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A CLASSIFICATION OF TETRAVALENT ONE-REGULAR GRAPHS OF ORDER $3p^2$

BY

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Abstract. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper, tetravalent one-regular graphs of order $3p^2$, where p is a prime, are classified.

1. Introduction. In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph X we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and its full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X. A graph X is said to be *vertex-transitive* and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. In particular, if $\mathrm{Aut}(X)$ acts regularly on $A(X)$, then X is said to be *one-regular* (or 1-regular).

The main result of the paper is Theorem 3.4 asserting that, given a prime p and a tetravalent 1-regular graph X of order $3p^2$, we have one of the following cases:

- (i) $p \in \{2, 3, 5, 7, 11, 13\}$;
- (ii) X is a Cayley graph over $\langle x, y \mid x^p = y^{3p} = [x, y] = 1\rangle$, with connection set $\{y, y^{-1}, xy, x^{-1}y^{-1}\}$;
- (iii) X is a connected arc-transitive circulant graph with respect to every connection set S;
- (iv) X is one of the graphs described in [GP2, Lemma 8.4].

Clearly, a one-regular graph is connected, and it is of valency 2 if and only if it is a cycle. In this sense the first non-trivial case is that of cubic graphs. The first example of a cubic one-regular graph was constructed by Frucht [F] and later on a lot of related work has been done (as part of the more general investigation of cubic arc-transitive graphs; see [FK1, FK2, FK3, FKW]). Tetravalent one-regular graphs have also received considerable attention. In [C], tetravalent one-regular graphs of prime order

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were constructed. In [M], an infinite family of tetravalent one-regular Cayley graphs on alternating groups is given. Tetravalent one-regular circulant graphs were classified in [X1] and tetravalent one-regular Cayley graphs on abelian groups were classified in [XX]. Next, one may deduce a classification of tetravalent one-regular Cayley graphs on dihedral groups from [KO, WX1, WZ]. Let p and q be primes. Then clearly every tetravalent one-regular graph of order p is a circulant graph. Also, by [CO, PWX, PX, WX2, X1, XX] every tetravalent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classifications of tetravalent one-regular graphs of order $4p^2$, $6p^2$ and $2pq$ are given in [FKMZ, GS, ZF]. Continuing this research, the aim of this paper is to classify tetravalent one-regular graphs of order $3p^2$ (see Theorem 3.4).

A referee has pointed out that the results and the technique used in the paper can find useful application in the study of signed graphs in the sense of Harary [Har] and Zaslavsky [Z], and in the Coxeter spectral analysis of connected simply-laced edge-bipartite graphs recently developed in [S1, S2, S3] (see also [Ino] and [SW]).

2. Preliminaries. In this section, we introduce some notation and definitions as well as some preliminary results which will be used later.

For a regular graph X, we use $d(X)$ to represent the valency of X, and for any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. Let X be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(X)$ be vertex-transitive on X. For a G-invariant partition β of $V(X)$, the *quotient graph* X_β is defined as the graph with vertex set β such that, for any two vertices $B, C \in \beta$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X. Let N be a normal subgroup of G. Then the set β of orbits of N in $V(X)$ is a G-invariant partition of $V(X)$. In this case, the symbol X_β will be replaced by X_N.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. We call C_n an *n-cycle*.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G,\ s \in S\}$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. The permutation group $R(G) = \{R(g) \mid g \in G\}$ on G is called the *right regular representation* of G. It is easy to see that $R(G)$ is isomorphic to G, and it is a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Also it is easy to see that X is connected if and only if $G = \langle S \rangle$, that is, S is a connection set. Furthermore, the group $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Actually, $\mathrm{Aut}(G,S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Xu [X2] proved that $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G,S))_1 = \mathrm{Aut}(G,S)$. Suppose that $\alpha \in \mathrm{Aut}(G)$. One can easily prove that $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,S^\alpha)$ is normal. Determining automorphism groups, or equivalently, studying normality of Cayley graphs, plays an important role in the investigation of various symmetry properties of graphs, and has become a very active topic in algebraic graph theory. The concept of normal Cayley graph was first introduced by Xu [X2], and later much related work was done (see [BFSX, FX, G, GZ, KO, WZ]).

For $u \in V(X)$, denote by $N_X(u)$ the *neighbourhood* of u in X, that is, the set of vertices adjacent to u in X. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A covering $\widetilde{X}$ of X with projection p is said to be *regular* (or a *K-covering*) if there is a semiregular subgroup K of $\mathrm{Aut}(\widetilde{X})$ such that X is isomorphic to the quotient graph $\widetilde{X}/K$, say via a map h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\widetilde{X}$ is called a *cyclic* or an *elementary abelian covering* of X, and if $\widetilde{X}$ is connected, K becomes the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre onto itself. All the fibre-preserving automorphisms form a group called the *fibre-preserving group*.

Let $\widetilde{X}$ be a K-covering of X with projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha}p = p\alpha$, we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$ respectively.

For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, we denote by $C_G(H)$ the centralizer of H in G, and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

PROPOSITION 2.1 ([Hup, Chapter I, Theorem 4.5]). *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

PROPOSITION 2.2 ([W, Chapter I, Theorem 4.5]). *Every transitive abelian group G on a set Ω is regular.*

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. For any $g \in G$, g is said to be *semiregular* if $\langle g \rangle$ is semiregular. The following proposition due to Praeger et al. (see [GP1, Theorem 1.1]) gives a characterization of Cayley graphs in terms of their automorphism groups.

PROPOSITION 2.3. *Let X be a connected tetravalent $(G,1)$-arc-transitive graph. For each normal subgroup N of G, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N acts transitively on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} on X_N;*
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected tetravalent G/N-symmetric graph, and X is a G-normal cover of X_N.*

Moreover, if X is also $(G,2)$-arc-transitive, then case (3) *cannot happen.*

The following classical result is due to Wielandt [W, Theorem 3.4].

PROPOSITION 2.4. *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $\omega \in \Omega$. If p^m divides the length of the G-orbit containing ω, then p^m also divides the length of the P-orbit containing ω.*

To state the next result we need to introduce a family of tetravalent graphs that were first defined in [GP2]. The graph $C^{\pm 1}(p;3p,1)$ is defined to have vertex set $\mathbb{Z}_p \times \mathbb{Z}_{3p}$ and edge set $\{(i,j)(i\pm 1, j+1) \mid i \in \mathbb{Z}_p,\ j \in \mathbb{Z}_{3p}\}$. Also from [GP2, Definition 2.2], the graphs $C^{\pm 1}(p;3p,1)$ are Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{3p}$ with connection set $\{(1,1),(-1,1),(-1,-1),(1,-1)\}$. In the proof of Theorem 3.4, we will need $C^{\pm 1}(p;3p,1)$ with $p > 13$. It can be readily checked from [GP2, Definition 2.2] that for $p > 13$ these graphs are actually normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{3p}$.

PROPOSITION 2.5 ([GP2, Theorem 1.1]). *Let X be a connected, G-symmetric, tetravalent graph of order $3p^2$, and let $N = \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p, where p is an odd prime. Let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_{3p}$ and $K_v \cong \mathbb{Z}_2$ then X is isomorphic to $C^{\pm 1}(p;3p,1)$.*

The graphs defined in [GP2, Lemma 8.4] are all one-regular (see [GP2, Section 8]) and therefore we refer to [GP2] for an intrinsic description of these families.

PROPOSITION 2.6 ([GP2, Theorem 1.2]). *Let X be a connected, G-symmetric, tetravalent graph of order $3p^2$, and let $N = \mathbb{Z}_p \times \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p^2, where p is an odd prime. Let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_3$ and $K_v \cong \mathbb{Z}_2$ then X is isomorphic to one of the graphs in* [GP2, *Lemma* 8.4].

Let A be a group that acts on the group G. Also let A or G be solvable. Then the action of A on G is *coprime* if $(|A|, |G|) = 1$. The following result can be deduced from [KS, 8.2.7, p. 187].

PROPOSITION 2.7. *Suppose that the action of A on G is coprime. Then $G = [G, A] \times C_G(A)$.*

Finally in the following example we introduce $G(3p, r)$, which was first defined in [CO].

EXAMPLE 2.8. For each positive divisor r of $p-1$ we use H_r to denote the unique subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$ of order r which is isomorphic to $\mathbb{Z}_r$. Define a graph $G(3p, r)$ by $V(G(3p, r)) = \{x_i \mid i \in \mathbb{Z}_3,\ x \in \mathbb{Z}_p\}$ and $E(G(3p, r)) = \{x_i y_{i+1} \mid i \in \mathbb{Z}_3,\ x, y \in \mathbb{Z}_p,\ y - x \in H_r\}$. Then $G(3p, r)$ is a connected symmetric graph of order $3p$ and valency $2r$. Also $\mathrm{Aut}(G(3p, p-1)) \cong S_p \times S_3$. For $r \neq p-1$, $\mathrm{Aut}(G(3p, r))$ is isomorphic to $(\mathbb{Z}_p.H_r).S_3$ and acts regularly on the arc set, where $X.Y$ denotes an extension of X by Y.

3. One-regular graphs of order $3p^2$. To prove the main theorem we need the following three lemmas.

LEMMA 3.1. *Let G be a non-abelian group of order p^2q, where p and q are primes. Also let $p > q$, and N be a normal subgroup of order p such that G/N is cyclic. Then G is isomorphic to $\langle x, y, z \mid x^p = y^q = z^p = [x, z] = [y, z] = 1,\ y^{-1}xy = x^i\rangle$, where $i^q \equiv 1 \pmod p$ and $(i, p) = 1$.*

Proof. Let P and Q be a Sylow p-subgroup and q-subgroup of G, respectively. Clearly $P \trianglelefteq G$, and since $G' \neq 1$, we have $N = G'$. Since the action of Q on P is coprime, it follows that $P = [P, Q] \times C_P(Q)$, by Proposition 2.7. If $C_P(Q) = 1$, then $P \leq G'$, a contradiction. Also since G is non-abelian, $[P, Q] \neq 1$. So $q \mid p-1$, and $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Thus $G = PQ = [P, Q]Q \times C_P(Q)$, and hence $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \times \mathbb{Z}_p$. Therefore $G = \langle x, y, z \mid x^p = y^q = z^p = [x, z] = [y, z] = 1,\ y^{-1}xy = x^i\rangle$, where $i^q \equiv 1 \pmod p$ and $(i, p) = 1$. ∎

LEMMA 3.2. *Let p be a prime, $p \geq 5$ and $G = \langle x, y, z \mid x^p = y^3 = z^p = [x, z] = [y, z] = 1,\ y^{-1}xy = x^i\rangle$, where $i^3 \equiv 1 \pmod p$ and $(i, p) = 1$. Then there is no tetravalent one-regular normal Cayley graph X of order $3p^2$ on G.*

Proof. Suppose to the contrary that X is a tetravalent one-regular normal Cayley graph $\mathrm{Cay}(G,S)$ on G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G,S)$ of $1 \in G$ is transitive on S and the elements in S are all of the same order. The elements of G of order 3 lie in $\langle x, y\rangle$ and those of order p lie in $\langle x, z\rangle$. Since X is connected, $G = \langle S\rangle$ and hence S consists of elements of order $3p$. Denote by $\mathcal{S}_{3p}$ the elements of G of order $3p$. Then

$$S \subseteq \mathcal{S}_{3p} = \{x^s y^t z^j \mid s \in \mathbb{Z}_p,\ t \in \mathbb{Z}_3^*,\ j \in \mathbb{Z}_p^*\}.$$

We now consider the action of $\mathrm{Aut}(G)$ on $\mathcal{S}_{3p}$. Clearly the action is transitive and hence we may assume that $yz \in S$. In particular

$$S = \{yz, y^{-1}z^{-1}, x^u y^v z^l, y^{-v}x^{-u}z^{-l}\}$$

for some $u \in \mathbb{Z}_p$, $v \in \mathbb{Z}_3^*$, and $l \in \mathbb{Z}_p^*$. Also since $\sigma : x \mapsto x^u, y \mapsto y, z \mapsto z$ is an automorphism of G, we may suppose that

$$S = \{yz, y^{-1}z^{-1}, xy^v z^l, y^{-v}x^{-1}z^{-l}\}.$$

Since $\mathrm{Aut}(G,S)$ acts transitively on S, there is an $\alpha \in \mathrm{Aut}(G,S)$ such that $(yz)^\alpha = xy^v z^l$. If $y^\alpha = y$, then $z^\alpha = y^{-1}xy^v z^l = x^i y^{v+2} z^l$. Since $o(z) = p$, it follows that $z^\alpha = x^i z^l$, and $v = -2$. Also $(y^{-1}z^{-1})^\alpha = y^{-1}z^{-l}x^{-i} = y^{-1}x^{-i}z^{-l} = y^2x^{-i}z^{-l}$. By considering S, one has $y^2x^{-i}z^{-l} = y^2x^{-1}z^{-l}$. So $i = 1$, a contradiction.

If $y^\alpha = y^{-1}$, then $z^\alpha = yxy^v z^l = x^{i^2} y^{v+1} z^l$. Thus $v = -1$ and $z^\alpha = x^{i^2} z^l$. Now $(y^{-1}z^{-1})^\alpha = yz^{-l}x^{-i^2} = yx^{-i^2}z^{-l}$. On the other hand $(y^{-1}z^{-1})^\alpha = yx^{-1}z^{-l}$. Thus $i^2 = 1$, a contradiction. ∎

LEMMA 3.3. *Let p be a prime and $G = \mathbb{Z}_p \times \mathbb{Z}_{3p} = \langle x, y \mid x^p = y^{3p} = [x,y] = 1\rangle$, where $p \geq 5$. Also let X be a connected normal tetravalent Cayley graph. Then X is one-regular if and only if $X = \mathrm{Cay}(G, \{y, y^{-1}, xy, x^{-1}y^{-1}\})$. Moreover $X \cong C^{\pm 1}(p; 3p, 1)$.*

Proof. Suppose that X is a tetravalent one-regular normal Cayley graph $\mathrm{Cay}(G,S)$ on G with respect to the generating set S. Since X is one-regular normal, and since G is an abelian group of exponent $3p$, we see that S contains an element of order $3p$. Denote by $\mathcal{S}_{3p}$ the set of all elements of G of order $3p$. Then

$$S \subseteq \mathcal{S}_{3p} = \{x^a y^b \mid a \in \mathbb{Z}_p,\ b \in \mathbb{Z}_{3p}^*\}.$$

It is clear that $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{3p}$ by conjugation. In particular, replacing S by a suitable $\mathrm{Aut}(G)$-conjugate, we may assume that $y \in S$. Therefore

$$S = \{y, y^{-1}, x^u y^v, x^{-u}y^{-v}\}$$

for some $u \in \mathbb{Z}_p^*$ and $v \in \mathbb{Z}_{3p}^*$.

Let $B = \{\phi \in \mathrm{Aut}(G) \mid y^\phi = y\}$. Given $\phi \in B$, we have $\phi : x \mapsto x^a y^{3b}$, $y \mapsto y$ with $a, b \in \mathbb{Z}_p$ and $a \neq 0$. Note that every invertible element of $\mathbb{Z}_{3p}$ is of the form $1 + 3b$ or $-1 + 3b$, for some $b \in \mathbb{Z}_p$. Therefore, we may choose $a, b \in \mathbb{Z}_p$ with $(xy)^\phi = x^u y^v$ or $(xy^{-1})^\phi = x^u y^v$. Thus, replacing S by a suitable B-conjugate, we may assume that either $xy \in S$ or $xy^{-1} \in S$, that is,

$$S = \{y, y^{-1}, xy, x^{-1}y^{-1}\} \quad \text{or} \quad S = \{y, y^{-1}, xy^{-1}, x^{-1}y\}.$$

Let α be the automorphism of G with $x^\alpha = x$ and $y^\alpha = y^{-1}$. Clearly, α maps the first possibility for S onto the second. Therefore, we may assume that

$$S = \{y, y^{-1}, xy, x^{-1}y^{-1}\}.$$

Also, [GP2, Definition 2.2], we see that X is isomorphic to $C^{\pm 1}(p; 3p, 1)$. ∎

The following classification theorem is the main result of this paper.

THEOREM 3.4. *Let p be a prime. A tetravalent graph X of order $3p^2$ is one-regular if and only if one of the following holds:*

(i) $p \in \{2, 3, 5, 7, 11, 13\}$;
(ii) *X is a Cayley graph over $\langle x, y \mid x^p = y^{6p} = [x, y] = 1\rangle$, with connection set $\{y, y^{-1}, xy, x^{-1}y^{-1}\}$;*
(iii) *X is a connected arc-transitive circulant graph with respect to every connection set S;*
(iv) *X is one of the graphs described in* [GP2, *Lemma* 8.4].

Proof. Let X be a tetravalent one-regular graph of order $3p^2$. If $p \leq 13$, then $|V(X)| = 12, 27, 75, 147, 363$, or 507. Now, a complete list of tetravalent arc-transitive graphs of order at most 640 has recently been obtained by Potočnik, Spiga and Verret [PSV1, PSV2]. A quick inspection of this list (with the invaluable help of `magma`, see [BCP]) gives the proof of the theorem for $p \leq 13$.

Now, suppose that $p > 13$. Let $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer of $v \in V(X)$ in A. Let P be a Sylow p-subgroup of A. Since A is one-regular, it follows that $|A| = 12p^2$. Clearly, P is normal in A.

Assume first that P is cyclic. Let X_P be the quotient graph of X relative to the orbits of P and let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, the orbits of P are of length p^2. Thus $|V(X_P)| = 3$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 2.3, we have $X_P \cong C_3$ and hence $A/K \cong D_6$, forcing that $|K| = 2p^2$.

If A/P is abelian then, since A/K is a quotient group of A/P, also A/K is abelian. But since A/K is vertex-transitive on X_P, Proposition 2.2 implies that it is regular on X_P, contradicting arc-transitivity of A/K on X_P. Thus A/P is a non-abelian group.

Clearly K is not semiregular on $V(X)$. Then $K_v \cong \mathbb{Z}_2$, where $v \in V(X)$. By Proposition 2.1, $A/C \lesssim \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is not abelian we find that P is a proper subgroup of C.

If $C \cap K \neq P$, then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus $C \cap K = P$ and $1 \neq C/P = C/C \cap K \cong CK/K \trianglelefteq A/K \cong D_6$.

If $C/P \cong \mathbb{Z}_2$, then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is abelian, a contradiction. It follows that $|C/P| \in \{3, 6\}$, and hence C/P has a characteristic subgroup of order 3, say H/P. Thus $|H| = 3p^2$, and $H/P \trianglelefteq A/P$ implies that $H \trianglelefteq A$. In addition since $H \leq C = C_A(P)$, we see that H is abelian. Clearly $|H_v| \in \{1, 3\}$.

If $|H_v| = 3$, then H_v is a Sylow 3-subgroup of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing $H_v = 1$, a contradiction.

If $H_v = 1$, then since $|H| = 3p^2$, H is regular on $V(X)$. It follows that X is a Cayley graph on an abelian group with a cyclic Sylow p-subgroup P. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{3p^2}$, where $p > 13$, is the only abelian group with a cyclic Sylow p-subgroup. Also by [X1, Theorem 7], X is one-regular.

Now assume that P is elementary abelian. Suppose first that P is a minimal normal subgroup of A, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, the orbits of P are of length p^2, and thus $|V(X_P)| = 3$. By Proposition 2.3, $X_P \cong C_3$, and hence $A/K \cong D_6$, forcing $|K| = 2p^2$ and thus $K_v = \mathbb{Z}_2$. Proposition 2.6 implies that X is isomorphic to one of the graphs described in [GP2, Lemma 8.4].

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is transitive on $V(X_N)$; moreover, we have $|V(X_N)| = 3p$. By Proposition 2.3, either X_N is a cycle of length $3p$, or N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N.

If $X_N \cong C_{3p}$, then $A/K \cong D_{6p}$. Thus $|K| = 2p$ and so $K_v \cong \mathbb{Z}_2$. Applying Proposition 2.5, we conclude that X is isomorphic to $C^{\pm 1}(p; 3p, 1)$.

If, however, X_N is a tetravalent connected G/N-symmetric graph, then, by Proposition 2.3, X is a covering graph of a symmetric graph of order $3p$. By [WX2], $G(3p, 2)$ is the tetravalent symmetric graph of order $3p$ (see Example 2.8). Observe that in this case a one-regular subgroup of automorphism contains a normal regular subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_p$. Let H be a one-regular subgroup of automorphisms of X_N. Since X is a one-regular graph, A is the lift of H. Since H contains a normal regular subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_p$, also A contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $3p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_3 \times \mathbb{Z}_p \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_3$. If G is an abelian group, then G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{3p}$, or $\mathbb{Z}_{3p^2}$. Also if G is not abelian, then by Lemma 3.1, G is isomorphic to $\langle x, y, z \mid x^p = y^3 = z^p = [x, z] = [y, z] = 1, y^{-1}xy = x^i\rangle$, where $i^3 \equiv 1 \pmod p$ and $(i, p) = 1$. If $G \cong \mathbb{Z}_{3p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_{3p}$ then by [X1, Theorem 7], and Lemma 3.3, X is one-regular. Also for the latter case, by Lemma 3.2, X is not one-regular. This completes the proof. ∎

REFERENCES

[BFSX] Y. G. Baik, Y. Q. Feng, H. S. Sim and M. Y. Xu, *On the normality of Cayley graph of abelian groups*, Algebra Colloq. 5 (1998), 297–304.

[BCP] W. Bosma, C. Cannon and C. Playoust, *The MAGMA algebra system I: the user language*, J. Symbolic Comput. 24 (1997), 235–265.

[C] C. Y. Chao, *On the classification of symmetric graphs with a prime number of vertices*, Trans. Amer. Math. Soc. 158 (1971), 247–256.

[CO] Y. Cheng and J. Oxley, *On weakly symmetric graphs of order twice a prime*, J. Combin. Theory Ser. B 42 (1987), 196–211.

[FKMZ] Y.-Q. Feng, K. Kutnar, D. Marušič and C. Zhang, *Tetravalent one-regular graphs of order* $4p^2$, submitted.

[FK1] Y.-Q. Feng and J. H. Kwak, *Classifying cubic symmetric graphs of order* $10p$ *or* $10p^2$, Sci. China A 49 (2006), 300–319.

[FK2] Y.-Q. Feng and J. H. Kwak, *Cubic symmetric graphs of order twice an odd prime power*, J. Austral. Math. Soc. 81 (2006), 153–164.

[FK3] Y.-Q. Feng and J. H. Kwak, *Cubic symmetric graphs of order a small number times a prime or a prime square*, J. Combin. Theory Ser. B 97 (2007), 627–646.

[FKW] Y.-Q. Feng, J. H. Kwak and K. S. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$, Eur. J. Combin. 26 (2005), 1033–1052.

[FX] Y. Q. Feng and M. Y. Xu, *Automorphism groups of tetravalent Cayley graphs on regular p-groups*, Discrete Math. 305 (2005), 354–360.

[F] R. Frucht, *A one-regular graph of degree three*, Canad. J. Math. 4 (1952), 240–247.

[G] M. Ghasemi, *Automorphism groups of tetravalent Cayley graphs on minimal non-abelian groups*, Algebra Discrete Math. 13 (2012), 52–58.

[GS] M. Ghasemi and P. Spiga, *Tetravalent one-regular graphs of order* $6p^2$, submitted.

[GZ] M. Ghasemi and J. X. Zhou, *Automorphisms of a family of cubic graphs*, Algebra Colloq., to appear.

[GP1] A. Gardiner and C. E. Praeger, *On tetravalent symmetric graphs*, Eur. J. Combin. 15 (1994), 375–381.

[GP2] A. Gardiner and C. E. Praeger, *A characterization of certain families of tetravalent symmetric graphs*, Eur. J. Combin. 15 (1994), 383–397.

[Har] F. Harary, *On the notion of balance of a signed graph*, Michigan Math. J. 2 (1953), 143–146.

[Hup] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

[Ino] T. Inohara, *Characterization of clusterability of signed graph in terms of Newcomb's balance of sentiments*, Appl. Math. Comput. 133 (2002), 93–104.

[KS] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups. An Introduction*, Springer, 2004.

[KO] J. H. Kwak and J. M. Oh, *One-regular normal Cayley graphs on dihedral groups of valency 4 or 6 with cyclic vertex stabilizer*, Acta Math. Sinica English Ser. 22 (2006), 1305–1320.

[M] D. Marušič, *A family of one-regular graphs of valency 4*, Eur. J. Combin. 18 (1997), 59–64.

[PSV1] P. Potočnik, P. Spiga and G. Verret, http://www.matapp.unimib.it/~spiga/

[PSV2] P. Potočnik, P. Spiga and G. Verret, *Cubic vertex-transitive graphs on up to 1280 vertices*, arXiv:1201.5317v1 [math.CO].

[PWX] C. E. Praeger, R. J. Wang and M. Y. Xu, *Symmetric graphs of order a product of two distinct primes*, J. Combin. Theory Ser. B 58 (1993), 299–318.

[PX] C. E. Praeger and M. Y. Xu, *Vertex-primitive graphs of order a product of two distinct primes*, J. Combin. Theory Ser. B 59 (1993), 216–45.

[S1] D. Simson, *Mesh algorithms for solving principal Diophantine equations, sandglasstubes and tori of roots*, Fund. Inform. 109 (2011), 425–462.

[S2] D. Simson, *Algorithms determining matrix morsifications, Weyl orbits, Coxeter polynomials and mesh geometries of roots for Dynkin diagrams*, Fund. Inform. 120 (2012), in press.

[S3] D. Simson, *A Coxeter–Gram classification of simply-laced edge-bipartite graphs*, SIAM J. Discrete Math. (2012), to appear.

[SW] D. Simson and M. Wojewódzki, *An algorithmic solution of a Birkhoff type problem*, Fund. Inform. 83 (2008), 389–410.

[WX1] C. Q. Wang and M. Y. Xu, *Non-normal one-regular and tetravalent Cayley graphs of dihedral groups* D_{2n}, Eur. J. Combin. 27 (2006), 750–766.

[WX2] R. J. Wang and M. Y. Xu, *A classification of symmetric graphs of order* $3p$, J. Combin. Theory Ser. B 58 (1993), 197–216.

[WZ] C. Q. Wang and Z. Y. Zhou, *Tetravalent one-regular normal Cayley graphs of dihedral groups*, Acta Math. Sinica Chinese Ser. 49 (2006), 669–678.

[W] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

[XX] J. Xu and M. Y. Xu, *Arc-transitive Cayley graphs of valency at most four on Abelian groups*, Southest Asian Bull. Math. 25 (2001), 355–363.

[X1] M. Y. Xu, *A note on one-regular graphs*, Chinese Sci. Bull. 45 (2000), 2160–2162.

[X2] M. Y. Xu, *Automorphism groups and isomorphisms of Cayley digraphs*, Discrete Math. 44 (2001), 1502–1508.

[Z] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. 4 (1982), 47–74.

[ZF] J.-X. Zhou and Y.-Q. Feng, *Tetravalent one-regular graphs of order 2pq*, J. Algebraic Combin. 29 (2009), 457–471.

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