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3-WEAK AMENABILITY OF $(2n)$TH DUALS OF BANACH ALGEBRAS

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Abstract. We show that under some conditions, 3-weak amenability of the $(2n)$th dual of a Banach algebra A for some $n \geq 1$ implies 3-weak amenability of A.

1. Introduction and preliminaries. Throughout this paper, A is a Banach algebra, and $A', A'', \dots, A^{(n)}$ denote the iterated duals of A. We always use the first Arens product on $A^{(2n)}$ $(n \geq 1)$, denoted by $\square$. We regard A as a subalgebra of A'' by the canonical embedding $i : A \to A''$ $(a \mapsto \hat{a})$ where $\langle \hat{a}, f \rangle = \langle f, a \rangle$ for $f \in A'$. We recall that A' is a Banach A-bimodule under the actions

$$\langle a.f, b \rangle = \langle f, ba \rangle, \quad \langle f.a, b \rangle = \langle f, ab \rangle \quad (a, b \in A,\ f \in A').$$

Also let E be a Banach A-bimodule. Then E'' is a Banach A''-bimodule under the actions

$$F.\Lambda = w^*\text{-}\lim_i w^*\text{-}\lim_j \widehat{a_i x_j}, \quad \Lambda.F = w^*\text{-}\lim_j w^*\text{-}\lim_i \widehat{x_j a_i} \tag{1.1}$$

where $F = w^*\text{-}\lim_i \widehat{a_i}$ and $\Lambda = w^*\text{-}\lim_j \widehat{x_j}$ are such that $(a_i) \subset A$ and $(x_j) \subset E$ are bounded nets, and the limits are in the weak* topology.

In Section 2 we investigate two A''-bimodule structures on $A^{(5)}$ given by $A^{(5)} = (((A')')')''$ and $A^{(5)} = (((A'')')')'$, and also two $A^{(4)}$-bimodule structures on $A^{(7)} = ((((A')')')'')''$ and $A^{(7)} = ((((A'')'')')')'$. In a similar work [6] we investigated two A''-bimodule structures on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$.

For a Banach A-bimodule E, a continuous linear map $D : A \to E$ is called a *derivation* if

$$D(ab) = a.D(b) + D(a).b \quad (a, b \in A).$$

For $x \in E$ the derivation $\delta_x : A \to E$ defined by $\delta_x(a) = a.x - x.a$ is called an *inner derivation*. The Banach algebra A is called *amenable* if every derivation $D : A \to E'$ is inner, for each Banach A-bimodule E (see [8]). If

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every derivation $D : A \to A'$ [$D : A \to A^{(n)}$, $n \in \mathbb{N}$] is inner, then A is called *weakly amenable* [*n-weakly amenable*]; see also [1, 4] for details.

THEOREM 1.1 ([3, Theorem 2.7.17]). *Let A be a Banach algebra, E a Banach A-bimodule and $D : A \to E$ a continuous derivation. Then $D'' : A'' \to E''$ is a continuous derivation.*

REMARK 1.2. In the above theorem, the A''-bimodule structure on E'' is as in (1.1).

It is known that every $(n+2)$-weakly amenable Banach algebra is n-weakly amenable for $n \geq 1$ (see [4]). Also it was shown in [7] that if A'' is n-weakly amenable and A is a dual Banach algebra, then A is n-weakly amenable. In [9] it was shown that if A is complete Arens regular and every derivation $D : A \to A'$ is weakly compact, then weak amenability of $A^{(2n)}$ for some $n \geq 1$ implies weak amenability of A. Recently in [2] the authors determined conditions guaranteeing that 3-weak amenability of A'' implies 3-weak amenability of A.

In this paper we introduce conditions implying that 3-weak amenability of $A^{(2n)}$ for some $n \geq 1$ implies 3-weak amenability of A, and hence weak amenability of A. We find these conditions by studying $A^{(2n)}$-module structures on $A^{(2n+3)}$ in Section 2, and then apply them in Section 3.

2. $A^{(2n)}$-bimodule structures on $A^{(2n+3)}$. First we consider the A''-bimodule structures on $A^{(5)} = (((A')')')''$ and $A^{(5)} = (((A'')')')'$ and then the $A^{(4)}$-bimodule structures on $A^{(7)} = ((((A')')')'')''$ and $A^{(7)} = ((((A'')'')')')'$. The results can be extended to $A^{(2n)}$-bimodule structures on $A^{(2n+3)}$.

Take $a^{(5)} \in A^{(5)}$, $a^{(4)} \in A^{(4)}$, and $a'' \in A''$ with bounded nets $(a'''_\alpha) \subset A'''$, $(a''_i) \subset A''$, and $(a_\beta) \subset A$ such that $a^{(5)} = w^*\text{-}\lim_\alpha \widehat{a'''_\alpha}$, $a^{(4)} = w^*\text{-}\lim_i \widehat{a''_i}$, and $a'' = w^*\text{-}\lim_\beta \widehat{a_\beta}$.

For the A''-bimodule structure on $A^{(5)} = (((A')')')''$ we have

$$\langle a''.a^{(5)}, a^{(4)}\rangle = \lim_\beta \lim_\alpha \langle a^{(4)}, a_\beta.a'''_\alpha\rangle = \lim_\beta \lim_\alpha \lim_i \langle a_\beta.a'''_\alpha, a''_i\rangle = \lim_\beta \lim_\alpha \lim_i \langle a'''_\alpha, a''_i \,\square\, \widehat{a_\beta}\rangle \tag{2.1}$$

and

$$\langle a^{(5)}.a'', a^{(4)}\rangle = \lim_\alpha \lim_\beta \langle a^{(4)}, a'''_\alpha.a_\beta\rangle = \lim_\alpha \lim_\beta \lim_i \langle a'''_\alpha.a_\beta, a''_i\rangle = \lim_\alpha \lim_\beta \lim_i \langle a'''_\alpha, \widehat{a_\beta} \,\square\, a''_i\rangle. \tag{2.2}$$

But the A''-bimodule structure on $A^{(5)} = (((A'')')')'$ is given as follows:

$$\begin{aligned}(2.3)\qquad \langle a''.a^{(5)}, a^{(4)}\rangle &= \langle a^{(5)}, a^{(4)}.a''\rangle = \lim_{\alpha} \langle a^{(4)}.a'', a'''_{\alpha}\rangle \\ &= \lim_{\alpha}\langle a^{(4)}, a''.a'''_{\alpha}\rangle = \lim_{\alpha}\lim_{i} \langle a''.a'''_{\alpha}, a''_{i}\rangle \\ &= \lim_{\alpha}\lim_{i} \langle a'''_{\alpha}, a''_{i} \,\square\, a''\rangle\end{aligned}$$

and

$$\begin{aligned}(2.4)\qquad \langle a^{(5)}.a'', a^{(4)}\rangle &= \langle a^{(5)}, a''.a^{(4)}\rangle = \lim_{\alpha} \langle a''.a^{(4)}, a'''_{\alpha}\rangle \\ &= \lim_{\alpha} \langle a^{(4)}, a'''_{\alpha}.a''\rangle = \lim_{\alpha}\lim_{i} \langle a'''_{\alpha}.a'', a''_{i}\rangle \\ &= \lim_{\alpha}\lim_{i} \langle a'''_{\alpha}, a'' \,\square\, a''_{i}\rangle.\end{aligned}$$

So the two A''-bimodule structures are not equal.

PROPOSITION 2.1. *Let A be a Banach algebra such that the following maps and A'' are Arens regular:*

$$\text{(i)}\ \begin{cases} A' \times A \to A', \\ (f, a) \mapsto f.a, \end{cases} \qquad \text{(ii)}\ \begin{cases} A'' \times A''' \to A''', \\ (F, \Lambda) \mapsto F.\Lambda. \end{cases}$$

Then the two A''-bimodule structures on $(((A'')')')'$ and $(((A')')')''$ coincide.

Proof. First we show $\varphi : \widehat{A} \to A''$ $(\widehat{a} \mapsto \widehat{a} \,\square\, G)$ is w^*-w-continuous for all $G \in A''$. For a net $(\widehat{a_\alpha})$ in $\widehat{A}$ and for $a''' = w^*\text{-}\lim_\beta \widehat{f_\beta} \in A'''$ such that (f_β) is a net in A' we have

$$\begin{aligned}\langle a''', (w^*\text{-}\lim_{\alpha} \widehat{a_\alpha}) \,\square\, G\rangle &= \langle a''', w^*\text{-}\lim_{\alpha}(\widehat{a_\alpha} \,\square\, G)\rangle = \lim_{\beta} \langle w^*\text{-}\lim_{\alpha}(\widehat{a_\alpha} \,\square\, G), f_\beta\rangle \\ &= \lim_{\beta}\lim_{\alpha} \langle G, f_\beta.a_\alpha\rangle = \lim_{\beta}\lim_{\alpha} \langle \widehat{f_\beta.a_\alpha}, G\rangle \\ &= \lim_{\alpha}\lim_{\beta} \langle \widehat{f_\beta.a_\alpha}, G\rangle = \lim_{\alpha}\lim_{\beta} \langle G, f_\beta.a_\alpha\rangle \\ &= \lim_{\alpha}\lim_{\beta} \langle \widehat{a_\alpha} \,\square\, G, f_\beta\rangle = \lim_{\alpha}\lim_{\beta} \langle \widehat{f_\beta}, \widehat{a_\alpha} \,\square\, G\rangle \\ &= \lim_{\alpha} \langle a''', \widehat{a_\alpha} \,\square\, G\rangle.\end{aligned}$$

This proves the w^*-w-continuity of φ. On the other hand for the nets (a_α) and (f_β) in A and A' respectively, by Arens regularity of the map in (ii), for $F \in A''$ we have

$$\begin{aligned}\langle w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} \widehat{a_\alpha.f_\beta}, F\rangle &= \lim_{\alpha}\lim_{\beta} \langle \widehat{a_\alpha.f_\beta}, F\rangle = \lim_{\alpha}\lim_{\beta} \langle \widehat{a_\alpha}.\widehat{f_\beta}, F\rangle \\ &= \lim_{\alpha}\lim_{\beta} \langle \widehat{\widehat{a_\alpha}.\widehat{f_\beta}}, \widehat{F}\rangle = \lim_{\beta}\lim_{\alpha} \langle \widehat{\widehat{a_\alpha}.\widehat{f_\beta}}, \widehat{F}\rangle \\ &= \langle w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} \widehat{a_\alpha.f_\beta}, F\rangle.\end{aligned}$$

This proves the Arens regularity of the map $A\times A'\to A'$ $((a,f)\mapsto a.f)$, and similarly we deduce the w^*-w-continuity of the map $\widehat{A}\to A''$ $(\widehat{a}\mapsto G\,\square\,\widehat{a})$ for all $G\in A''$.

For the rest of proof we continue equality (2.3):

$$\begin{aligned}\langle a''.a^{(5)},a^{(4)}\rangle &= \lim_\alpha\lim_i\langle a'''_\alpha, a''_i\,\square\,a''\rangle=\lim_\alpha\lim_i\lim_\beta\langle a'''_\alpha,a''_i\,\square\,\widehat{a_\beta}\rangle\\ &=\lim_\alpha\lim_\beta\lim_i\langle a'''_\alpha,a''_i\,\square\,\widehat{a_\beta}\rangle=\lim_\alpha\lim_\beta\lim_i\langle a_\beta.a'''_\alpha,a''_i\rangle\\ &=\lim_\alpha\lim_\beta\langle\widehat{a_\beta.a'''_\alpha},a^{(4)}\rangle=\lim_\beta\lim_\alpha\langle\widehat{a_\beta.a'''_\alpha},a^{(4)}\rangle\\ &=\lim_\beta\lim_\alpha\lim_i\langle a'''_\alpha,a''_i\,\square\,\widehat{a_\beta}\rangle;\end{aligned}$$

this proves the equality of (2.1) and (2.3). Similarly we continue equality (2.4):

$$\begin{aligned}\langle a^{(5)}.a'',a^{(4)}\rangle &= \lim_\alpha\lim_i\langle a'''_\alpha,a''\,\square\,a''_i\rangle=\lim_\alpha\lim_i\lim_\beta\langle a'''_\alpha,\widehat{a_\beta}\,\square\,a''_i\rangle\\ &=\lim_\alpha\lim_\beta\lim_i\langle a'''_\alpha,\widehat{a_\beta}\,\square\,a''_i\rangle,\end{aligned}$$

which proves the equality of (2.2) and (2.4). ∎

We need the following lemma to extend our results to $A^{(7)}$.

LEMMA 2.2. *Let A be a Banach algebra such that the following maps and $A^{(4)}$ are Arens regular:*

$$\text{(i)}\ \begin{cases}A'''\times A''\to A''',\\ (\varLambda,F)\mapsto\varLambda.F,\end{cases}\qquad \text{(ii)}\ \begin{cases}A^{(4)}\times A^{(5)}\to A^{(5)},\\ (\varLambda,F)\mapsto\varLambda.F.\end{cases}$$

Then the following maps and A'' are Arens regular:

$$\text{(a)}\ \begin{cases}A'\times A\to A',\\ (f,a)\mapsto f.a,\end{cases}\qquad \text{(b)}\ \begin{cases}A''\times A'''\to A''',\\ (F,\varLambda)\mapsto F.\varLambda.\end{cases}$$

Proof. For Arens regularity of A'' take nets (F_α) and (G_β) in A'' and $a'''\in A'''$, so $(\widehat{F_\alpha})$ and $(\widehat{G_\beta})$ are nets in $A^{(4)}$ and $\widehat{a'''}\in A^{(5)}$. By Arens regularity of $A^{(4)}$ we have

$$\begin{aligned}\lim_\alpha\lim_\beta\langle a''',F_\alpha\,\square\,G_\beta\rangle&=\lim_\alpha\lim_\beta\langle\widehat{a'''},\widehat{F_\alpha}\,\square\,\widehat{G_\beta}\rangle=\lim_\beta\lim_\alpha\langle\widehat{a'''},\widehat{F_\alpha}\,\square\,\widehat{G_\beta}\rangle\\ &=\lim_\beta\lim_\alpha\langle a''',F_\alpha\,\square\,G_\beta\rangle.\end{aligned}$$

This proves the Arens regularity of A''. Now suppose that (f_α) and (a_β) are nets in A' and A respectively, and let $F\in A''$, so $(\widehat{f_\alpha})$ and $(\widehat{a_\beta})$ are nets in

A''' and A'' respectively and $\widehat{F} \in A^{(4)}$. By Arens regularity of (i) we have

$$\begin{aligned}\langle w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta \widehat{f_\alpha.a_\beta}, F\rangle &= \lim_\alpha \lim_\beta \langle \widehat{f_\alpha}.\widehat{a_\beta}, F\rangle = \lim_\alpha \lim_\beta \langle \widehat{\widehat{f_\alpha}.\widehat{a_\beta}}, \widehat{F}\rangle \\ &= \lim_\beta \lim_\alpha \langle \widehat{\widehat{f_\alpha}.\widehat{a_\beta}}, \widehat{F}\rangle \\ &= \langle w^*\text{-}\lim_\beta w^*\text{-}\lim_\alpha \widehat{f_\alpha.a_\beta}, F\rangle,\end{aligned}$$

which proves the Arens regularity of (a). Similarly the Arens regularity of (ii) implies the Arens regularity of (b). ■

Now we are ready to consider two $A^{(4)}$-bimodule structures on $A^{(7)}$. Take $a^{(7)} \in A^{(7)}$, $a^{(6)} \in A^{(6)}$ and $a^{(4)} \in A^{(4)}$ with bounded nets $(a_\beta^{(5)}) \subset A^{(5)}$, $(a_i^{(4)}) \subset A^{(4)}$ and $(a''_\alpha) \subset A''$ such that $a^{(7)} = w^*\text{-}\lim_\beta \widehat{a_\beta^{(5)}}$, $a^{(6)} = w^*\text{-}\lim_i \widehat{a_i^{(4)}}$ and $a^{(4)} = w^*\text{-}\lim_\alpha \widehat{a''_\alpha}$.

For the $A^{(4)}$-bimodule structure on $A^{(7)} = ((((A')')')'')''$ we can write

$$\begin{aligned}\langle a^{(4)}.a^{(7)}, a^{(6)}\rangle &= \lim_\alpha \lim_\beta \langle a^{(6)}, a''_\alpha.a_\beta^{(5)}\rangle \\ &= \lim_\alpha \lim_\beta \lim_i \langle a''_\alpha.a_\beta^{(5)}, a_i^{(4)}\rangle,\end{aligned}\tag{2.5}$$

$$\begin{aligned}\langle a^{(7)}.a^{(4)}, a^{(6)}\rangle &= \lim_\beta \lim_\alpha \langle a^{(6)}, a_\beta^{(5)}.a''_\alpha\rangle \\ &= \lim_\beta \lim_\alpha \lim_i \langle a_\beta^{(5)}.a''_\alpha, a_i^{(4)}\rangle.\end{aligned}\tag{2.6}$$

But for the $A^{(4)}$-bimodule structure on $A^{(7)} = ((((A'')'')')')'$ we have

$$\begin{aligned}\langle a^{(4)}.a^{(7)}, a^{(6)}\rangle &= \langle a^{(7)}, a^{(6)}.a^{(4)}\rangle = \lim_\beta \langle a^{(6)}.a^{(4)}, a_\beta^{(5)}\rangle \\ &= \lim_\beta \langle a^{(6)}, a^{(4)}.a_\beta^{(5)}\rangle = \lim_\beta \lim_i \langle a^{(4)}.a_\beta^{(5)}, a_i^{(4)}\rangle \\ &= \lim_\beta \lim_i \langle a_\beta^{(5)}, a_i^{(4)} \square a^{(4)}\rangle,\end{aligned}\tag{2.7}$$

$$\begin{aligned}\langle a^{(7)}.a^{(4)}, a^{(6)}\rangle &= \langle a^{(7)}, a^{(4)}.a^{(6)}\rangle = \lim_\beta \langle a^{(4)}.a^{(6)}, a_\beta^{(5)}\rangle \\ &= \lim_\beta \langle a^{(6)}, a_\beta^{(5)}.a^{(4)}\rangle = \lim_\beta \lim_i \langle a_\beta^{(5)}.a^{(4)}, a_i^{(4)}\rangle \\ &= \lim_\beta \lim_i \langle a_\beta^{(5)}, a^{(4)} \square a_i^{(4)}\rangle.\end{aligned}\tag{2.8}$$

PROPOSITION 2.3. *Let A be a Banach algebra as in the hypothesis of Lemma 2.2. Then the two $A^{(4)}$-bimodule structures on $((((A')')')'')''$ and $((((A'')'')')')'$ coincide.*

Proof. By Lemma 2.2 the hypothesis of Proposition 2.1 holds, and so we can use the equality of the two A''-bimodule structures on $A^{(5)} = (((A'')')')'$ and $A^{(5)} = (((A')')')''$. Our proof is similar to the proof of Proposition 2.1. First we continue equality (2.5):

$$\begin{aligned}
\langle a^{(4)}.a^{(7)}, a^{(6)}\rangle &= \lim_\alpha \lim_\beta \lim_i \langle a''_\alpha.a^{(5)}_\beta, a^{(4)}_i\rangle \\
&= \lim_\alpha \lim_\beta \lim_i \langle a^{(5)}_\beta, a^{(4)}_i \,\square\, \widehat{a''_\alpha}\rangle \qquad \text{(by Proposition 2.1)} \\
&= \lim_\beta \lim_\alpha \lim_i \langle a^{(5)}_\beta, a^{(4)}_i \,\square\, \widehat{a''_\alpha}\rangle \\
&= \lim_\beta \lim_i \lim_\alpha \langle a^{(5)}_\beta, a^{(4)}_i \,\square\, \widehat{a''_\alpha}\rangle \\
&= \lim_\beta \lim_i \langle a^{(5)}_\beta, a^{(4)}_i \,\square\, a^{(4)}\rangle.
\end{aligned}$$

This proves the equality of (2.5) and (2.7). The proof of the equality of (2.6) and (2.8) is similar. ∎

REMARK 2.4. There are many other A''-bimodule [$A^{(4)}$-bimodule] structures on $A^{(5)}$ [$A^{(7)}$] that we do not need to mention.

The following corollary is about a similar work in [6] for two A''-bimodule structures on A'''.

COROLLARY 2.5. *Let A be a Banach algebra as in the hypothesis of Proposition 2.1. Then the two A''-bimodule structures on $A''' = (A')''$ and $A''' = (A'')'$ coincide.*

Proof. The Arens regularity of A'' implies the Arens regularity of A, and also the Arens regularity of the map in (ii) of Proposition 2.1 implies the Arens regularity of $\varphi : A \times A' \to A'$ $((a, f) \mapsto a.f)$. Thus the assertion holds by Theorem 2.1 of [6]. ∎

3. 3-weak amenability of $A^{(2n)}$.

We recall that by Theorem 1.1, for a continuous derivation $D : A \to E$, the second transpose $D'' : A'' \to E''$ and hence the fourth transpose $D^{(4)} : A^{(4)} \to E^{(4)}$ are continuous derivations. In this section we consider a continuous derivation $D : A \to A^{(3)} = ((A')')'$ and its second and fourth transposes.

LEMMA 3.1. *Let A be a Banach algebra as in the hypothesis of Proposition 2.1. If the second transpose of a continuous derivation $D : A \to A''' = ((A')')'$ is inner, then D is inner.*

Proof. Since $D'' : A'' \to (((A')')')'' = A^{(5)}$ is inner, there is an $a^{(5)} \in A^{(5)}$ such that $D''(a'') = a''.a^{(5)} - a^{(5)}.a''$ $(a'' \in A'')$, where the A''-bimodule structure is as in (2.1) and (2.2). Let $a^{(3)} = i^*(a^{(5)})$ where $i : A'' \to (A'')'' =$

$A^{(4)}$ is the natural map and so $i^* : ((A'')'')' = A^{(5)} \to A^{(3)} = (A'')'$. Then for $a'' \in A''$ we have

$$\begin{aligned}\langle D(a), a''\rangle = \langle \widehat{D(a)}, \widehat{a''}\rangle &= \langle D''(\hat{a}), \widehat{a''}\rangle = \langle \hat{a}.a^{(5)} - a^{(5)}.\hat{a}, \widehat{a''}\rangle \\ &= \langle a^{(5)}, \widehat{a''}.\hat{a} - \hat{a}.\widehat{a''}\rangle = \langle a^{(5)}, \widehat{a''.a - a.a''}\rangle \\ &= \langle i^*(a^{(5)}), a''.a - a.a''\rangle = \langle a.a^{(3)} - a^{(3)}.a, a''\rangle.\end{aligned}$$

Thus $D(a) = a.a^{(3)} - a^{(3)}.a$. ∎

By using Lemma 2.2 we can similarly prove the following lemma:

LEMMA 3.2. *Let A be a Banach algebra as in the hypothesis of Lemma 2.2. If the fourth transpose of a continuous derivation $D : A \to A''' = ((A')')'$ is inner then D'' and D are inner.*

PROPOSITION 3.3. *Let A be a Banach algebra as in the hypothesis of Proposition 2.1. If A'' is 3-weakly amenable then so is A.*

Proof. Suppose that $D : A \to A^{(3)} = ((A')')'$ is a continuous derivation. Then $D'' : A'' \to A^{(5)} = (((A')')')''$ is again a derivation by Theorem 1.1. We know that the two A''-bimodule structures on $A^{(5)} = (((A')')')''$ and $A^{(5)} = (((A'')')')'$ are equal by Proposition 2.1, so $D'' : A'' \to A^{(5)} = (((A'')')')'$ is also a derivation. Thus D'' is inner because A'' is 3-weakly amenable. Finally D is inner by Lemma 3.1, and this proves the 3-weak amenability of A. ∎

PROPOSITION 3.4. *Let A be a Banach algebra as in the hypothesis of Lemma 2.2. If $A^{(4)}$ is 3-weakly amenable then so is A.*

Proof. This is a consequence of Proposition 2.3 and Lemma 3.2. ∎

Now we can extend Propositions 3.3 and 3.4 as follows:

THEOREM 3.5. *Let $n \geq 1$ and A be a Banach algebra such that the following maps and $A^{(2n)}$ are Arens regular:*

(i) $\begin{cases} A^{(2n-1)} \times A^{(2n-2)} \to A^{(2n-1)}, \\ (\Lambda, F) \mapsto \Lambda.F, \end{cases}$ (ii) $\begin{cases} A^{(2n)} \times A^{(2n+1)} \to A^{(2n+1)}, \\ (F, \Lambda) \mapsto F.\Lambda. \end{cases}$

If $A^{(2n)}$ is 3-weakly amenable then A is 3-weakly amenable and hence it is weakly amenable.

EXAMPLE 3.6. C^*-algebras are standard examples of Banach algebras that are Arens regular and have a bounded approximate identity. The second dual A'' of a C^*-algebra A is itself a C^*-algebra and a von Neumann algebra [3, Corollary 3.2.37]. Every C^*-algebra is n-weakly amenable for each n (see [4]), so the conclusions of Theorem 3.5 hold for any C^*-algebras, but the assumptions only hold for finite-dimensional ones [5, Corollary 4.6].

According to Corollary 2.5 and Proposition 4.5 in [5], it seems that an example of the conditions in Proposition 2.1, with a non-reflexive Banach algebra, can only be obtained when the algebra has no two sided bounded approximate identity.

EXAMPLE 3.7. Assume that A is a non-reflexive complex Banach space and $\varphi : A \to \mathbb{C}$ is a bounded linear functional. Define a multiplication on A by $ab = \langle \varphi, a \rangle b$. This makes A into a Banach algebra which is called the ideally factored algebra associated to φ. It is easy to check that φ is multiplicative and also

$$a.f = \langle f, a \rangle \varphi, \quad f.a = \langle \varphi, a \rangle f, \quad f.F = \langle F, \varphi \rangle f, \quad F.f = \langle F, f \rangle \varphi,$$
$$F \,\Box\, G = F \,\Diamond\, G = \langle F, \varphi \rangle G, \quad F.\Lambda = \langle \Lambda, F \rangle \widehat{\varphi}, \quad \Lambda.F = \langle F, \varphi \rangle \Lambda,$$

for $a \in A$, $f \in A'$, $\varphi \in A'''$ and $F, G \in A''$. Now for bounded nets (a_i) and (f_j) in A and A' respectively, we have

$$\begin{aligned} w^*\text{-}\lim_i w^*\text{-}\lim_j \widehat{f_j a_i} &= w^*\text{-}\lim_i w^*\text{-}\lim_j \langle \varphi, a_i \rangle \widehat{f_j} \\ &= w^*\text{-}\lim_i \langle \varphi, a_i \rangle \, w^*\text{-}\lim_j \widehat{f_j}. \end{aligned}$$

This proves the Arens regularity of the map $A' \times A \to A'$ $((f, a) \mapsto f.a)$. Since A is not reflexive, the map $A \times A' \to A'$ $((a, f) \mapsto a.f)$ is not Arens regular, because

$$\begin{aligned} w^*\text{-}\lim_i w^*\text{-}\lim_j \widehat{a_i f_j} &= w^*\text{-}\lim_i w^*\text{-}\lim_j \langle f_j, a_i \rangle \varphi \\ &\neq w^*\text{-}\lim_j w^*\text{-}\lim_i \langle f_j, a_i \rangle \varphi. \end{aligned}$$

Similarly we can check that A and A'' and the map $A''' \times A'' \to A'''$ $((\Lambda, F) \mapsto \Lambda.F)$ are Arens regular, but the map $A'' \times A''' \to A'''$ $((F, \Lambda) \mapsto F.\Lambda)$ is not Arens regular. Now for $a, b \in A$ we have

$$\varphi(ab) = \varphi(\langle \varphi, a \rangle b) = \langle \varphi, a \rangle \langle \varphi, b \rangle = \varphi(ba),$$

so if φ is one-to-one then $ab = ba$, that is, A is commutative. In this situation the assumptions of Proposition 2.1 hold.

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REFERENCES

[1] W. G. Bade, P. C. Curtis and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebra*, Proc. London Math. Soc. 55 (1987), 359–377.

[2] A. Bodaghi, M. Ettefagh, M. E. Gordji and A. Medghalchi, *Module structures on iterated duals of Banach algebras*, An. Ştiinţ. Univ. Ovidius Constanţa 18 (2010), 63–80.

[3] H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford Univ. Press, 2000.

[4] H. G. Dales, F. Ghahramani, and N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Studia Math. 128 (1998), 19–54.
[5] H. D. Dales, A. Rodríguez-Palacios and M. V. Velasco, *The second transpose of a derivation*, J. London Math. Soc. (2) 64 (2001), 707–721.
[6] M. Ettefagh, *The third dual of a Banach algebra*, Studia Sci. Math. Hungar. 45 (2008), 1–11.
[7] A. Jabbari, M. S. Moslehian, and H. R. E. Vishki, *Constructions preserving n-weak amenability of Banach algebras*, to appear.
[8] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).
[9] A. Medghalchi and T.Yazdanpanah, *Problems concerning n-weak amenability of a Banach algebra*, Czechoslovak Math. J. 55 (130) (2005), 863–876.

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