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THE DIMENSION OF HYPERSPACES OF NON-METRIZABLE CONTINUA

BY

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Abstract. We prove that, for any Hausdorff continuum X, if $\dim X \geq 2$ then the hyperspace $C(X)$ of subcontinua of X is not a C-space; if $\dim X = 1$ and X is hereditarily indecomposable then either $\dim C(X) = 2$ or $C(X)$ is not a C-space. This generalizes some results known for metric continua.

1. Introduction. Throughout the paper all spaces are normal. A continuum is a compact, connected Hausdorff space. By dimension we always mean the covering dimension dim. A continuum X is hereditarily indecomposable iff for any subcontinua $A, B \subseteq X$ we have $A \subseteq B$, $B \subseteq A$ or $A \cap B = \emptyset$. For a compact X denote by $K(X)$ the hyperspace of all non-empty subcompacta of X, equipped with the Vietoris topology. By $C(X)$ we denote the hyperspace of all non-empty subcontinua of X, with the topology inherited from $K(X)$.

DEFINITION 1.1. A space X is a *C-space* (or has *property C*) if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ such that each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X, $\mathcal{V}_i \prec \mathcal{U}_i$ ($\mathcal{V}_i$ refines $\mathcal{U}_i$, i.e. for each $V \in \mathcal{V}_i$ there is $U \in \mathcal{U}_i$ such that $V \subseteq U$) and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

We refer to [3] for basic properties of C-spaces. It is easy to observe that C-spaces are weakly infinite-dimensional. The class of C-spaces contains all finite-dimensional spaces and countable-dimensional metric spaces.

We prove the following theorem:

THEOREM 1.2.

(i) *Suppose X is a continuum of dimension ≥ 2. Then $C(X)$ is not a C-space.*

(ii) *Suppose X is a 1-dimensional hereditarily indecomposable continuum. Then either $\dim C(X) = 2$ or $C(X)$ is not a C-space.*

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The theorem is already known for metric continua. Part (i) was stated by M. Levin and J. T. Rogers, Jr. in [8]. Part (ii) can be obtained using methods from [2, 8, 9] (see [10, Theorem 3.1]).

To prove it for non-metric spaces we use the technique of lattices and Wallman representations as well as some set-theoretical methods, as in [1]. We refer to [11] for the definition of a lattice and preliminary facts on Wallman spaces. We consider only distributive and separative lattices.

2. Lattices and Wallman spaces. For a compact space X we consider the lattice 2^X of closed subsets of X with $\cup$ and $\cap$ as lattice operations, and $\emptyset$ and X as the minimal and maximal elements. Each lattice L corresponds to the Wallman space wL consisting of all ultrafilters on L. For $a \in L$ let $\widehat{a} = \{u \in wL \colon a \in u\}$. We define the topology in wL by taking the family $\{\widehat{a} \colon a \in L\}$ as a base for closed sets.

It is easy to show that $w2^X$ is homeomorphic to X. More generally, the following fact holds true:

FACT 2.1. *If $\mathcal{F}$ is a base for closed sets in X which is closed under finite unions and intersections* (*so $\mathcal{F}$ is a lattice*), *then $w\mathcal{F}$ is homeomorphic to X.*

Proof. We define $h\colon X \to w\mathcal{F}$ in the natural way: $h(x) = \{F \in \mathcal{F} \colon x \in F\}$. It is not difficult but tedious to verify that h is a well-defined homeomorphism; we leave it as an exercise. ∎

DEFINITION 2.2. A lattice L is *normal* if

$$L \models \forall a, b\ (a \cap b = 0_L \to \exists c, d\ (c \cup d = 1_L \wedge c \cap a = 0_L \wedge d \cap b = 0_L)).$$

We now collect some well-known observations.

FACT 2.3 (see, e.g., [11]). *L is normal if and only if wL is Hausdorff.*

FACT 2.4 ([11, Theorem 2.6]). *If L is a countable normal lattice then wL is a compact metric space.*

REMARK 2.5. *A sublattice L of L^* yields the continuous surjection $q : wL^* \to wL$, given by $q(u) = u \cap L$.*

3. Proof of Theorem 1.2. The proof is rather simple, but it uses some set-theoretic framework. We deal with some inner model of (a large enough fragment of) ZFC and its countable elementary submodel.

Our strategy is to reduce the non-metric case to the metric one. Suppose X is a non-metric continuum. We will find a countable sublattice $L \subseteq 2^X$ such that wL is a metric continuum, $\dim wL = \dim X$ and $\dim C(wL) = \dim C(X)$. Moreover, wL $[C(wL)]$ is hereditarily indecomposable if and only if so X $[C(X)]$, and wL $[C(wL)]$ is a C-space if and only if so is X $[C(X)]$.

To find the sublattice L, we apply the technique used in [1].

For an infinite cardinal κ, $H(\kappa)$ is the set of all sets x with $|\mathrm{TC}(x)| < \kappa$ (TC is the transitive closure, i.e. $\mathrm{TC}(x) = x \cup \bigcup x \cup \bigcup\bigcup x \cup \cdots$). If κ is regular then $H(\kappa)$ is a model of ZFC without the Power Set Axiom (see [7, p. 162]). But if κ is large enough, then there are power sets in $H(\kappa)$ for all sets we need.

Let X be a (non-metric) continuum. Fix a suitably large regular cardinal κ (it is enough if $\mathcal{P}(\mathcal{P}(X)) \in H(\kappa)$) and take a countable elementary submodel $\mathcal{M} \prec H(\kappa)$ such that $X \in \mathcal{M}$ (use the Löwenheim–Skolem theorem). Then $\mathcal{M}$ also models enough of ZFC. Moreover, every finite subset of $\mathcal{M}$ belongs to $\mathcal{M}$. Denote $L = 2^X \cap \mathcal{M}$. By elementarity, L is a normal sublattice of 2^X. Since L is countable, applying Fact 2.4 and Remark 2.5, we obtain:

FACT 3.1. *wL is a metric continuum.* ∎

Let us recall two well-known facts.

PROPOSITION 3.2 (see [5, Subsection 4.1]). $\dim X = \dim wL$. *More generally, let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then* $\dim wK^* = \dim wK$.

PROPOSITION 3.3. *A continuum X is hereditarily indecomposable if and only if so is wL.*

The "if" part is straightforward. For the "only if" see [6, Lemma 2.2].

Now we prove a similar fact about property C.

THEOREM 3.4. *The space X is a C-space if and only if so is wL. More generally, let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then wK^* is a C-space if and only if so is wK.*

Proof. We provide the proof for the first part of the proposition. It can be easily adapted to give the more general statement.

Denote $\mathcal{B} = \{wL \setminus \widehat{F} \colon F \in L\}$ (an open base for wL, closed under finite unions and intersections).

($\Leftarrow$) We will show that if X is not a C-space then neither is wL. Assume X is not a C-space. Then by compactness there exists a sequence $(\mathcal{U}_i)_{i=1}^{\infty}$ of finite open covers of X such that for every $m \geq 1$ and finite families $\mathcal{V}_1, \dots, \mathcal{V}_m$ of open disjoint sets which satisfy $\mathcal{V}_i \prec \mathcal{U}_i$, their union is not a cover of X (compactness allows us to consider only finite families). In terms of the lattice 2^X we find that $H(\kappa)$ models the following sentence φ:

(φ) There exists a sequence $(\mathcal{F}_i)_{i=1}^{\infty}$ of finite subsets of 2^X such that for each $i \geq 1$ the intersection $\bigcap \mathcal{F}_i$ is empty and for every $m \geq 1$ and finite $\mathcal{G}_1, \dots, \mathcal{G}_m \subseteq 2^X$ the following holds:

(∗) If for any $j \leq m$ and $G \in \mathcal{G}_j$ there exists $F \in \mathcal{F}_j$ such that $F \subseteq G$ and for any distinct $G, G' \in \mathcal{G}_j$ we have $G \cup G' = X$, then $\bigcap(\mathcal{G}_1 \cup \dots \cup \mathcal{G}_m) \neq \emptyset$.

We have $\mathcal{M} \models \varphi$ by elementarity. So there is $(\mathcal{F}_i)_{i=1}^{\infty} \in \mathcal{M}$ as in φ, such that $(*)$ holds for every $m < \omega$ and $\mathcal{G}_1, \ldots, \mathcal{G}_m \in \mathcal{M}$.

The sequence $(\mathcal{F}_i)_{i=1}^{\infty}$ gives rise to a sequence $(\mathcal{U}_i)_{i=1}^{\infty}$ of open covers of wL (namely $\mathcal{U}_i = \{wL \setminus \widehat{F} \colon F \in \mathcal{F}_i\}$), which witnesses that wL is not a C-space. Indeed, suppose we have a finite sequence $\mathcal{V}_1, \ldots, \mathcal{V}_m$ of finite families of open disjoint sets such that $\mathcal{V}_i \prec \mathcal{U}_i$ and their union is a cover of wL. Then we can produce $\mathcal{V}_1', \ldots, \mathcal{V}_m'$ which are additionally contained in the base $\mathcal{B}$: shrink each $V \in \bigcup_{i=1}^{m} \mathcal{V}_i$ to a closed set C_V so that $\bigcup_{i=1}^{m}\{C_V \colon V \in \mathcal{V}_i\}$ forms a closed cover of wL. Since C_V is compact, it can be covered by finitely many sets $B_1^V, \ldots, B_{j(V)}^V \subseteq V$ from the basis $\mathcal{B}$. Let $V' = B_1^V \cup \cdots \cup B_{j(V)}^V$. We have $V' \in \mathcal{B}$, since $\mathcal{B}$ is closed under finite unions. Define $\mathcal{V}_i' = \{V' \colon V \in \mathcal{V}_i\}$.

Having $\mathcal{V}_1', \ldots, \mathcal{V}_m'$ it is easy to produce $\mathcal{G}_1, \ldots, \mathcal{G}_m \in \mathcal{M}$ which do not satisfy $(*)$. Indeed, each $V' \in \mathcal{V}_i'$ is given by some $F_{V'} \in L$ via $V' = wL \setminus \widehat{F}_{V'}$. Then we set $\mathcal{G}_i = \{F_{V'} \colon V' \in \mathcal{V}_i'\}$. Since $\mathcal{V}_i' \subseteq \mathcal{B}$, we have $\mathcal{G}_i \in \mathcal{M}$.

$(\Rightarrow)$ Suppose $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a sequence of finite open covers of wL, say $\mathcal{U}_i = \{U_{i1}, \ldots, U_{ik_i}\}$. Without loss of generality we may assume that each $\mathcal{U}_i$ consists of sets from $\mathcal{B}$, i.e. for each $i \in \mathbb{N}$ and $j \le k_i$ there is some $F_{ij} \in \mathcal{M}$ closed in X such that $U_{ij} = wL \setminus \widehat{F_{ij}}$.

Define $U_{ij}' = X \setminus F_{ij}$ and $\mathcal{U}_i' = \{U_{i1}', \ldots, U_{ik_i}'\}$. Note that $\mathcal{U}_i'$ is an open cover of X since $F_{i1} \cap \cdots \cap F_{ik_i} = \emptyset$ ($\mathcal{U}_i$ is a cover of wL).

Since X is a compact C-space there exist $n \in \mathbb{N}$ and finite families $\mathcal{V}_1', \ldots, \mathcal{V}_n'$ of pairwise disjoint open sets such that each $\mathcal{V}_i'$ refines $\mathcal{U}_i'$ and $\bigcup_{i=1}^{n} \mathcal{V}_i'$ is a cover of X. Let us code this in terms of the lattice 2^X. First denote $\mathcal{V}_i' = \{V_{i1}', \ldots, V_{il_i}'\}$ and $G_{ij}' = X \setminus V_{ij}'$ for $i \le n$ and $j \le l_i$. The following sentence ψ is true in $H(\kappa)$:

$$(\psi)\quad \begin{cases} \text{There exist } G_{11}', G_{12}', \ldots, G_{1l_1}', G_{21}', G_{22}', \ldots G_{2l_2}', \ldots, G_{n1}', G_{n2}', \ldots \\ \ldots, G_{nl_n}' \text{ such that:} \\ (1)\ \bigwedge_{i=1}^{n}\big(\bigwedge_{1 \le j < j' \le l_i}(G_{ij}' \cup G_{ij'}' = X)\big), \\ (2)\ \bigwedge_{i=1}^{n}\big(\bigwedge_{j=1}^{l_i}\big(\bigvee_{j'=1}^{k_i}(G_{ij}' \cap F_{ij'} = F_{ij'})\big)\big), \\ (3)\ \bigcap_{i=1}^{n} \bigcap_{j=1}^{l_i} G_{ij}' = \emptyset. \end{cases}$$

The symbols $\bigwedge$ and $\bigvee$ stand for finite conjuctions and disjunctions. Note that F_{ij}'s appear in ψ as parameters from $\mathcal{M}$.

We have $H(\kappa) \models \psi$ and by elementarity $\mathcal{M} \models \psi$. Hence, for $i \le n$ and $j \le l_i$ there are $G_{ij} \in \mathcal{M}$ which satisfy (1)–(3) when placed in ψ instead of G_{ij}'. Take $V_{ij} = wL \setminus \widehat{G_{ij}}$ and $\mathcal{V}_i = \{V_{i1}, \ldots, V_{il_k}\}$. Then $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are families of pairwise disjoint sets (by (1)), open in wL. For $i \le n$ the family $\mathcal{V}_i$ refines $\mathcal{U}_i$ (by (2)) and $\bigcup_{i=1}^{n} \mathcal{V}_i$ is a cover of wL (by (3)). ∎

Now we will link the space X with its hyperspace $C(X)$ in terms of lattices. Namely, having the lattice 2^X we define a lattice $K^* \in \mathcal{M}$ such that wK^* is homeomorphic to $C(X)$. Then, taking $K = K^* \cap \mathcal{M}$ we will show that wK is homeomorphic to $C(wL)$.

We define the set $C(X)$ in terms of the lattice 2^X:

$$C(X) = \{F \in 2^X : \neg(\exists G_1, G_2 \in 2^X)(G_1 \cup G_2 = F \wedge G_1 \cap G_2 = \emptyset)\}.$$

Define K^* as a sublattice of $(\mathcal{P}(C(X)), \cup, \cap, \emptyset, C(X))$ generated by the family $\{\mathcal{F}^* : \mathcal{F} \in [2^X]^{<\omega}\}$, where

$$\mathcal{F}^* = C(X) \setminus \Big\{G \in C(X) \colon G \cap \bigcap \mathcal{F} = \emptyset \wedge (\forall F \in \mathcal{F})(F \cup G \neq F)\Big\}.$$

The lattice K^* is the closure under finite unions and intersections of the family of sets $\mathcal{F}^*$ for all finite $\mathcal{F} \subseteq 2^X$. It is easy to verify that the sets $\mathcal{F}^*$ form a closed base for $C(X)$. Hence, K^* is a closed base and a lattice simultaneously. By Fact 2.1, we get:

Remark 3.5. *$C(X)$ is homeomorphic to wK^*.*

Since $X \in \mathcal{M}$, it follows directly by the definition of K^* that $K^* \in \mathcal{M}$. Take $K = K^* \cap \mathcal{M}$. The only thing we still lack is:

Proposition 3.6. *wK is homeomorphic to $C(wL)$.*

Proof. We know that K^* is generated by the family $\{\mathcal{F}^* : \mathcal{F} \in [2^X]^{<\omega}\}$. By elementarity, K is generated by $\{\mathcal{F}^* : \mathcal{F} \in [L]^{<\omega}\}$. Note that a basic closed set in $C(wL)$ is determined by $\mathcal{F} \in [L]^{<\omega}$ via the formula $\mathcal{C}_{\mathcal{F}} = C(wL) \setminus \{C \in C(wL) \colon C \cap \bigcap\{\widehat{F} \colon F \in \mathcal{F}\} = \emptyset \wedge (\forall F \in \mathcal{F})(\widehat{F} \cup C \neq \widehat{F})\}$ (since L is isomorphic to a closed base for wL). Hence, the lattice K is isomorphic to the lattice generated by $\{\mathcal{C}_{\mathcal{F}} \colon \mathcal{F} \in [L]^{<\omega}\}$, which forms a closed base for $C(wL)$. By Fact 2.1, wK is homeomorphic to $C(wL)$. ∎

Now we have all ingredients to prove Theorem 1.2.

Proof of Theorem 1.2. (i) Suppose that $\dim X \geq 2$. Proposition 3.2 gives $\dim wL \geq 2$. By the results of M. Levin, J. T. Rogers, Jr. for metric continua [8] we know that $C(wL)$ is not a C-space. But $C(wL)$ is homeomorphic to wK (Proposition 3.6). Hence wK^* is not a C-space either (Theorem 3.4). By Remark 3.5, $C(X)$ is homeomorphic to wK^*, so it is not a C-space.

(ii) Similarly, suppose that X is a 1-dimensional, hereditarily indecomposable continuum. Then wL is also 1-dimensional (Proposition 3.2) and hereditarily indecomposable (Proposition 3.3). By the result for metric continua ([10, Theorem 3.1]), either $C(wL)$ is 2-dimensional, or it is not a C-space. By Proposition 3.6, $C(wL)$ is homeomorphic to wK. Therefore, wK^* is either 2-dimensional (Proposition 3.2) or not a C-space (Theorem 3.4). But wK^* is homeomorphic to $C(X)$ by Remark 3.5. ∎

4. Remarks on m-C-spaces

Definition 4.1 ([4]). For $m \geq 2$ a space X is said to be an *m-C-space* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of m-element open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ such that each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X, $V_i \prec \mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

Observe that

$$2\text{-}C\text{-spaces} \supseteq 3\text{-}C\text{-spaces} \supseteq \cdots \supseteq m\text{-}C\text{-spaces} \supseteq \cdots \supseteq C\text{-spaces}.$$

Moreover, the following holds:

Fact 4.2 ([4, Proposition 2.11]). *A space is weakly infinite-dimensional if and only if it is a 2-C-space.*

One can easily adapt the proof of Theorem 3.4 to obtain the following:

Proposition 4.3. *Let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then wK^* is an m-C-space if and only if so is wK.* ∎

Let us recall two definitions and one question from [11]:

Definition 4.4 ([11, Definition 2.7]). We will say that a property $\mathcal{P}$ of a compact space is *elementarily reflected* if whenever some compact space X has property $\mathcal{P}$ then the Wallman representation wL of any elementary sublattice L of 2^X also has property $\mathcal{P}$.

Definition 4.5 ([11, Definition 2.8]). A property $\mathcal{P}$ of a compact space is *elementarily reflected by submodels* if whenever some compact space X has property $\mathcal{P}$ then the Wallman representation wL of any elementary sublattice L of the form $L = 2^X \cap \mathcal{M}$, where $2^X \in \mathcal{M}$ and $\mathcal{M} \prec H(\kappa)$ (for a large enough regular κ), also has property $\mathcal{P}$.

Question 4.6 ([11, Question 2.32]). *Is having strong infinite dimension elementarily reflected, and is not having strong infinite dimension elementarily reflected?*

Recall that, by definition, a space is strongly infinite-dimensional if it is not weakly infinite-dimensional.

Proposition 4.3 gives a partial answer to Question 4.6. In particular it says that that both these properties are elementarily reflected by submodels (use the characterization of weak infinite dimension from Fact 4.2). Moreover, following the proof of Theorem 3.4, one can observe that the model $\mathcal{M} \prec H(\kappa)$ is not needed for the left-to-right implication. This means property C is elementarily reflected and the opposite is elementarily reflected by submodels. Properties m-C and non-m-C behave in the same way. Summarizing, we can say that having strong infinite dimension is elementarily reflected by submodels, and not having strong infinite dimension is elementarily reflected.

It is not known if the notions of property C and weak infinite dimension coincide within the class of compact spaces. However, since both properties are elementarily reflected by submodels, there exists a metric counterexample which distinguishes these two notions if and only if there exists a non-metric one.

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