

Observe that all the exponents of the polynomials on the right-hand side of the above equality are even numbers. Thus we may replace t^2 by x. Furthermore we consider the least common multiple in the ring $k[x]$. Obviously α_0 is a partial derivative.

PROPOSITION 4. *If* $n \neq 0$, *then* α_n *is a local derivation of* $k[t]$ *which is not a derivation.*

Proof. Let $f \in k[t]$. If $\deg_t(f) \leq n$, then $\alpha_n(f) = 0 = d(f)$ where d is the zero derivation of $k[t]$. Assume that $\deg_t(f) = m > n$. Let $U := \{n < u \leq m;\, 2 \,|\, (u-1)\}$ and $J_n(g) := \operatorname{lcm}\{g + h;\, h \in A_n\}$ for $g \in k[t]$ such that $\deg_t(g) \geq n$. If $f = \sum_{i \leq m} a_i t^i$, where $a_i \in k$ and $a_m \neq 0$, then

$$\alpha_n(f) = \sum_{i \in U} a_i J_n(t^{i-1}).$$

Suppose that $\alpha_n(f) = f'g$ (we denote by f' the derivative $\partial f/\partial t$) for some $g \in k[t]$. Let $d : k[t] \to k[t]$ be the derivation such that $d(t) = g$. Then $d(f) = f'd(t) = \alpha_n(f)$ and consequently α_n is a local derivation. Hence, it remains to prove that f' divides $\alpha_n(f)$. Since

$$f' \in \Big\{\sum_{i \in U} a_i t^{i-1} + h;\, h \in A_n\Big\},$$

it suffices to show that

$$\sum_{i\in U} a_i J_n(t^{i-1}) \quad \text{is divisible by} \quad J_n\Big(\sum_{i\in U} a_i t^{i-1}\Big).$$

Again, all the exponents of the polynomials of these two expressions are even numbers and we may replace t^2 by x. Then the above expressions are respectively of the forms

$$\sum_{l\le j\le s} c_j H_l(x^j), \qquad H_l\Big(\sum_{l\le j\le s} c_j x^j\Big)$$

for some l and s, where $c_j \in k$ (more precisely $j = (i-1)/2$ and $c_j = a_i$). Obviously

$$H_l\Big(\sum_{l\le j\le s} c_j x^j\Big) = H_l\Big(\sum_{l\le j\le s} \frac{c_j}{c_s}\, x^j\Big).$$

We now proceed to show that

$$\sum_{l\le j\le s} c_j H_l(x^j) = c_s H_l\Big(\sum_{l\le j\le s} \frac{c_j}{c_s}\, x^j\Big),$$

which completes the proof. By Lemma 1 it remains to prove that

$$\sum_{l\le j\le s} c_j G_l(x^j) = c_s G_l\Big(\sum_{l\le j\le s} \frac{c_j}{c_s}\, x^j\Big).$$

This equality is a consequence of Corollary 3.

The map α_n is not a derivation since $\alpha_n(t) = 0 \ne \alpha_n(t^{n+1})$. ■

We denote by $\sum_{2|n} f_n\alpha_n$, for $f_0, f_2, \ldots \in k[t]$, a map from $k[t]$ to $k[t]$ such that

$$\Big(\sum_{2|n} f_n\alpha_n\Big)(f) = \sum_{2|n} f_n\alpha_n(f)$$

for every $f \in k[t]$. Observe that the map is well defined since for each f of degree s we have $\alpha_n(f) = 0$ for $n \ge s$. The map $\sum_{2|n} f_n\alpha_n$ is a local derivation of $k[t]$ since for each f of degree s,

$$\sum_{2|n} f_n\alpha_n(f) = \sum_{n<s,\, 2|n} f_n\alpha_n(f)$$

and since a linear combination, with coefficients in $k[t]$, of local derivations of $k[t]$ is a local derivation of $k[t]$.

THEOREM 5. *A map β is a local derivation of $k[t]$ iff there exist unique polynomials $f_0, f_2, \ldots \in k[t]$ such that $\beta = \sum_{2|n} f_n\alpha_n$.*

Proof. Assume that $\beta : k[t] \to k[t]$ is a local derivation. Then $\beta(t^m) = 0$ for all even m. Define

$$f_0 := \beta(t), \qquad \beta_0 := \beta - f_0\alpha_0 = \beta - f_0\frac{\partial}{\partial t}.$$

Then β_0 is a local derivation and $\beta_0(t) = 0$. Suppose we have defined polynomials $f_0, f_2, \ldots, f_n$ and local derivations $\beta_0, \beta_2, \ldots, \beta_n$ where n is an even integer, and suppose $\beta_i(t^m) = 0$ for $m \leq i+2$. Since β_n is a local derivation and for every derivation $d : k[t] \to k[t]$ and polynomial $f \in k[t]$ we have $d(f) = f'd(t)$, we conclude that

$$f' \mid \beta_n(f)$$

for all $f \in k[t]$. In particular this is valid for all f of the form $t^{n+3} + g$ where $\deg_t(g) < n+3$. Since in this case $\beta_n(g) = 0$ and $(\partial/\partial t)t^m = 0$ for even m, we obtain

$$t^{n+2} + h \mid \beta_n(t^{n+3})$$

for all $h \in A_{n+2}$. Then

$$J_{n+2}(t^{n+2}) \mid \beta_n(t^{n+3}).$$

In the case $\beta_n(t^{n+3}) = 0$ we put $f_{n+2} := 0$. If $\beta_n(t^{n+3}) \neq 0$, then

$$\beta_n(t^{n+3}) = f_{n+2}J_{n+2}(t^{n+2})$$

for some $f_{n+2} \in k[t]$. Define $\beta_{n+2} := \beta_n - f_{n+2}\alpha_{n+2}$. Thus β_{n+2} is a local derivation and $\beta_{n+2}(t^{n+3}) = 0$. Moreover,

$$\beta - \beta_{n+2} = \sum_{m<n+3,\, 2|m} f_m\alpha_m.$$

We have obtained the sequence $(f_0, f_2, \ldots)$ of polynomials in $k[t]$ (in some cases f_n may be 0 for almost all n). Then $\beta = \sum_{2|n} f_n\alpha_n$ since for each $f \in k[t]$ of degree s we have

$$\sum_{2|n} f_n\alpha_n(f) = \sum_{n<s,\, 2|n} f_n\alpha_n(f) = \beta(f).$$

The sequence $(f_0, f_2, \ldots)$ is unique since the value of $\beta = \sum_{2|n} f_n\alpha_n$ at the monomials $t, t^3, t^5, \ldots$ determines successively $f_0, f_2, \ldots$ ∎

A consequence of the fact that $(f_0, f_2, \ldots)$ is unique, is the linear independence of the set $\{\alpha_n\}_{2|n}$ over $k[t]$. Unfortunately the methods used in the paper in characteristic two are not valid for higher characteristics. Instead it is possible to generalize Nowicki's example of [2].

Let k be a finite field of characteristic p and of cardinality $q := p^r$ with $r \geq 1$. Let $v(t) \in k[t]$ be a polynomial of the form

$$v(t) = c_m t^{q^m} + c_{m-1}t^{q^{m-1}} + \ldots + c_1 t^q + c_0 t,$$

where $c_0, \ldots, c_m \in k$. Define a map $\gamma : k[t] \to k[t]$ by

$$\gamma(f) = v(f')$$

for all $f \in k[t]$. Obviously if $\deg_t v(t) = 1$, then γ is a derivation. A modification of the proof of Proposition 3 from [2] shows that if $\deg_t v(t) > 1$, then γ is a local derivation of $k[t]$ which is not a derivation. The example from [2] in the case of one variable is obtained for $v(t) = t - t^q$.

As in [2] we may generalize the construction of the map γ to the ring $k[x_1, \ldots, x_n]$. Unfortunately even for one variable and characteristic two the family of local derivations obtained above for all v is not sufficient for description of all local derivations. A simple calculation shows that if $k = \mathbb{Z}_2$, then the local derivation $\alpha_4 : \mathbb{Z}_2[t] \to \mathbb{Z}_2[t]$ and the set of the local derivations γ for all v are linearly independent over $\mathbb{Z}_2[t]$. It is also impossible to represent the map α_4 as an infinite sum of these local derivations.

4. Local derivations in the power series rings. Let k be an arbitrary field and $A = k[[t]]$ be the formal power series ring over k. In [2] Nowicki proved that the map $\gamma : A \to A$ such that if $f \in A$, then $\gamma(f)$ is the coefficient of the monomial t in f, is a local derivation of A which is not a derivation. First we generalize this result. Denote by p the characteristic of k.

Let n be a positive integer. Define $h_n : A \to A$ by

$$h_n(f) = na_n t^{n-1} \quad \text{for } f = \sum_{i=0}^{\infty} a_i t^i.$$

If $p > 0$, then we denote by B the set of the maps h_n for n not divisible by p. If $p = 0$, then put $B := \{h_1, h_2, \ldots\}$.

PROPOSITION 6. *The map $h_n : A \to A$ is a local derivation of A for each $n \geq 1$. If $h_n \in B$, then h_n is not a derivation. The set $\{\partial/\partial t\} \cup B$ is linearly independent over A.*

Proof. Let m be an integer such that $m > n$ and if $p > 0$, then m is not divisible by p. By definition $h_n(t^n) = nt^{n-1}$ and $h_n(t^m) = 0$. Suppose h_n is a derivation. Then $h_n(t^n) = nt^{n-1}h_n(t)$ and $h_n(t^m) = mt^{m-1}h_n(t)$. Thus $h_n(t) = 1$ and $h_n(t) = 0$. We obtain a contradiction.

Obviously h_n is k-linear. Let $f = \sum_{i=0}^{\infty} a_i t^i \in A$. If $na_n = 0$, then $h_n(f) = 0 = d(f)$ where d is the zero derivation of A. Suppose that $na_n \neq 0$. Assume sa_s is the first non-zero element of the sequence $(a_1, 2a_2, \ldots, na_n)$. Then the element $g := \sum_{i=s}^{\infty} ia_i t^{i-s}$ is invertible. Define the derivation d as follows:

$$d(t) = g^{-1} na_n t^{n-s}.$$

Hence

$$d(f) = f' d(t) = t^{s-1} na_n t^{n-s} = h_n(f).$$

Therefore h_n is a local derivation.

Let $c\partial/\partial t + c_1 h_{i_1} + \ldots + c_s h_{i_s} = 0$. Let m be a positive integer such that $m > \max\{i_1, \ldots, i_s\}$ and m is not divisible by p in the case $p > 0$. Then

$$0 = \left(c\frac{\partial}{\partial t} + c_1 h_{i_1} + \ldots + c_s h_{i_s}\right)(t^m) = cmt^{m-1},$$

hence $c = 0$. Suppose $i_r = \max\{i_j;\ c_j \neq 0\}$. Then

$$0 = (c_1 h_{i_1} + \ldots + c_s h_{i_s})(t^{i_r}) = c_r i_r t^{i_r - 1},$$

and thus $c_r = 0$. This contradiction proves that $c_j = 0$ for all i_j. ■

Note that the γ from Nowicki's example is equal to h_1. Define now $M_n := \{f \in A;\ f = \sum_{i=n}^{\infty} a_i t^i\}$ for $n \geq 0$. It is evident that if $n \geq 1$, then $h_n(A) \subseteq M_{n-1}$. Let $f_n \in A$ for all $n \geq 1$. We denote by $\sum_{n=1}^{\infty} f_n h_n$ the map from A to A defined by

$$\Big(\sum_{n=1}^{\infty} f_n h_n\Big)(f) = \sum_{n=1}^{\infty} f_n h_n(f)$$

for every $f \in A$. Note that this map is well defined since for each $f \in A$ we have $f_n h_n(f) \in M_{n-1}$. Thus the coefficient of every monomial t^i is a sum of a finite number of summands.

PROPOSITION 7. *Every local derivation $h : A \to A$ is determined by its values at t^i for all $i \geq 1$, that is, by the set $\{h(t^i);\ i \geq 1\}$.*

Proof. Let $f = \sum_{i=0}^{\infty} a_i t^i \in A$ and $h(f) = \sum_{i=0}^{\infty} b_i t^i$. Clearly $h(a_0) = 0$. Then

$$h(f) = a_1 h(t) + h\Big(\sum_{i=2}^{\infty} a_i t^i\Big).$$

Observe that $h(M_n) \subseteq M_{n-1}$ for all $n \geq 1$. Indeed, if $g = \sum_{i=n}^{\infty} c_i t^i$, then

$$h(g) = d_g(g) = \Big(\sum_{i=n}^{\infty} i c_i t^{i-1}\Big) d_g(t) \subseteq M_{n-1}.$$

Since $h(\sum_{i=2}^{\infty} a_i t^i) \in M_1$, the coefficient b_0 is determined by $h(t)$. Suppose b_j for $j < n$ is determined by $h(t^i)$ for $1 \leq i \leq n$. Then

$$h(f) = \sum_{i=1}^{n+1} a_i h(t^i) + h\Big(\sum_{i=n+2}^{\infty} a_i t^i\Big).$$

Since $h(\sum_{i=n+2}^{\infty} a_i t^i) \in M_{n+1}$, the coefficient b_n is determined by $h(t^i)$ for $1 \leq i \leq n+1$. ■

THEOREM 8. *A map $h : A \to A$ is a local derivation iff it is of the form $h = \sum_{n=1}^{\infty} f_n h_n$ where $f_1, f_2, \ldots \in A$.*

Proof. The map $\sum_{n=1}^{\infty} f_n h_n$ is k-linear since, by the fact that $h_n(A) \subseteq M_{n-1}$ for $n \geq 1$, the coefficient of every monomial t^i is determined only by a finite number of summands. Thus k-linearity follows from the fact that a linear combination of k-linear maps is k-linear.

Let $f = \sum_{i=0}^{\infty} a_i t^i \in A$. If $(\sum_{n=1}^{\infty} f_n h_n)(f) = 0$, then d_f may be the zero derivation. Assume $(\sum_{n=1}^{\infty} f_n h_n)(f) = \sum_{i=r}^{\infty} b_i t^i$ and $b_r \neq 0$. Let $f' = \sum_{i=s}^{\infty} i a_i t^{i-1}$ and $s a_s \neq 0$. We first prove that $s \leq r+1$. Suppose, contrary to our claim, that $s > r+1$. Then

$$\Big(\sum_{n=1}^{\infty} f_n h_n\Big)(f) = \sum_{n=1}^{r+1} f_n h_n(f) + \sum_{n=r+2}^{\infty} f_n h_n(f) = \sum_{n=r+2}^{\infty} f_n h_n(f)$$

since $h_n(f) = n a_n t^{n-1} = 0$ for $n \leq r+1$. However $\sum_{n=r+2}^{\infty} f_n h_n(f) \in M_{r+1}$ and we obtain a contradiction.

Since $s a_s \neq 0$, the element $h := \sum_{i=s}^{\infty} i a_i t^{i-s}$ is invertible. Define a derivation d of A by

$$d(t) = h^{-1} \sum_{i=r}^{\infty} b_i t^{i+1-s}.$$

Hence

$$d(f) = f' d(t) = t^{s-1} \sum_{i=r}^{\infty} b_i t^{i+1-s} = \sum_{i=r}^{\infty} b_i t^i = \Big(\sum_{n=1}^{\infty} f_n h_n\Big)(f).$$

Let $h : A \to A$ be a local derivation. Then

$$h(t^i) = d_{t^i}(t^i) = i t^{i-1} d_{t^i}(t)$$

for all $i \geq 1$. Define $f_i := d_{t^i}(t)$ for each $i \geq 1$. By Proposition 7, the local derivations h and $\sum_{n=1}^{\infty} f_n h_n$ are equal if and only if they have the same values at t^i for all $i \geq 1$. Indeed,

$$\Big(\sum_{n=1}^{\infty} f_n h_n\Big)(t^i) = f_i h_i(t^i) = f_i i t^{i-1} = i t^{i-1} d_{t^i}(t) = h(t^i). \ \blacksquare$$

The proof above gives more, namely a k-linear map $h : A \to A$ is a local derivation iff for each $f = \sum_{i=0}^{\infty} a_i t^i \in A$ we have $s \leq r+1$, where $h(f) = \sum_{i=r}^{\infty} b_i t^i$, $b_r \neq 0$ and $f' = \sum_{i=s}^{\infty} i a_i t^{i-1}$, $s a_s \neq 0$. Note that as a consequence of Proposition 7 we find that the polynomials f_n in Theorem 8 are unique for any n in the case $p = 0$, and for n not divisible by p in the case $p > 0$.

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