COLLOQUIUM MATHEMATICUM

VOL. 128 2012 NO. 1

EXTENDED DERDZIŃSKI–SHEN THEOREM FOR CURVATURE TENSORS

BY

CARLO ALBERTO MANTICA and LUCA GUIDO MOLINARI (Milano)

Abstract. We extend a remarkable theorem of Derdziński and Shen, on the restrictions imposed on the Riemann tensor by the existence of a nontrivial Codazzi tensor. We show that the Codazzi equation can be replaced by a more general algebraic condition. The resulting extension applies both to the Riemann tensor and to generalized curvature tensors.

1. Introduction. On a Riemannian manifold with metric g_{ij} and Riemann connection ∇_i, a *Codazzi tensor* is a symmetric tensor that satisfies the Codazzi equation

$$\nabla_j b_{kl} - \nabla_k b_{jl} = 0. \tag{1.1}$$

In terms of differential forms, (1.1) states the closedness of the 1-form $b_{jk}dx^k$ [2, 11]. Codazzi tensors are of great interest in geometry and have been studied by several authors, including Berger and Ebin [1], Bourguignon [2], Derdziński [5, 6], Derdziński and Shen [7], Ferus [8], Simon [15, 16]; a survey of results is given in Besse's book [3] (pp. 436–440). Codazzi tensors occur in the study of Riemannian manifolds with harmonic curvature. For example, the Ricci tensor is a Codazzi tensor if and only if $\nabla_m R_{jkl}{}^m = 0$, i.e. the manifold has harmonic Riemann curvature [3, p. 435]. The Weyl 1-form $\left[R_{kj} - \frac{R}{2(n-1)} g_{kj}\right] dx^k$ is closed if and only if $\nabla_m C_{jkl}{}^m = 0$, i.e. the manifold has harmonic conformal curvature [3, p. 435].

In [2] important geometric and topological consequences of the existence of a nontrivial Codazzi tensor are examined, in particular the restrictions this imposes on the structure of the curvature operator. Derdziński and Shen improved them and stated the following remarkable theorem (recorded in Besse's book [3, p. 438]):

THEOREM 1.1 (Derdziński–Shen, [7]). *Let b_{ij} be a Codazzi tensor on a Riemannian manifold M, x a point of M, and λ and μ two eigenvalues of*

2010 *Mathematics Subject Classification*: Primary 53B20; Secondary 53B21.
Key words and phrases: Codazzi tensor, Riemann tensor, generalized curvature tensor.

DOI: 10.4064/cm128-1-1

the operator $b_i{}^j(x)$*, with eigenspaces* V_λ *and* V_μ *in* T_xM*. Then the subspace* $V_\lambda \wedge V_\mu$ *is invariant under the action of the curvature operator* R_x*.*

They also obtained the following result: in an n-dimensional Riemannian manifold with a Codazzi tensor having n distinct eigenvalues almost everywhere, all real Pontryagin classes vanish.

We point out that the Codazzi equation is a *sufficient* condition for the theorem to hold. A more general one is suggested by the following lemma:

LEMMA 1.2. *Any symmetric Codazzi tensor* b_{kl} *satisfies the algebraic identity*

$$R_{jkl}{}^m b_{im} + R_{kil}{}^m b_{jm} + R_{ijl}{}^m b_{km} = 0. \tag{1.2}$$

Proof. The following identity among commutators is true for a Codazzi tensor:

$$[\nabla_j, \nabla_k] b_{il} + [\nabla_k, \nabla_i] b_{lj} + [\nabla_i, \nabla_j] b_{kl} = 0. \tag{1.3}$$

Each commutator is then evaluated: $[\nabla_i, \nabla_j] b_{kl} = R_{ijk}{}^m b_{ml} + R_{ijl}{}^m b_{km}$. Cancellations occur by the first Bianchi identity and the result is obtained. ∎

Our first extension of the theorem states that if a symmetric tensor b_{kl} satisfies the algebraic condition (1.2), then the same conclusions of Derdziński and Shen's theorem are valid for the Riemann tensor. It turns out that the proof of the extended theorem is much simpler than Derdziński and Shen's proof.

The replacement of the Codazzi equation by an algebraic condition allows for a further natural extension of the theorem to generalized curvature tensors. It includes well known tensors such as conformal, concircular and conharmonic tensors [14, 9, 12].

2. Extension of the Derdziński–Shen theorem for the Riemann tensor. As in the original theorem, we need an auxiliary tensor and a lemma to prove that it is a generalized curvature tensor, a concept introduced by Kobayashi and Nomizu [10, p. 198]. The algebraic condition (1.2) is here used rather than Codazzi's equation to prove both the lemma and the extended theorem.

DEFINITION 2.1. A tensor K_{ijlm} is a *generalized curvature tensor* if it has the symmetries of the Riemann curvature tensor:

(a) $K_{ijkl} = -K_{jikl} = -K_{ijlk}$,
(b) $K_{ijkl} = K_{klij}$,
(c) $K_{ijkl} + K_{jkil} + K_{kijl} = 0$ (first Bianchi identity).

LEMMA 2.2. *If a symmetric tensor* b_{kl} *satisfies* (1.2)*, then* $K_{ijkl} = R_{ijrs} b_k{}^r b_l{}^s$ *is a generalized curvature tensor.*

Proof. Properties (a) are shown easily, for example

$$K_{ijlk} = R_{ijrs}b_l{}^r b_k{}^s = R_{ijsr}b_l{}^s b_k{}^r = -R_{ijrs}b_l{}^s b_k{}^r = -K_{ijkl}.$$

Property (c) follows from (1.2):

$$\begin{aligned} K_{ijkl} + K_{jkil} + K_{kijl} &= R_{ijrs}b_k{}^r b_l{}^s + R_{jkrs}b_i{}^r b_l{}^s + R_{kirs}b_j{}^r b_l{}^s \\ &= (R_{jis}{}^r b_{kr} + R_{kjs}{}^r b_{ir} + R_{iks}{}^r b_{jr})b_l{}^s = 0. \end{aligned}$$

Property (b) follows from (c): $K_{ijkl} + K_{jkil} + K_{kijl} = 0$. Sum the identity over cyclic permutations of all indices i, j, k, l and use the symmetries (a) (this fact was pointed out in [10]).

It is easy to see that a first Bianchi identity also holds for the last three indices: $K_{ijkl} + K_{iklj} + K_{iljk} = 0$. ∎

THEOREM 2.3. *Let M be an n-dimensional Riemannian manifold with a symmetric tensor b_{kl} that satisfies the algebraic equation*

$$b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m = 0.$$

If X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point x of the manifold, with eigenvalues λ, μ and ν, then

$$X^i Y^j Z^k R_{ijkl} = 0 \tag{2.1}$$

provided that λ and μ are different from ν.

Proof. Consider the first Bianchi identity for the Riemann tensor, the condition (1.2) and the first Bianchi identity for the curvature $K_{lijk} = R_{lirs}b_j{}^r b_k{}^s$, and apply them to the three eigenvectors. The three algebraic relations can be put in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \mu\nu & \lambda\nu & \lambda\mu \end{bmatrix} \begin{bmatrix} R_{lijk}X^iY^jZ^k \\ R_{ljki}X^iY^jZ^k \\ R_{lkij}X^iY^jZ^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the matrix is $(\lambda - \mu)(\lambda - \nu)(\nu - \mu)$. If the eigenvalues are all different then $R_{lijk}X^iY^jZ^k = 0$; by the symmetries of the Riemann tensor the statement is true for the contraction of any three indices.

Suppose now that $\lambda = \mu \neq \nu$, i.e. X and Y belong to the same eigenspace; then the system of equations implies that $R_{lkij}X^iY^jZ^k = 0$. ∎

This completes the first extension of the theorem, involving the Riemann tensor. The question arises about non-Codazzi symmetric tensor fields that fulfill (1.2), and make the theorem applicable. We give some examples.

DEFINITION 2.4. A Riemannian manifold with symmetric tensor b_{ij} is *weakly b-symmetric* if

$$\nabla_i b_{kl} = A_i b_{kl} + B_k b_{il} + D_l b_{ik} \tag{2.2}$$

where A, B and D are 1-forms.

Weakly Ricci-symmetric manifolds [17] (see [4] and [12] for an overview) satisfying $\nabla_i R_{kl} = A_i R_{kl} + B_k R_{il} + D_l R_{ik}$, and the more general weakly Z-symmetric manifolds [13], are of this sort.

If $A-B$ is closed the evaluation of the three commutators on the left-hand side of (1.3) yields zero, so b_{ij} satisfies (1.2).

3. Extension of the Derdziński–Shen theorem to generalized curvature tensors. The theorem can be generalized further, and continues to hold if the Riemann tensor R is replaced by a generalized curvature tensor K, and a symmetric tensor field exists such that the condition (1.2) is valid with R replaced by K.

DEFINITION 3.1. Let K be a generalized curvature tensor. A symmetric tensor b_{ij} is *K-compatible* if

$$K_{jkl}{}^m b_{im} + K_{kil}{}^m b_{jm} + K_{ijl}{}^m b_{km} = 0. \tag{3.1}$$

The metric tensor is trivially K-compatible, by the first Bianchi identity for K.

The following lemma is needed. Its proof is identical to that of Lemma 2.2, with (3.1) being used:

LEMMA 3.2. *If K is a generalized curvature tensor and b_{kl} is K-compatible, then $\hat{K}_{ijkl} = K_{ijrs} b_k{}^r b_l{}^s$ is a generalized curvature tensor.*

The following fact can be proven by exactly the same argument as Theorem 2.3.

THEOREM 3.3. *Let M be an n-dimensional Riemannian manifold with a generalized curvature tensor K and a K-compatible tensor b. If X, Y and Z are three eigenvectors of the matrix $b_r{}^s$ at a point x of M, with eigenvalues λ, μ and ν, then*

$$X^i\, Y^j\, Z^k\, K_{ijkl} = 0 \tag{3.2}$$

provided that λ and μ are different from ν.

Consider the following family of curvature tensors K:

$$\begin{aligned} K_{jkl}{}^m = R_{jkl}{}^m &+ \varphi(\delta_j^m R_{kl} - \delta_k^m R_{jl} + R_j{}^m g_{kl} - R_k{}^m g_{jl}) \\ &+ \chi(\delta_j^m g_{kl} - \delta_k^m g_{jl}), \end{aligned} \tag{3.3}$$

where φ, χ are scalar functions. An appropriate choice of the scalars gives conformal, concircular or conharmonic curvature tensors.

For this family, (1.2) is a sufficient condition for the extended theorem to apply:

PROPOSITION 3.4. *Let K be a tensor of the form* (3.3). *If a symmetric tensor b_{kl} is Riemann compatible, i.e.* (1.2) *holds, then it is K-compatible, and Theorem* 3.3 *applies.*

Proof. The proof is based on the following identity, which holds for curvature tensors of the form (3.3):

$$b_{im}K_{jkl}{}^m + b_{jm}K_{kil}{}^m + b_{km}K_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m$$
$$+\varphi[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)].$$

K-compatibility requires the right-hand side to be zero. If b_{kl} is Riemann compatible, then also $b_{im}R_j^m - b_{jm}R_i^m = 0$ (this is obtained by transvecting (1.2)), and all terms on the right-hand side vanish. ∎

REFERENCES

[1] M. Berger and D. Ebin, *Some characterizations of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geom. 3 (1969), 379–392.

[2] J.-P. Bourguignon, *Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein*, Invent. Math. 63 (1981), 263–286.

[3] A. L. Besse, *Einstein Manifolds*, Springer, 1987.

[4] U. C. De, *On weakly symmetric structures on Riemannian manifolds*, Facta Univ. Ser. Mech. Automatic Control Robotics 3 (2003), 805–819.

[5] A. Derdziński, *Some remarks on the local structure of Codazzi tensors*, in: Global Differential Geometry and Global Analysis, Lecture Notes in Math. 838, Springer, 1981, 251–255.

[6] A. Derdziński, *On compact Riemannian manifolds with harmonic curvature*, Math. Ann. 259 (1982), 145–152.

[7] A. Derdziński and C. L. Shen, *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. 47 (1983), 15–26.

[8] D. Ferus, *A remark on Codazzi tensors in constant curvature spaces*, in: Global Differential Geometry and Global Analysis, Lecture Notes in Math. 838, Springer, 1981, 257.

[9] Q. Khan, *On recurrent Riemannian manifolds*, Kyungpook Math. J. 44 (2004), 269–276.

[10] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience, New York, 1963.

[11] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles*, Dover, 1988.

[12] C. A. Mantica and L. G. Molinari, *A second order identity for the Riemann tensor and applications*, Colloq. Math. 122 (2011), 69–82.

[13] C. A. Mantica and L. G. Molinari, *Weakly Z-symmetric manifolds*, Acta Math. Hungar. 135 (2012), 80–96.

[14] M. M. Postnikov, *Geometry VI, Riemannian Geometry*, Encyclopaedia Math. Sci. 91, Springer, 2001.

[15] U. Simon, *Codazzi tensors*, in: Global Differential Geometry and Global Analysis, Lecture Notes in Math. 838, Springer, 1981, 289–296.

[16] U. Simon, A. Schwenk-Schellschmidt and L. Vrancken, *Codazzi-equivalent Riemannian metrics*, Asian J. Math. 14 (2010), 291–302.

[17] L. Tamássy and T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N.S.) 53 (1993), 140–148.

Carlo Alberto Mantica
I.I.S. Lagrange
Via L. Modignani 65
20161, Milano, Italy
E-mail: carloalberto.mantica@libero.it

Luca Guido Molinari
Physics Department
Università degli Studi di Milano
and
I.N.F.N. sezione di Milano
Via Celoria 16
20133, Milano, Italy
E-mail: luca.molinari@mi.infn.it

Received 20 January 2012;
revised 11 September 2012 (5615)