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$\frac{1}{2}$-HOMOGENEOUS HYPERSPACE SUSPENSIONS

BY

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Abstract. We continue the study of $\frac{1}{2}$-homogeneity of the hyperspace suspension of continua. We prove that if X is a decomposable continuum and its hyperspace suspension is $\frac{1}{2}$-homogeneous, then X must be continuum chainable. We also characterize $\frac{1}{2}$-homogeneity of the hyperspace suspension for several classes of continua, including: continua containing a free arc, atriodic and decomposable continua, and decomposable irreducible continua about a finite set.

1. Introduction. Hyperspace suspensions are introduced in [30] in order to obtain a class of disk-like continua with the fixed point property. Hyperspace suspensions are studied further in [5], [17], and [19]. We note that $\frac{1}{2}$-homogeneity has been studied for the hyperspace of subcontinua of a continuum in [32] and for topological suspensions of continua in [13] and [34]. Hence, it is natural to study $\frac{1}{2}$-homogeneity of the hyperspace suspension of a continuum. Here we continue the investigation initiated in [23].

This paper is divided into seven sections. After the introduction and definitions, in Section 3 we give the results we use later. In Section 4, we present our main results, for example we prove that if X is a decomposable continuum and its hyperspace suspension is $\frac{1}{2}$-homogeneous, then X must be continuum chainable (Theorem 4.4). We also characterize $\frac{1}{2}$-homogeneity of the hyperspace suspension for several classes of continua, including: continua containing a free arc (Theorem 4.19), atriodic and decomposable continua (Corollary 4.20), decomposable continua which are irreducible about a finite set (Theorem 4.25), and pseudo-linear continua (Corollary 4.27). In Section 5, we study dendroids whose hyperspace suspension is $\frac{1}{2}$-homogeneous; we strongly believe that the arc is the only dendroid with this property, and in this section we give several conditions that help to support our conjecture: see Corollaries 5.4, 5.5 and Theorem 5.21. In Section 6 we give a result analogous to Theorem 5.21 for $\mathcal{C}(X)$ (Theorem 6.2). In Section 7, we present a list of open questions.

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2. Definitions. If (Z,d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}_\varepsilon^d(A)$, the interior of A is denoted by $\mathrm{Int}_Z(A)$, the boundary of A is denoted by $\mathrm{Bd}_Z(A)$, and the closure of A is denoted by $\mathrm{Cl}_Z(A)$. Also, $Z \setminus A$ denotes the complement of A in Z and Z/A denotes the quotient space of Z modulo A with the quotient topology. A subset A of Z is a *closed domain* provided that $A = \mathrm{Cl}_Z(\mathrm{Int}_Z(A))$.

An *arc* is any space homeomorphic to $[0,1]$, and a *simple closed curve* is any space homeomorphic to $\mathcal{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

If S is a topological space, then $H(S)$ denotes the group of homeomorphisms of S. For a point s of S, the *orbit of s in S* is the set $\mathcal{O}_S(s) = \{h(s) \mid h \in H(S)\}$. Note that the family of orbits of S forms a decomposition of S. A topological space S is $\frac{1}{n}$*-homogeneous* if S has exactly n orbits.

A *continuum* is a compact, connected, metric space. A continuum X is *unicoherent* provided that for each pair of subcontinua A and B of X such that $X = A \cup B$, the intersection $A \cap B$ is connected. The continuum X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent.

A continuum X is said to be *bicoherent* if for any two proper subcontinua A and B whose union is X, the set $A \cap B$ has exactly two components.

A continuum X is *irreducible about a finite set* provided that there exists a finite subset F of X such that if Z is a proper subcontinuum of X, then $F \setminus Z \neq \emptyset$.

A continuum X is *continuum chainable* if for each $\varepsilon > 0$ and each pair of points $p, q \in X$ such that $p \neq q$, there exists a finite sequence $\{M_1, \dots, M_n\}$ of subcontinua of X such that $\mathrm{diam}(M_j) < \varepsilon$, $p \in M_1 \setminus \bigcup_{j=2}^n M_j$, $q \in M_n \setminus \bigcup_{j=1}^{n-1} M_j$, and $M_j \cap M_k \neq \emptyset$ if and only if $|j-k| \leq 1$.

Let X be a continuum, and let A and B be two subcontinua of X such that $A \subset B$. We say that *A is terminal in B* provided that if C and D are two subcontinua of B such that $A \subset C \cap D$, then either $C \subset D$ or $D \subset C$.

A continuum X is said to be *pseudo-linear* provided that there exist two proper subcontinua X_1 and X_2 of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2$ is connected and is terminal in both X_1 and X_2, and each subcontinuum of X intersecting both $X \setminus X_1$ and $X \setminus X_2$ must contain $X_1 \cap X_2$.

A continuum X is said to be *pseudo-circular* provided that there exist two proper subcontinua X_1 and X_2 of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2$ has exactly two components K_1 and K_2 each of which is terminal in both X_1 and X_2, each subcontinuum of X intersecting both K_1 and K_2 must contain either X_1 or X_2, and there exists $\varepsilon > 0$ such that if L is a subcontinuum of X and $X \subset \mathcal{V}_\varepsilon^d(L)$, then either $K_1 \subset L$ or $K_2 \subset L$.

A continuum X is a *triod* provided that it contains a subcontinuum N, called a *core*, such that $X \setminus N$ is the union of three nonempty mutually separated sets. A *simple triod* is a continuum homeomorphic to

$\{(0,y) \in \mathbb{R}^2 \mid 0 \le y \le 1\} \cup \{(x,1) \in \mathbb{R}^2 \mid -1 \le x \le 1\}$. Let X be a continuum and let A_1, A_2 and A_3 be subcontinua of X. We say that A_1, A_2 and A_3 form a *weak triod* if:

(i) $A_1 \cap A_2 \cap A_3 \neq \emptyset$, and
(ii) $A_i \setminus (A_j \cup A_k) \neq \emptyset$ whenever $\{i,j,k\} = \{1,2,3\}$.

Given a continuum X we consider the following *hyperspaces*:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\},$$
$$\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is a subcontinuum of } X\},$$
$$\mathcal{F}_1(X) = \{\{x\} \mid x \in X\}.$$

We topologize 2^X with the Hausdorff metric $\mathcal{H}$ [29, (0.1)]. It is known that 2^X and $\mathcal{C}(X)$ are continua [29, (1.13)]. Hence, we may consider the hyperspaces 2^{2^X} and $\mathcal{C}(2^X)$ topologized with the Hausdorff metric $\mathcal{H}^2$ induced by $\mathcal{H}$. Given $A \in \mathcal{C}(X)$, we let $\mathcal{C}(A,X) = \{K \in \mathcal{C}(X) \mid A \subset K\}$. If $f\colon X \to Y$ is a map between continua, then $\mathcal{C}(f)\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ given by $\mathcal{C}(f)(A) = f(A)$ is the *induced map by f* [29, (0.49)].

We also consider the quotient space

$$\mathrm{HS}(X) = \mathcal{C}(X)/\mathcal{F}_1(X),$$

with the quotient topology. $\mathrm{HS}(X)$ is called the *hyperspace suspension* of X and was originally defined in [30]. Let $q_X\colon \mathcal{C}(X) \to \mathrm{HS}(X)$ be the quotient map. We denote by T_X the point $q_X(X)$ and by F_X the point corresponding to $q_X(\mathcal{F}_1(X))$. In view of the notation above, note that $\mathcal{O}_{\mathrm{HS}(X)}(\chi)$ denotes the orbit of χ in $\mathrm{HS}(X)$.

A continuum X has the *property of Kelley at a point* $x_1 \in X$ provided that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if x_2 is a point of X with $d(x_1,x_2) < \delta$ and $A \in \mathcal{C}(X)$ is such that $x_1 \in A$, then there exists $B \in \mathcal{C}(X)$ such that $x_2 \in B$ and $\mathcal{H}(A,B) < \varepsilon$. The continuum X has the *property of Kelley* if it has the property of Kelley at each of its points. A continuum X has the *property of Kelley weakly* provided that there exists a dense subset $\mathcal{A}$ of $\mathcal{C}(X)$ such that X has the property of Kelley at some point of each $A \in \mathcal{A}$ [22].

2.1. Remark. Note that the sets $\mathrm{HS}(X) \setminus \{F_X\}$ and $\mathrm{HS}(X) \setminus \{T_X, F_X\}$ are homeomorphic to $\mathcal{C}(X) \setminus \mathcal{F}_1(X)$ and $\mathcal{C}(X) \setminus (\{X\} \cup \mathcal{F}_1(X))$, respectively, using the appropriate restriction of q_X.

3. Preliminary results. We begin by noting that as a consequence of [35, Theorem 1.8] and [1, Theorem 1, p. 654], we have:

3.1. Theorem. *Let X be an arc-like or circle-like continuum. Then X does not contain weak triods.*

The following lemma appears as Corollary 2.8 of [27], but it is supposed to be a consequence of a proposition that does not exist. Hence, we decided to include a proof of it.

3.2. Lemma. *Let X be a continuum, let p be a point of X and let $A \in \mathcal{C}(\{p\}, X)$. If X is connected im kleinen at p, then $\mathcal{C}(X)$ is connected im kleinen at A.*

Proof. Let $\varepsilon > 0$. Since X is connected im kleinen at p, there exists a subcontinuum W of X such that $p \in \mathrm{Int}_X(W) \subset W \subset \mathcal{V}^d_\varepsilon(p)$. Let $\delta \in (0, \varepsilon)$ be such that $\mathcal{V}^d_\delta(p) \subset W$. Let $B \in \mathcal{V}^{\mathcal{H}}_\delta(A)$. Then $B \cap \mathcal{V}^d_\delta(p) \neq \emptyset$ and $B \cap \mathcal{V}^d_\delta(p) \subset B \cap W$. Let $\alpha_1, \alpha_2\colon [0,1] \to \mathcal{C}(X)$ be order arcs such that $\alpha_1(0) = B$, $\alpha_1(1) = A \cup B \cup W$, $\alpha_2(0) = A$ and $\alpha_2(1) = A \cup B \cup W$ [29, (1.8)]. Let $\mathcal{D}_B = \alpha_1([0,1]) \cup \alpha_2([0,1])$. Then $\mathcal{D}_B$ is a subcontinuum of $\mathcal{C}(X)$ and $\{A, B\} \subset \mathcal{D}_B \subset \mathcal{V}^{\mathcal{H}}_\varepsilon(A)$. Therefore, by [18, 1.7.9], $\mathcal{C}(X)$ is connected im kleinen at A. ■

As a consequence of [36, (12.1), p. 18] and [7, Theorem 2], we have:

3.3. Lemma. *Let X be a continuum, let p be a point of X and let $\delta > 0$. If X is not connected im kleinen at p, then there exists a nondegenerate subcontinuum Y of X such that $p \in Y \subset \mathcal{V}^d_\delta(p)$ and $\mathcal{C}(X)$ is not connected im kleinen at Y.*

3.4. Lemma. *Let X be a continuum and let $\chi \in \mathrm{HS}(X) \setminus \{F_X\}$ be such that $\mathrm{HS}(X) \setminus \{F_X, \chi\}$ is not arcwise connected. If $h\colon \mathrm{HS}(X) \to \mathrm{HS}(X)$ is a homeomorphism such that $h(\chi) \neq F_X$, then $h(F_X) = F_X$.*

Proof. Since $\mathrm{HS}(X) \setminus \{F_X, \chi\}$ is not arcwise connected, we see that $\mathrm{HS}(X) \setminus \{h(F_X), h(\chi)\}$ is not arcwise connected. By [20, 4.4], we obtain $F_X \in \{h(F_X), h(\chi)\}$. Hence, since $h(\chi) \neq F_X$, $h(F_X) = F_X$. ■

3.5. Lemma. *If X is a continuum, then $\mathrm{HS}(X)$ is locally connected at both T_X and F_X.*

Proof. Since $\mathcal{C}(X)$ is locally connected at X [31, (1.136)], $\mathrm{HS}(X)$ is locally connected at T_X by Remark 2.1; and it is locally connected at F_X by [17, 3.2]. ■

A *Whitney map* is a map $\mu\colon \mathcal{C}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ for all $A, B \in \mathcal{C}(X)$ such that $A \subsetneq B$.

3.6. Lemma. *Let X be a continuum. If F_X has an open neighborhood $\mathfrak{U}$ in $\mathrm{HS}(X)$ such that $\mathrm{HS}(X)$ is locally connected at each point of $\mathfrak{U}$, then X is locally connected.*

Proof. Suppose X is not locally connected. Then there exists a point p in X such that X is not connected im kleinen at p. Hence, there exists $\delta > 0$ such that if V is a neighborhood of p and $V \subset \mathcal{V}^d_\delta(p)$, then V is not connected. Without loss of generality, we assume that $\langle \mathcal{V}_\delta^d(p)\rangle \subset q_X^{-1}(\mathfrak{U})$, where $\langle \mathcal{V}_\delta^d(p)\rangle = \{B \in \mathcal{C}(X) \mid B \subset \mathcal{V}_\delta^d(p)\}$; it is known [29, (0.13)] that $\langle \mathcal{V}_\delta^d(p)\rangle$ is an open subset of $\mathcal{C}(X)$.

Let V be a neighborhood of p such that $\mathrm{Cl}_X(V) \subset \mathcal{V}_\delta^d(p)$, and let A' be the component of $\mathrm{Cl}_X(V)$ that contains p. Then, by [36, (12.1), p. 18], we have:

(1) $p \in A'$;
(2) $A' \subset \mathcal{V}_\delta^d(p)$;
(3) there exist a nondegenerate subcontinuum A of A' and a sequence $\{A_n\}_{n=1}^\infty$ of subcontinua of X such that $\lim A_n = A$, $A_n \cap A_m = \emptyset$ if $n \neq m$ and $A_n \cap A = \emptyset$; in fact, no A_n is contained in the component of $\mathcal{V}_\delta^d(p)$ that contains p.

Since $A \in \langle \mathcal{V}_\delta^d(p)\rangle \setminus \mathcal{F}_1(X) \subset q_X^{-1}(\mathfrak{U}\setminus\{F_X\})$, and $q_X^{-1}(\mathfrak{U}\setminus\{F_X\})$ is locally connected at each of its points, there exists an open connected subset $\mathcal{K}$ of $q_X^{-1}(\mathfrak{U} \setminus \{F_X\})$ such that $A \in \mathcal{K} \subset \mathrm{Cl}_{\mathcal{C}(X)}(\mathcal{K}) \subset \langle \mathcal{V}_\delta^d(p)\rangle$. Since $\lim A_n = A$, there exists $N \in \mathbb{N}$ such that $A_n \in \mathcal{K}$ for all $n \geq N$. Let $W = \bigcup \mathrm{Cl}_{\mathcal{C}(X)}(\mathcal{K})$. Then W is a subcontinuum of X [29, (1.49)] and $W \subset \mathcal{V}_\delta^d(p)$. Observe that for each $n \geq N$, $A_n \subset W$, contrary to the construction of the sequence $\{A_n\}_{n=1}^\infty$. Therefore, X is locally connected. ■

3.7. Lemma. *Let X be a decomposable continuum and let $\mathcal{U}$ be a neighborhood of X in $\mathcal{C}(X)$. Then there exists a neighborhood $\mathcal{W}$ of X in $\mathcal{C}(X)$ such that $\mathcal{W} \setminus \{X\}$ is arcwise connected and $\mathcal{W} \subset \mathcal{U}$.*

Proof. Let A and B be proper subcontinua of X such that $X = A \cup B$ and let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map. Let $t \in (0,1)$ be such that $t > \max\{\mu(A), \mu(B)\}$ and $\mu^{-1}([t,1]) \subset \mathcal{U}$. We show first that $\mu^{-1}([t,1))$ is arcwise connected.

Let $\alpha\colon [0,1] \to \mathcal{C}(X)$ be an order arc such that $\alpha(0) = A$ and $\alpha(1) = X$. Also, let $s \in [0,1]$ be such that $\mu(\alpha(s)) = t$. Since $\alpha([s,1))$ is arcwise connected, it is enough to consider $L \in \mu^{-1}([t,1)) \setminus \alpha([s,1))$ and show that there exists an arc $\mathcal{A} \subset \mu^{-1}([t,1))$ that contains L and intersects $\alpha([s,1))$. To this end, let $r \in [s,1)$ be such that $\mu(\alpha(r)) = \mu(L)$. Since $\mu(L) > \mu(B)$, we have $L \not\subset B$. Thus, $L \cap A \neq \emptyset$ and $L \cap A \subset L \cap \alpha(r)$. By [29, (14.8.1)], there exists an arc $\mathcal{A} \subset \mu^{-1}(\mu(L))$ containing both $\alpha(r)$ and L. This proves that $\mu^{-1}([t,1))$ is arcwise connected. Then $\mathcal{W} = \mu^{-1}([t,1])$ has the required properties. ■

As a consequence of Lemma 3.7, we have the following:

3.8. Corollary. *Let X be a decomposable continuum and let $\mathcal{U}$ be a neighborhood of T_X in $\mathrm{HS}(X)$. Then there exists a neighborhood $\mathcal{V}$ of T_X in $\mathrm{HS}(X)$ such that $\mathcal{V} \setminus \{T_X\}$ is arcwise connected and $\mathcal{V} \subset \mathcal{U}$.*

Proof. By Lemma 3.7, there exists a neighborhood $\mathcal{W}$ of X in $\mathcal{C}(X) \setminus \mathcal{F}_1(X)$ such that $\mathcal{W} \subset q_X^{-1}(\mathcal{U})$ and $\mathcal{W} \setminus \{X\}$ is arcwise connected. Now, let $\mathcal{V} = q_X(\mathcal{W})$. Then $\mathcal{V}$ is a neighborhood of T_X in $\mathrm{HS}(X)$, $\mathcal{V} \subset \mathcal{U}$ and $\mathcal{V} \setminus \{T_X\}$ is arcwise connected. ∎

3.9. LEMMA. *If X is a pseudo-linear continuum, then X is irreducible.*

Proof. Let X_1 and X_2 be two proper subcontinua of X satisfying the definition of pseudo-linearity. For each $j \in \{1,2\}$, let $\alpha_j \colon [0,1] \to \mathcal{C}(X)$ be an order arc such that $\alpha_j(0) = X_1 \cap X_2$, $\alpha_j(1) = X_j$.

Let $j \in \{1,2\}$. Since $X_1 \cap X_2$ is a terminal subcontinuum in both X_1 and X_2, we find that α_j is unique in the sense that whenever $M \in \mathcal{C}(X_j)$ and $X_1 \cap X_2 \subset M$, we have $M = \alpha_j(t)$ for some $t \in [0,1]$. Note that, by [16, 3.1], there exists $x_j \in \alpha_j(1) \setminus \bigcup_{t<1} \alpha_j(t)$.

We prove that X is irreducible between x_1 and x_2. To this end, let $L \in \mathcal{C}(X)$ be such that $\{x_1, x_2\} \subset L$. Since $L \cap (X \setminus X_j) \neq \emptyset$ for $j \in \{1,2\}$, $X_1 \cap X_2 \subset L$. Note that for each $j \in \{1,2\}$, $L \cap X_j$ is connected because $X_1 \cap X_2 \subset L \cap X_j$ and each component of $L \cap X_j$ intersects $X_1 \cap X_2$. Hence, from what we said in the previous paragraph, we deduce that there exists $t_j \in [0,1]$ such that $L \cap X_j = \alpha(t_j)$, $j \in \{1,2\}$. Since $x_j \in L \cap X_j$, this implies that $t_j = 1$. Hence, $L \cap X_j = X_j$. Therefore, $L = X$, and X is irreducible. ∎

3.10. LEMMA. *Let X be a decomposable, proper circle-like continuum. Then there exist $Y_1, Y_2 \in \mathcal{C}(X)$ such that $X = Y_1 \cup Y_2$, $Y_j = \mathrm{Cl}(\mathrm{Int}(Y_j))$ and $\mathrm{Int}(Y_j) \cap Y_k = \emptyset$ whenever $\{j,k\} = \{1,2\}$. Moreover, $Y_1 \cap Y_2$ has exactly two components.*

Proof. By [1, Theorem 4, p. 655], no subcontinuum of X separates X. Hence, by [1, Theorem 5, p. 656] we know that X is bicoherent. Since X is decomposable, there exist two proper subcontinua X_1 and X_2 of X such that $X = X_1 \cup X_2$. Let $Y_1 = \mathrm{Cl}_X(X \setminus X_2)$ and $Y_2 = \mathrm{Cl}_X(X \setminus Y_1) = \mathrm{Cl}_X(X_2 \setminus Y_1)$. It follows from [1, Theorem 4, p. 655] that Y_1 and Y_2 are subcontinua of X. Moreover, note that

$$X = (X_1 \setminus X_2) \cup X_2 \subset Y_1 \cup X_2 = Y_1 \cup (X_2 \setminus Y_1) \subset Y_1 \cup Y_2.$$

It is easy to see that Y_j is a closed domain for each $j \in \{1,2\}$. Furthermore, $\mathrm{Int}_X(Y_2) \cap Y_1 \subset \mathrm{Int}_X(X_2) \cap \mathrm{Cl}_X(X \setminus X_2) = \emptyset$. Similarly, $\mathrm{Int}_X(Y_1) \cap Y_2 = \emptyset$. Since Y_1 and Y_2 have disjoint (nonempty) interiors, it follows that both are proper subcontinua of X. Therefore, since X is bicoherent, we conclude that $Y_1 \cap Y_2$ has exactly two components. ∎

3.11. LEMMA. *Let X be a decomposable, proper circle-like continuum. Then X is pseudo-circular.*

Proof. By Lemma 3.10, there exist two subcontinua Y_1 and Y_2 of X such that

$$X = Y_1 \cup Y_2, \quad Y_j \text{ is a closed domain} \quad \mathrm{Int}_X(Y_j) \cap Y_k = \emptyset,$$

whenever $\{j,k\} = \{1,2\}$, and $Y_1 \cap Y_2$ has exactly two components K_1 and K_2.

Let $r, j \in \{1,2\}$ and let $s, t \in \{1,2\}$ be such that $\{r,s\} = \{1,2\} = \{j,t\}$. We show that K_r is terminal in Y_j. Let B_1 and B_2 be two subcontinua of Y_j such that $K_r \subset B_1 \cap B_2$, and suppose that $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. Let $\alpha\colon [0,1] \to \mathcal{C}(Y_t)$ be an order arc such that $\alpha(0) = K_r$ and $\alpha(1) = Y_t$. Let $t_0 \in (0,1)$ be such that $\alpha(t_0) \cap K_s = \emptyset$. It follows that B_1, B_2 and $\alpha(t_0)$ form a weak triod, contradicting Theorem 3.1. This proves that K_r is terminal in both Y_1 and Y_2.

Next, let $L \in \mathcal{C}(X)$ be such that $L \cap K_1 \neq \emptyset$ and $L \cap K_2 \neq \emptyset$. We show that L contains either Y_1 or Y_2. As a consequence of [31, 5.2], there exist $k \in \{1,2\}$ and a component M of $L \cap Y_k$ such that $M \cap K_1 \neq \emptyset$ and $M \cap K_2 \neq \emptyset$. We assume without loss of generality that $k = 1$, and prove that $Y_1 \subset M$. Suppose that $Y_1 \not\subseteq M$. Thus, since Y_1 is a closed domain, we see that $\mathrm{Int}(Y_1) \not\subseteq M$. Hence, there exists $y \in \mathrm{Int}(Y_1) \setminus (M \cup Y_2)$. This implies that $M \cup Y_2$ is a proper subcontinuum of X. Moreover, note that $M \cup Y_2$ is not unicoherent; however, since X is circle-like, this is not possible. Therefore, $Y_1 \subset M \subset L$.

Now, let $\mu\colon C(X) \to [0,1]$ be a Whitney map and let $\varepsilon > 0$ be such that if $Z \in C(X)$ and $X \subset \mathcal{V}_\varepsilon(Z)$, then $\mu(Z) > \max\{\mu(Y_1), \mu(Y_2)\}$. We show that either $K_1 \subset Z$ or $K_2 \subset Z$. Assume that neither Y_1 nor Y_2 is contained in Z (otherwise we are done). By the choice of ε and Z, we have $Z \setminus Y_1 \neq \emptyset$ and $Z \setminus Y_2 \neq \emptyset$. Hence, since Z is connected, it follows that $Z \cap K_l \neq \emptyset$ for some $l \in \{1,2\}$. We assume without loss of generality that $l = 1$, and prove that $K_1 \subset Z$. By the preceding paragraph, $Z \cap K_2 = \emptyset$. Next we show that $Z \cap Y_1$ is connected. Since Y_2 is a closed domain, $\mathrm{Int}(Y_2) \setminus Z \neq \emptyset$. Hence, $Z \cup Y_1 \neq X$. Thus, $Z \cup Y_1$ is a proper subcontinuum of X. This implies that $Z \cup Y_1$ is unicoherent. Therefore, $Z \cap Y_1$ is connected. Similarly, $Z \cap Y_2$ is connected. Finally, if $K_1 \not\subseteq Z$, then $Z \cap Y_1$, $Z \cap Y_2$ and K_1 form a weak triod, contrary to Theorem 3.1. This proves that $K_1 \subset Z$ and we conclude that X is pseudo-circular. ∎

3.12. LEMMA. *Let X be a continuum and let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map. Suppose there exists $t \in (0,1)$ such that $A \cap B \neq \emptyset$ whenever $A, B \in \mu^{-1}([t,1))$. Then $\mathcal{C}(X)$ is locally connected at each point of $\mu^{-1}((t,1])$.*

Proof. It is known that $\mathcal{C}(X)$ is locally connected at X [29, (1.136)], and $\{X\} = \mu^{-1}(1)$. Let $A \in \mu^{-1}((t,1))$, and let $\delta > 0$ be such that $\mathcal{V}^{\mathcal{H}}_\delta(A) \subset \mu^{-1}((t,1))$. Let $B \in \mathcal{V}^{\mathcal{H}}_\delta(A)$. Then $\mu(B) > t$, $A \cap B \neq \emptyset$ and $A \cup B \in \mathcal{V}^{\mathcal{H}}_\delta(A)$.

Let $\alpha, \beta\colon [0,1] \to \mathcal{C}(X)$ be order arcs such that $\alpha(0) = A$, $\alpha(1) = A \cup B$, $\beta(0) = B$ and $\beta(1) = A \cup B$ [29, (1.8)]. Then $\alpha([0,1]) \cup \beta([0,1])$ is a locally connected continuum such that $\{A, B\} \subset \alpha([0,1]) \cup \beta([0,1]) \subset \mathcal{V}_\delta^{\mathcal{H}}(A)$. Therefore, $\mathcal{C}(X)$ is locally connected at each point of $\mu^{-1}((t,1])$. ■

3.13. COROLLARY. *Let X be a continuum and let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map. Suppose there exists $t \in (0,1)$ such that $A \cap B \neq \emptyset$ whenever $A, B \in \mu^{-1}([t,1))$. Then* HS(X) *is locally connected at each point of $q_X(\mu^{-1}((t,1]))$.*

The following result is a consequence of the proof of [29, (1.100)]:

3.14. LEMMA. *Let X be a triod with core K. Then both X and K belong to a 3-cell in $\mathcal{C}(X)$.*

3.15. LEMMA. *Let X be a continuum. Assume that for each $\varepsilon > 0$ and for each pair $p, q \in X$ with $p \neq q$ there exists a finite sequence $\{Y_1, \dots, Y_m\}$ of subcontinua of X such that* $\mathrm{diam}(Y_i) < \varepsilon$, *$p \in Y_1$, $q \in Y_m$ and $Y_i \cap Y_{i+1} \neq \emptyset$ for each $i < m$. Then X is continuum chainable.*

Proof. Let $\varepsilon > 0$ and let $p, q \in X$ be such that $p \neq q$. Let $\{Y_1, \dots, Y_m\}$ be such that $\mathrm{diam}(Y_i) < \varepsilon$, $p \in Y_1$, $q \in Y_m$ and $Y_i \cap Y_{i+1} \neq \emptyset$ for each $i < m$. Let $n_p = \max\{j \in \{1, \dots, m\} \mid p \in Y_j\}$ and let $Q = \{j \in \{1, \dots, m\} \mid q \in Y_j\}$. We may assume that $n_p \notin Q$.

Define a function $\varphi\colon \{1, \dots, m\} \to \{1, \dots, m\}$ by

$$\varphi(k) = \max\{j \in \{k, \dots, m\} \mid Y_k \cap Y_j \neq \emptyset\}.$$

As usual, φ^0 denotes the identity function and $\varphi^{n+1} = \varphi \circ \varphi^n$ for each $n \in \mathbb{N}$. Note that $k + 1 \leq \varphi(k)$ for each $k < m$. Hence:

(a) $\varphi(k) < \varphi(\varphi(k))$ whenever $\varphi(k) < m$, and
(b) there exists $r \in \{1, \dots, m\}$ such that $\varphi^s(n_p) \notin Q$ for each $s < r$ and $\varphi^r(n_p) \in Q$.

Moreover, it follows directly from the definition of φ that:

(c) $Y_k \cap Y_{\varphi(k)} \neq \emptyset$ for each k, and
(d) $Y_k \cap Y_j = \emptyset$ for each $j > \varphi(k)$.

Thus, using (a), (b), (c) and (d) it is not difficult to see that the sequence $\{Y_{n_p}, Y_{\varphi(n_p)}, \dots, Y_{\varphi^r(n_p)}\}$ satisfies the conditions of continuum chainability. ■

4. General results

4.1. EXAMPLE. If $X = [0,1]$, then HS(X) is a 2-cell. Hence, HS(X) is $\frac{1}{2}$-homogeneous. Observe that T_X and F_X belong to the manifold boundary of HS(X). Thus, T_X and F_X belong to the same orbit of HS(X).

4.2. Example. If $X = \mathcal{S}^1$, then $\mathcal{C}(X)$ is a 2-cell and $\mathcal{F}_1(X)$ is the manifold boundary of $\mathcal{C}(X)$. Hence, $\mathrm{HS}(X)$ is a 2-sphere, which is homogeneous. Therefore, $\mathrm{HS}(X)$ is not $\frac{1}{2}$-homogeneous.

4.3. Lemma. *If X is a nonlocally connected continuum such that* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous, then $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$.*

Proof. Since X is not locally connected, there exists $\chi \in \mathrm{HS}(X)$ such that $\mathrm{HS}(X)$ is not locally connected at χ. By Lemma 3.5, $\mathrm{HS}(X)$ is locally connected at both T_X and F_X. Hence, since $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, we have $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. ∎

4.4. Theorem. *Let X be a decomposable continuum. If* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous, then X is continuum chainable.*

Proof. If X is a locally connected continuum, then X is arcwise connected [29, 8.23]. Thus, X is continuum chainable. Hence, we assume that X is not locally connected.

Let $\varepsilon > 0$ and let $p, q \in X$ with $p \neq q$. Let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map and let $t \in (0,1)$ be such that if $L \in \mu^{-1}([0,t))$, then $\operatorname{diam}(L) < \varepsilon/3$.

Since $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, by Lemma 4.3, we have $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. Hence, by Corollary 3.8, there exists a neighborhood $\mathcal{V}$ of F_X in $\mathrm{HS}(X)$ such that $\mathcal{V}\setminus\{F_X\}$ is arcwise connected. Without loss of generality, we assume that $\mathcal{V} \subset q_X(\mu^{-1}([0,t)))$.

Let $P, Q \in q_X^{-1}(\mathcal{V})\setminus F_1(X)$ be such that $p \in P$ and $q \in Q$. We can assume that $P \neq Q$. Note that there exists an arc $\mathcal{A} \subset \mathcal{V} \setminus \{F_X\}$ from $q_X(P)$ to $q_X(Q)$. Let $h\colon [0,1] \to q_X^{-1}(\mathcal{A})$ be a homeomorphism such that $h(0) = P$ and $h(1) = Q$.

Recall that the union map $\bigcup\colon 2^{2^X} \to 2^X$ is uniformly continuous [29, (1.48)]. Then there exists $\delta > 0$ such that if $\mathcal{B}, \mathcal{D} \in 2^{2^X}$ and $\mathcal{H}^2(\mathcal{B}, \mathcal{D}) < \delta$, then $\mathcal{H}(\bigcup\mathcal{B}, \bigcup\mathcal{D}) < \varepsilon/3$.

Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$ be such that $\operatorname{diam}(h([a_{j-1}, a_j])) < \delta$ for each j. Let $j \in \{1, \ldots, n\}$ and let $M_j = \bigcup h([a_{j-1}, a_j])$. Then $p \in h(0) \subset M_1$ and $q \in h(1) \subset M_n$. Moreover, since $\mathcal{H}^2(h([a_{j-1}, a_j]), \{h(a_j)\}) < \delta$, we obtain that $\mathcal{H}(M_j, h(a_j)) < \varepsilon/3$. Since $h(a_j) \in q_X^{-1}(\mathcal{A}) \subset q_X^{-1}(\mathcal{V}) \subset \mu^{-1}([0,t))$, it follows, by the choice of t, that $\operatorname{diam}(M_j) < \varepsilon$. By [29, (1.49)], M_j is a subcontinuum of X and $h(a_{j-1}) \subset M_{j-1} \cap M_j$. Therefore, by Lemma 3.15, X is continuum chainable. ∎

4.5. Corollary. *Let X be a decomposable plane continuum with only finitely many complementary domains. If* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous, then X is arcwise connected.*

Proof. By Theorem 4.4, X is continuum chainable. Hence, by [8, Theorem 2], X is arcwise connected. ∎

4.6. LEMMA. *If X is an irreducible or an atriodic continuum such that* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous, then* $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$.

Proof. If X is not locally connected, the result follows from Lemma 4.3. If X is locally connected and irreducible, then X is an arc and the assertion follows from Example 4.1. If X is locally connected an atriodic, then X is an arc or a simple closed curve [31, 8.40(b)] and the lemma follows from Examples 4.1 and 4.2. ∎

4.7. THEOREM. *Let X be an atriodic continuum such that* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous. If $\{T_X, F_X\}$ is an orbit of* $\mathrm{HS}(X)$*, then X is indecomposable.*

Proof. Assume $\{T_X, F_X\}$ is an orbit of $\mathrm{HS}(X)$. Suppose X is decomposable. Hence, there exists a proper subcontinuum A of X such that $\mathrm{Int}_X(A) \neq \emptyset$. By [7, Theorem 3], $\mathcal{C}(X)$ is locally connected at A. Thus, $\mathrm{HS}(X)$ is locally connected at $q_X(A)$. Hence, since $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous and $\{T_X, F_X\}$ is an orbit of $\mathrm{HS}(X)$, $\mathrm{HS}(X) \setminus \{T_X, F_X\}$ is locally connected. Thus, $\mathrm{HS}(X)$ is locally connected (Lemma 3.5). This implies that X is locally connected [5, 5.1]. Hence, X is an arc or a simple closed curve [31, 8.40(b)]. Since the hyperspace suspension of a simple closed curve is homogeneous (Example 4.2), we see that X is an arc, a contradiction because $\{T_X, F_X\}$ is not an orbit of $\mathrm{HS}(X)$ when X is an arc. Therefore, X is indecomposable. ∎

As a consequence of Theorem 4.7 and [23, 6.8], we obtain:

4.8. THEOREM. *Let X be an atriodic continuum with the property of Kelley weakly such that* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous. Then X is indecomposable if and only if $\{T_X, F_X\}$ is an orbit of* $\mathrm{HS}(X)$.

4.9. COROLLARY. *If X is an indecomposable atriodic continuum with the weak property of Kelley such that* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous, then every proper nondegenerate subcontinuum of X is decomposable.*

Proof. Observe that, by Theorem 4.8, $\{T_X, F_X\}$ is an orbit of $\mathrm{HS}(X)$. Suppose there exists a proper nondegenerate indecomposable subcontinuum Y of X. Then $\mathcal{C}(X)\setminus\{Y\}$ has uncountably many arc components [32, 2.8]. It is easy to see that $\mathcal{C}(X) \setminus (\{Y\} \cup \mathcal{F}_1(X))$ has uncountably many arc components. Hence, $\mathrm{HS}(X) \setminus \{F_X, q_X(Y)\}$ has uncountably many arc components [20, 3.3]. Since $\mathcal{O}_{\mathrm{HS}(X)}(q_X(Y)) = \mathrm{HS}(X) \setminus \{T_X, F_X\}$, $\mathrm{HS}(X) \setminus \{F_X, \chi\}$ has uncountably many arc components for each $\chi \in \mathrm{HS}(X)\setminus\{T_X, F_X\}$. This implies that for each such χ, $q_X^{-1}(\chi)$ is an indecomposable subcontinuum of X [20, 4.2]. Hence, X is a hereditarily indecomposable continuum [21, 3.1], a contradiction to [23, 6.6]. ∎

4.10. THEOREM. *If X is a decomposable continuum such that* $\mathrm{HS}(X)$ *is finite-dimensional and* $\frac{1}{2}$*-homogeneous, then X is hereditarily decomposable.*

Proof. Suppose X contains an indecomposable subcontinuum Y. Then $\mathcal{C}(X) \setminus \{Y\}$ is not arcwise connected [21, 3.4]. Hence, $\mathrm{HS}(X) \setminus \{F_X, q_X(Y)\}$ is not arcwise connected [20, 3.3]. Note that $\mathrm{HS}(X) \setminus \{T_X, \chi\}$ is arcwise connected for each $\chi \in \mathrm{HS}(X) \setminus \{F_X\}$ [19, 4.3]. Since X is decomposable, $\mathrm{HS}(X) \setminus \{T_X, F_X\}$ is arcwise connected [5, 3.3]. Hence, by Lemma 3.4, there does not exist a homeomorphism $h\colon \mathrm{HS}(X) \to \mathrm{HS}(X)$ such that $h(q_X(Y)) = T_X$. Thus, $\mathcal{O}_{\mathrm{HS}(X)}(T_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(q_X(Y))$.

Now, we show that $\mathcal{O}_{\mathrm{HS}(X)}(T_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. Suppose $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$ then there exists a homeomorphism $h_0\colon \mathrm{HS}(X) \to \mathrm{HS}(X)$ such that $h_0(F_X) = T_X$. Let $h_1\colon \mathrm{HS}(X) \to \mathrm{HS}(X)$ be a homeomorphism. If $h_1(q_X(Y)) = F_X$, then $\mathcal{O}_{\mathrm{HS}(X)}(T_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. Assume that $h_1(q_X(Y)) \neq F_X$. Then $h_1(F_X) = F_X$, by Lemma 3.4. Note that $h_0 \circ h_1$ is a homeomorphism such that $h_0 \circ h_1(F_X) = T_X$. Since $\mathrm{HS}(X) \setminus \{F_X, q_X(Y)\}$ is not arcwise connected, $\mathrm{HS}(X) \setminus \{h_0 \circ h_1(F_X), h_0 \circ h_1(q_X(Y))\} = \mathrm{HS}(X) \setminus \{T_X, h_0 \circ h_1(q_X(Y))\}$ is not arcwise connected. Hence, $h_0 \circ h_1(q_X(Y)) = F_X$ by Lemma 3.4. This implies that X is indecomposable [5, 3.3], a contradiction. Thus, $\mathcal{O}_{\mathrm{HS}(X)}(T_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. This contradicts [23, 6.5]. Therefore, X is hereditarily decomposable. ∎

4.11. Theorem. *If X is an indecomposable continuum with the property of Kelley weakly such that* $\mathrm{HS}(X)$ *is finite-dimensional and $\frac{1}{2}$-homogeneous, then every proper nondegenerate subcontinuum of X is decomposable.*

Proof. It is the same proof as the one given for Corollary 4.9, using [23, 6.10] and [21, 3.4] instead of Theorem 4.8 and [32, 2.8], respectively. ∎

4.12. Theorem. *Let X be a continuum and let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map. Suppose there exists $t \in (0,1)$ such that $A \cap B \neq \emptyset$ whenever $A, B \in \mu^{-1}([t,1))$. If* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous, then X is locally connected.*

Proof. By Corollary 3.13, $\mathrm{HS}(X)$ is locally connected at each point of $q_X(\mu^{-1}((t,1]))$. Note that $q_X(\mu^{-1}((t,1]))$ is an open neighborhood of T_X in $\mathrm{HS}(X)$.

Suppose X is not locally connected. Then there exists $\chi \in \mathrm{HS}(X)$ such that $\mathrm{HS}(X)$ is not connected im kleinen at χ. Note that by Lemma 4.3, we have $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(\chi)$. This implies, by the first paragraph, that F_X has an open neighborhood $\mathfrak{U}$ such that $\mathrm{HS}(X)$ is locally connected at each point of $\mathfrak{U}$. Thus, by Lemma 3.6, X is locally connected, a contradiction. Therefore, X is locally connected. ∎

4.13. Corollary. *Let X be a continuum with a cut point. If* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous, then X is locally connected.*

Proof. Let p be a cut point of X, and let $A, B \in \mathcal{C}(X)$ be such that $X = A \cup B$ and $A \cap B = \{p\}$ [18, 1.7.18]. Let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map. Let $t \in (0,1)$ be such that $\max\{\mu(A), \mu(B)\} < t$. Thus, if

$D \in \mathcal{C}(X)$ and $\mu(D) > t$, then $p \in D$. Hence, by Theorem 4.12, X is locally connected. ■

4.14. Remark. Let X be a continuum and let T be a simple triod in X. Then F_X is in the manifold boundary of a 3-cell, since HS(T) is homeomorphic to $\mathcal{C}(T)$ [5, 3.2]. In particular, $\dim_{F_X}(\mathrm{HS}(X)) \geq 3$.

4.15. Remark. Let X be a locally connected continuum such that HS(X) is $\frac{1}{2}$-homogeneous. If X does not contain simple triods, then X is an arc of a simple closed curve [31, 8.40(b)]. Thus, since the hyperspace suspension of a simple closed curve is a 2-sphere (Example 4.2), X is an arc.

By a *loop* in a continuum X we mean a simple closed curve C such that $\mathrm{Bd}_X(C)$ has exactly one point. Also, for a continuum X define

$$\mathcal{J}(X) = \{A \subset X \mid A \text{ is a maximal free arc or a loop}\},$$
$$T(X) = \{x \in X \mid x \text{ is the core of a simple triod in } X\}.$$

The following lemma is easy to prove.

4.16. Lemma. *Let X be a locally connected continuum and assume $J \in \mathcal{J}(X)$. If $b \in \mathrm{Bd}_X(J)$, then $b \in \mathrm{Cl}(T(X))$.*

4.17. Lemma. *Let X be a locally connected continuum and assume $J \in \mathcal{J}(X)$. If $D \in \mathcal{C}(J)$ and $D \cap \mathrm{Bd}_X(J) \neq \emptyset$, then D does not have planar neighborhoods in $\mathcal{C}(X)$.*

Proof. Let $\varepsilon > 0$. It is enough to show that there exists $D_\varepsilon \in \mathcal{C}(X)$ such that D_ε belongs to a 3-cell in $\mathcal{C}(X)$ and $\mathcal{H}(D, D_\varepsilon) < \varepsilon$. We consider two cases.

Case (1): J is a loop. In this case $\mathrm{Bd}_X(J)$ consists of a single point, say p. Since X is locally connected, X is arcwise connected [31, 8.23]. Let A be an arc in X such that $A \cap J = \{p\}$. Let $D_\varepsilon \in \mathcal{C}(J)$ be an arc such that $p \in D_\varepsilon$ and $\mathcal{H}(D, D_\varepsilon) < \varepsilon$ (if $D \neq J$, simply take $D_\varepsilon = D$). Let B be a subarc of J such that $D_\varepsilon \subset B$ and $B \setminus D_\varepsilon$ has two components. It follows that $A \cup B$ is a triod with core D_ε. Hence, by Lemma 3.14, we conclude that D_ε belongs to a 3-cell in $C(X)$.

Case (2): J is a maximal free arc. Let $p \in D \cap \mathrm{Bd}_X(J)$. We assume that the end points of J are p and e for some $e \in X$. Let J_ε be a subarc of J such that $p \in J_\varepsilon \subset J \setminus \{e\}$ and $\mathcal{H}(D, J_\varepsilon) < \varepsilon$ (if $D \neq J$, simply let $J_\varepsilon = D$). By Lemma 4.16, we see that $p \in \mathrm{Cl}(T(X))$. Suppose $p \notin T(X)$; since X is locally arcwise connected [31, 8.25], we may take $q \in T(X) \setminus J$ and an arc L in X whose endpoints are q and p, $L \cap J = \{p\}$ and $\mathrm{diam}(L) < \varepsilon$. If $p \in T(X)$, we let $q = p$ and $L = \{p\}$.

Let $D_\varepsilon = J_\varepsilon \cup L$. Then $\mathcal{H}(D, D_\varepsilon) < \varepsilon$. Also, let T be a simple triod with core $\{q\}$ such that $e \notin T$. This implies that $(J \cup L \cup T) \setminus D_\varepsilon$ has at least three components; i.e., $J \cup L \cup T$ is a triod with core D_ε. Hence, by Lemma 3.14, D_ε belongs to a 3-cell in $\mathcal{C}(X)$. ∎

A proof of the following lemma may be found in [34, Lemma 4.24].

4.18. Lemma. *Let X be a locally connected continuum such that X is not a simple closed curve. If L is a free arc in X, then there exists $J \in \mathcal{J}(X)$ such that $L \subset J$.*

Given an n-cell A, $i(A)$ denotes the manifold interior of A.

4.19. Theorem. *If X is a continuum containing a free arc, then* HS(X) *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. If X is an arc, then the result is clear by Example 4.1.

Suppose HS(X) is $\frac{1}{2}$-homogeneous. Note that X is not a simple closed curve by Example 4.2. Let

$$\mathcal{L}(X) = \{\chi \in \mathrm{HS}(X) \mid \chi \text{ has a 2-cell neighborhood in } \mathrm{HS}(X)\},$$
$$\mathfrak{K}(X) = \{\mathcal{K} \subset \mathcal{L}(X) \mid \mathcal{K} \text{ is a component of } \mathcal{L}(X)\}.$$

We consider two cases.

Case (1): $F_X \in \mathcal{L}(X)$. Since F_X has a 2-cell neighborhood in HS(X), by Lemma 3.6, X is locally connected. By Remark 4.14, X does not contain simple triods. Hence, by Remark 4.15, X is an arc.

Case (2): $F_X \notin \mathcal{L}(X)$. Let A be a free arc in X and let $B \in i(\mathrm{HS}(A))$. Note that $B \in \mathcal{L}(X)$ and that $\mathcal{L}(X)$ is invariant under homeomorphisms of HS(X) onto itself. Thus, since in this case $\mathcal{L}(X)$ is a proper and nonempty subset of HS(X) and HS(X) is $\frac{1}{2}$-homogeneous, it follows that the two orbits of HS(X) are $\mathcal{L}(X)$ and HS$(X) \setminus \mathcal{L}(X)$. Moreover, since HS$(X)$ is locally connected at both F_X and any point of $\mathcal{L}(X)$, we deduce that HS(X) is locally connected. Hence, X is locally connected [5, 5.1].

By Lemma 4.18, there exists $J \in \mathcal{J}(X)$ such that $A \subset J$. Since X is locally connected we have

$$\mathrm{Bd}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \setminus \{F_X\} = \{Z \in \mathrm{HS}(J) \setminus \{F_X\} \mid q_X^{-1}(Z) \cap \mathrm{Bd}_X(J) \neq \emptyset\}.$$

Hence, by Lemma 4.17, we find that $\mathrm{Bd}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \cap \mathcal{L}(X) = \emptyset$. Thus, since $\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J))$ is a connected subset of $\mathcal{L}(X)$, it follows that

$$\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \in \mathfrak{K}(X).$$

Moreover, since $\mathrm{Cl}_{\mathrm{HS}(X)}(\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J))) \subset \mathrm{HS}(J)$, which is either a 2-cell or a 2-sphere, we see that

$$\mathrm{Cl}_{\mathrm{HS}(X)}(\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J))) \text{ is 2-dimensional.}$$

Since $\mathcal{L}(X)$ is an orbit of HS(X), and since $\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \in \mathfrak{K}(X)$, we deduce that $\mathcal{K}$ is open in HS(X) for each $\mathcal{K} \in \mathfrak{K}(X)$. Hence, $\mathfrak{K}(X)$ is countable. Furthermore, using again the fact that $\mathcal{L}(X)$ is an orbit of HS(X), since

$\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \in \mathfrak{K}(X)$ and $\mathrm{Cl}_{\mathrm{HS}(X)}(\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)))$ is 2-dimensional, we find that $\mathrm{Cl}_{\mathrm{HS}(X)}(\mathcal{K})$ is 2-dimensional for each $\mathcal{K} \in \mathfrak{K}(X)$.

Now, we show that

$$\mathrm{HS}(X) = \bigcup_{\mathcal{K}\in\mathfrak{K}(X)} \mathrm{Cl}_{\mathrm{HS}(X)}(\mathcal{K}).$$

To this end, let $\chi \in \mathrm{HS}(X) \setminus \mathcal{L}(X)$ and let $\chi_0 \in \mathrm{Bd}_{\mathrm{HS}(X)}(\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)))$. Since $\mathrm{HS}(X) \setminus \mathcal{L}(X)$ is an orbit of $\mathrm{HS}(X)$ that contains both χ and χ_0, and since $\mathrm{Int}_{\mathrm{HS}(X)}(\mathrm{HS}(J)) \in \mathfrak{K}(X)$, there exists $\mathcal{K}_0 \in \mathfrak{K}(X)$ such that $\chi \in \mathrm{Cl}_{\mathrm{HS}(X)}(\mathcal{K}_0)$. This proves the equation.

Finally, since $\mathrm{Cl}_{\mathrm{HS}(X)}(\mathcal{K})$ is 2-dimensional for each $\mathcal{K} \in \mathfrak{K}(X)$ and $\mathfrak{K}(X)$ is countable, by the Sum Theorem [10, III 2, p. 30], we deduce that $\mathrm{HS}(X)$ is 2-dimensional. This implies that X contains no simple triods (Remark 4.14). Therefore, by Remark 4.15, X is an arc. ■

4.20. Corollary. *Let X be an atriodic, decomposable continuum. Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. If X an arc, then the result is clear by Example 4.1.

Suppose $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. By Theorem 4.4, X is continuum chainable. Hence, since X is atriodic, X is arcwise connected [9, Theorem 3]. Thus, by [15, Theorem 11], X contains a free arc. Therefore, by Theorem 4.19, X is an arc. ■

4.21. Corollary. *Let X be a compactification of a ray (or the real line). Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

4.22. Corollary. *Let X be a hereditarily decomposable C-H continuum. Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. By [28, (1.1)], each hereditarily decomposable *C-H* continuum contains a free arc. Now the corollary follows from Theorem 4.19. ■

4.23. Corollary. *Let X be a locally connected continuum satisfying either*

(1) $\dim(\mathrm{HS}(X)) < \infty$, *or*
(2) *X is contractible.*

Then $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. If X an arc, then the result is clear by Example 4.1.

Suppose $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. Note that if X does not contain a free arc, then $\dim(\mathrm{HS}(X)) = \infty$ by [5, 3.4] and [29, (1.98)]. Hence, if we assume that $\dim(\mathrm{HS}(X)) < \infty$, then the corollary follows from Theorem 4.19.

Assume now that X is contractible. If X does not contain a free arc, then $\mathrm{HS}(X)$ is homeomorphic to the Hilbert cube [5, 5.4], which is homogeneous [25, 6.1.6], contradicting our assumption. Thus, X contains a free arc and the corollary follows from Theorem 4.19. ■

As a consequence of Corollaries 4.13 and 4.23, we have the following:

4.24. COROLLARY. *Let X be a continuum with a cut point. If either*

(1) $\dim(\mathrm{HS}(X)) < \infty$, *or*
(2) *X is contractible,*

then $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

4.25. THEOREM. *If X is a decomposable continuum which is irreducible about a finite set, then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. If X is an arc, then the result is clear by Example 4.1.

We show the converse. By Theorem 4.4, X is continuum chainable. Hence, by [9, Theorem 3], X is a tree. Now the result follows from Theorem 4.19. ■

Note that Theorem 4.25 cannot be extended to the whole class of continua which are irreducible about a finite set, because if Σ is a solenoid, then Σ is indecomposable, and hence it is a continuum which is irreducible about a finite set and such that $\mathrm{HS}(\Sigma)$ is $\frac{1}{2}$-homogeneous [23, 6.2].

Since chainable continua are irreducible [31, 12.5], we obtain:

4.26. COROLLARY. *If X is a decomposable chainable continuum, then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

As a consequence of Lemma 3.9 and Theorem 4.25, we have:

4.27. COROLLARY. *If X is a pseudo-linear continuum, then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

4.28. THEOREM. *If X is a pseudo-circular continuum, then* $\mathrm{HS}(X)$ *is not $\frac{1}{2}$-homogeneous.*

Proof. Suppose $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. Since X is pseudo-circular, it has a 2-cell neighborhood in $\mathcal{C}(X)$ [14, Corollary 9]. Hence, T_X has a 2-cell neighborhood in $\mathrm{HS}(X)$. We show that X is locally connected. Suppose it is not. Then there exists $\chi \in \mathrm{HS}(X)$ such that $\mathrm{HS}(X)$ is not locally connected at χ [5, 5.1]. By Lemma 4.3 we deduce that $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(\chi)$. Hence, F_X has a 2-cell neighborhood in $\mathrm{HS}(X)$. This implies by Lemma 3.6 that X is locally connected, a contradiction. Therefore, X is locally connected.

Let X_1, X_2, K_1 and K_2 be subcontinua of X satisfying the pseudo-circularity of X. Since X is locally connected, there exists an arc A in X intersecting both K_1 and K_2. Hence, either $X_1 \subset A$ or $X_2 \subset A$. In either case, it follows that either X_1 or X_2 is an arc with nonempty interior. Then X is an arc by Theorem 4.19, a contradiction, since, clearly, an arc is not pseudo-circular. ∎

As a consequence of Theorem 4.28 and Lemma 3.11, we have:

4.29. COROLLARY. *If X is a decomposable proper circle-like continuum, then* $\mathrm{HS}(X)$ *is not* $\frac{1}{2}$*-homogeneous.*

5. Dendroids. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. A point p in a dendroid X is a *ramification point* if p is the common part of three otherwise disjoint arcs. The set of ramification points of X is denoted by $R(X)$. A point e of a dendroid X is and *end point* of X if e does not separate any arc that contains it. The set of end points of X is denoted by $E(X)$. A *fan* is a dendroid with only one ramification point. Given two points x and x' in a dendroid, xx' denotes the unique arc joining x and x'. A *dendrite* is a locally connected dendroid.

A continuum X is *hereditarily unicoherent at a point* p provided that the intersection of any two subcontinua of X each containing p is connected.

We start with a consequence of Theorem 4.4.

5.1. COROLLARY. *Let X be a decomposable continuum that is hereditarily unicoherent at a point p. If* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous, then X is a dendroid.*

Proof. By Theorem 4.4, X is continuum chainable. Hence, by [9, Corollary, p. 252], X is a dendroid. ∎

A continuum X is *semi-hereditarily unicoherent* provided that, if A and B are subcontinua of X such that $\mathrm{Int}_X(A \setminus B) \neq \emptyset$ and $\mathrm{Int}_X(B \setminus A) \neq \emptyset$, then $A \cap B$ is connected.

5.2. LEMMA. *Let X be a semi-hereditarily unicoherent, decomposable continuum. If* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous, then X is a dendroid.*

Proof. By Theorem 4.4, X is continuum chainable. Hence, since X is semi-hereditarily unicoherent, by [3, Lemma 1.1 and Theorem 1.4], X is a dendroid. ∎

A λ-*dendroid* is a hereditarily decomposable and hereditarily unicoherent continuum.

5.3. COROLLARY. *Let X be a λ-dendroid. If* $\mathrm{HS}(X)$ *is* $\frac{1}{2}$*-homogeneous, then X is a dendroid.*

Let X be a dendroid and let $p \in X$. We say that p is a *strong center* if there exist two open subsets U and V of X such that every arc from U to V contains p.

5.4. COROLLARY. *Let X be a dendroid that contains a strong center. Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. Suppose $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. Let p be a strong center and let U and V be open subsets of X such that every arc from U to V contains p. Let $u \in U$, $v \in V$ and let $\varepsilon > 0$ be such that $\mathcal{V}^d_\varepsilon(u) \subset U$ and $\mathcal{V}^d_\varepsilon(v) \subset V$.

Let $\mu\colon \mathcal{C}(X) \to [0,1]$ be a Whitney map and let $t \in (0,1)$ be such that $\mu^{-1}([t,1]) \subset \mathcal{V}^{\mathcal{H}}_\varepsilon(X)$. Let $A \in \mu^{-1}([t,1])$. Then $\mathcal{H}(A,X) < \varepsilon$. Hence, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. Let $a_1 \in A \cap U$ and let $a_2 \in A \cap V$. Since both X and A are dendroids, the arc from a_1 to a_2 is contained in A. Thus, since p is a strong center, we obtain that $p \in A$. Thus, by Theorem 4.12, we deduce that X is locally connected. Hence, X is a dendrite. In particular, X is contractible. Therefore, by Corollary 4.23, we conclude that X is an arc.

The reverse implication follows from Example 4.1. ∎

5.5. COROLLARY. *Let X be a planar dendroid. Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. The corollary follows from the fact that every planar dendroid contains a strong center [26, Theorem 3.11] and Corollary 5.4. ∎

5.6. LEMMA. *If X is a fan, then X has a strong center.*

Proof. Let v be the ramification point of X. Since X is a fan, there exist two subfans X_1 and X_2 of X such that $X = X_1 \cup X_2$ and $v \in X_1 \cap X_2$. Let $x_1 \in X_1 \setminus X_2$ and let $x_2 \in X_2 \setminus X_1$. Let $\varepsilon > 0$ be such that $\mathcal{V}^d_\varepsilon(x_1) \subset X_1 \setminus X_2$ and $\mathcal{V}^d_\varepsilon(x_2) \subset X_2 \setminus X_1$. Note that if $z_1 \in \mathcal{V}^d_\varepsilon(x_1)$ and $z_2 \in \mathcal{V}^d_\varepsilon(x_2)$, then, by the construction of z_1 and z_2, the only arc joining z_1 and z_2 must contain v. Therefore, v is a strong center of X. ∎

As a consequence of Corollary 5.4 and Lemma 5.6, we have:

5.7. COROLLARY. *If X is a fan, then* $\mathrm{HS}(X)$ *is not $\frac{1}{2}$-homogeneous.*

Recall that the *null comb* is the union of the line segments in the plane from $(0,0)$ to $(1,0)$ and from $(1/n,0)$ to $(1/n,1/n)$ for each positive integer n. The point $q = (0,0)$ is called the *follicle* of the null comb.

5.8. NOTATION. If Z is a continuum, then

$$\Delta(Z) = \{\mathcal{D} \subset Z \mid \mathcal{D} \text{ is an } n\text{-cell for some } n \geq 3, \text{ or a Hilbert cube}\}.$$

If X is a dendroid, then:

$$\mathcal{A}(X) = \{B \in \mathcal{C}(X) \mid B \text{ is a maximal arc such that } i(B) \cap R(X) = \emptyset\},$$
$$\Omega_1(X) = \{D \in \mathcal{C}(X) \mid D \in \mathcal{D}, \text{ for some } \mathcal{D} \in \Delta(\mathcal{C}(X))\},$$
$$\Omega_2(X) = \mathcal{C}(X) \setminus \Omega_1(X),$$
$$\Lambda_1(X) = \{D \in \mathcal{C}(X) \mid \mathcal{C}(X) \text{ is connected im kleinen at } D\},$$
$$\Lambda_2(X) = \mathcal{C}(X) \setminus \Lambda_1(X),$$
$$\mathrm{OR}_L(X) = \{x \in X \mid x \in i(B) \text{ for some } B \in \mathcal{A}(X)\},$$
$$N(X) = \{x \in X \mid x \text{ is the follicle of a null comb in } X\}.$$

5.9. Remark. Let X be a dendroid. Note that if $R(X) \neq \emptyset$ and $B \in \mathcal{A}(X)$, then $B \cap (N(X) \cup R(X)) \subset E(B)$ and $(N(X) \cup R(X)) \cap E(B) \neq \emptyset$. Moreover, $X \setminus (R(X) \cup E(X)) = \mathrm{OR}_L(X) \cup N(X)$.

Recall the following theorem by Sorgenfrey [35, 3.2]:

5.10. Theorem. *Every nondegenerate unicoherent continuum X which is not a triod is irreducible.*

5.11. Remark. Since arcwise connected irreducible continua are arcs, it follows by Theorem 5.10 that if Y is a dendroid which is not an arc, then Y is a triod. Thus, by Lemma 3.14, $Y \in \Omega_1(Y)$. Hence, if X is a dendroid and $Y \in \Omega_2(X)$, then Y is an arc or a point.

5.12. Lemma. *Let X be a dendroid and let $x \in N(X)$. If $A \in \mathcal{C}(\{x\}, X)$, then $A \in \Omega_1(X)$.*

Proof. By Remark 5.11, we may assume that A is an arc or a one-point set. Let K be a null comb in X such that x is the follicle of K. Note that $K \cup A$ is a locally connected continuum [12, Theorem 1, p. 230] and that x is not in the interior (relative to $K \cup A$) of any finite graph in $K \cup A$. Hence, by [4, Theorem 4], we deduce that $\mathcal{C}(\{x\}, K \cup A)$ is a Hilbert cube. Therefore, $A \in \Omega_1(X)$. ∎

5.13. Lemma. *Let X be a dendroid, let $B \in \mathcal{A}(X)$ and let $Z \in \mathcal{C}(B)$. If $Z \cap (N(X) \cup R(X)) = \emptyset$, then $Z \in \Omega_2(X)$.*

Proof. Suppose that $Z \in \Omega_1(X)$. Let $\mathcal{D} \in \Delta(\mathcal{C}(X))$ be such that $Z \in \mathcal{D}$. By Remark 5.9, we assume that $\mathcal{D}$ is small enough so that $D \cap (N(X) \cup R(X)) = \emptyset$ for each $D \in \mathcal{D}$. By [29, (1.49)], $\bigcup \mathcal{D}$ is a subcontinuum of X that contains Z. Moreover, note that $(\bigcup \mathcal{D}) \cap (N(X) \cup R(X)) = \emptyset$. Hence, by Remark 5.9, $\bigcup \mathcal{D} \subset B$. Thus, $\mathcal{D} \subset \mathcal{C}(B)$, which is impossible since $\mathcal{C}(B)$ is a 2-cell. Therefore, $Z \in \Omega_2(X)$. ∎

5.14. Lemma. *Let X be a nonlocally connected dendroid such that $\mathcal{A}(X) \neq \emptyset$. If $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, then $q_X(\Omega_1(X)) \cup \{F_X\} = q_X(\Lambda_1(X))$*

$\cup\{F_X\}$, $q_X(\Omega_2(X))\setminus\{F_X\} = q_X(\Lambda_2(X))\setminus\{F_X\}$ *and these are the two orbits of* HS(X).

Proof. By [29, (1.208.2)], there exists $Z \in \Lambda_2(X) \setminus F_1(X)$. Then we have $q_X(\Lambda_2(X)) \setminus \{F_X\} \neq \emptyset$. Since $X \in \Lambda_1(X)$ [29, (1.136)], it follows that $q_X(\Lambda_1(X)) \neq \emptyset$. Since HS$(X)$ is connected im kleinen at χ if and only if $\chi \in q_X(\Lambda_1(X)) \cup \{F_X\}$, we obtain that the two orbits of HS(X) are $q_X(\Lambda_1(X)) \cup \{F_X\}$ and $q_X(\Lambda_2(X)) \setminus \{F_X\}$.

Since X is not locally connected, by Remark 5.11, we have $X \in \Omega_1(X)$. Then $q_X(\Omega_1(X)) \neq \emptyset$. Let $B \in \mathcal{A}(X)$ and let L be an arc such that $L \subset B \setminus E(B) \subset B \setminus (N(X) \cup R(X))$ (Remark 5.9). Thus, by Lemma 5.13, $L \in \Omega_2(X)$. This implies that $q_X(\Omega_2(X)) \setminus \{F_X\} \neq \emptyset$.

Observe that the elements of $q_X(\Omega_1(X)) \setminus \{F_X\}$ belong to elements of $\Delta(\mathrm{HS}(X))$, and that the elements of $q_X(\Omega_2(X)) \setminus \{F_X\}$ do not have this property. Moreover, by Lemma 4.3, $\mathcal{O}_{\mathrm{HS}(X)}(T_X) = \mathcal{O}_{\mathrm{HS}(X)}(F_X)$. Therefore, since HS$(X)$ is $\frac{1}{2}$-homogeneous, the two orbits of HS(X) are $q_X(\Omega_1(X)) \cup \{F_X\}$ and $q_X(\Omega_2(X)) \setminus \{F_X\}$. Now the theorem follows from the first paragraph. ∎

5.15. Lemma. *Let X be a dendroid such that* HS(X) *is* $\frac{1}{2}$*-homogeneous and $\mathcal{A}(X) \neq \emptyset$. If $A \in \mathcal{C}(X)$ and $A \cap R(X) \neq \emptyset$, then $A \in \Lambda_1(X)$.*

Proof. Let $r \in A \cap R(X)$. Then there exist three arcs A_1, A_2, A_3 in X such that $A_i \cap A_j = \{r\}$ whenever $i \neq j$. We prove first that X is connected im kleinen at r. Suppose that it is not, and let W be a neighborhood of r such that no connected neighborhood of r is contained in W. Without loss of generality, we assume that $A_j \setminus W \neq \emptyset$ for each $j \in \{1, 2, 3\}$. By Lemma 3.3, there exists a nondegenerate subcontinuum Y of X such that $r \in Y \subset W$ and $\mathcal{C}(X)$ is not connected im kleinen at Y. Now, note that $(\bigcup_{j=1}^{3} A_j) \setminus Y$ has at least three components. Thus, $Y \cup \bigcup_{j=1}^{3} A_j$ is a triod with core Y. By Lemma 3.14, this implies that $Y \in \Omega_1(X)$. Hence, as a consequence of Lemma 5.14, $Y \in \Lambda_1(X)$, a contradiction. Therefore, X is connected im kleinen at r. It follows from Lemma 3.2 that $A \in \Lambda_1(X)$. ∎

5.16. Lemma. *Let X be a dendroid and let $B \in \mathcal{A}(X)$. Suppose there exists $e \in E(X) \cap E(B)$, let $L \in \mathcal{C}(\{e\}, B) \setminus \{B, \{e\}\}$, and let $M \in \mathcal{C}(B) \setminus \mathcal{F}_1(X)$ be such that $M \cap E(B) = \emptyset$. Then $q_X(L)$ does not belong to the manifold interior of any* 2*-cell in* HS(X)*, but $q_X(M)$ does.*

Proof. Suppose that $q_X(L) \in i(\mathcal{D})$ for some 2-cell $\mathcal{D}$ in HS(X). We may assume that $F_X \notin \mathcal{D}$. By Remark 5.9, we assume that $\mathcal{D}$ is small enough so that $D \cap (N(X) \cup R(X)) = \emptyset$ for each $D \in q_X^{-1}(\mathcal{D})$. By [29, (1.49)], we know that $\bigcup q_X^{-1}(\mathcal{D})$ is a subcontinuum of X that contains L. Moreover, note that $(\bigcup q_X^{-1}(\mathcal{D})) \cap (N(X) \cup R(X)) = \emptyset$. Hence, by Remark 5.9, $\bigcup q_X^{-1}(\mathcal{D}) \subset B$.

Thus, $q_X^{-1}(\mathcal{D}) \subset \mathcal{C}(B)$. Furthermore, $L \in q_X^{-1}(i(\mathcal{D})) = i(q_X^{-1}(\mathcal{D})) \subset i(\mathcal{C}(B))$, a contradiction since $e \in L \cap E(B)$. Therefore, $q_X(L) \notin i(\mathcal{D})$.

Observe that $M \in i(\mathcal{C}(B))$. Then $q_X(M)$ belongs to the manifold interior of the 2-cell $\mathrm{HS}(B)$. ∎

5.17. Lemma. *Let X be a dendroid, let B be an arc in X such that $i(B) \cap R(X) = \emptyset$ and let $e \in E(X) \cap E(B)$. If X is not connected im kleinen at any point of $i(B) \cup \{e\}$, then there exist $L, M \in \mathcal{C}(B) \setminus \mathcal{F}_1(X)$ such that $e \in L$, $e \notin M$ and $\mathcal{C}(X)$ is connected im kleinen at neither L nor M.*

Proof. Let e and b be the end points of B. Let $w \in i(B)$ and let $\delta > 0$ be such that $\delta < \min\{d(e, b), d(w, E(B))\}$. By Lemma 3.3, there exist nondegenerate subcontinua L and M of X such that:

(1) $e \in L \subset \mathcal{V}^d_\delta(e)$,
(2) $w \in M \subset \mathcal{V}^d_\delta(w)$, and
(3) $\mathcal{C}(X)$ is connected im kleinen at neither L nor M.

It follows from the choice of δ that $b \notin L \cup M$. Hence, since X is a dendroid, we have $L, M \in \mathcal{C}(B) \setminus \mathcal{F}_1(X)$. Now, since $\delta < d(w, e)$, by (2), we conclude that $e \notin M$. ∎

5.18. Lemma. *Let X be a dendroid and let $B \in \mathcal{A}(X)$. If there exists $e \in E(X) \cap E(B)$ and $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, then X is an arc.*

Proof. Suppose first that X is not connected im kleinen at any point of $i(B) \cup \{e\}$. Let $b \in E(B) \setminus E(X)$ and let $L, M \in \mathcal{C}(B) \setminus \mathcal{F}_1(X)$ be such that $e \in L$, $e \notin M$, and $\mathcal{C}(X)$ is connected im kleinen at neither L nor M (Lemma 5.17). Also assume that $b \notin L \cup M$. By Lemma 5.13, we know that $L, M \in \Omega_2(X)$. Moreover, Lemma 5.16 implies that $\mathcal{O}_{\mathrm{HS}(X)}(q_X(L)) \neq \mathcal{O}_{\mathrm{HS}(X)}(q_X(M))$. Since $\mathrm{HS}(X)$ is locally connected at F_X by Lemma 3.5, we obtain $\mathcal{O}_{\mathrm{HS}(X)}(q_X(L)) \neq \mathcal{O}_{\mathrm{HS}(X)}(F_X) \neq \mathcal{O}_{\mathrm{HS}(X)}(q_X(M))$. This contradicts the $\frac{1}{2}$-homogeneity of $\mathrm{HS}(X)$. Hence, X is connected im kleinen at some point of $i(B) \cup \{e\}$. Therefore, X contains a free arc and the result follows from Theorem 4.19. ∎

5.19. Lemma. *Let X be a dendroid such that $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. If X is not locally connected and $\mathcal{A}(X) \neq \emptyset$, then $q_X(\Omega_2(X)) \setminus \{F_X\} = \{\chi \in \mathrm{HS}(X) \setminus \{F_X\} \mid q_X^{-1}(\chi) \subset B \setminus (N(X) \cup R(X))$ for some $B \in \mathcal{A}(X)\}$.*

Proof. Let $L \in \mathcal{C}(X)$ be such that $q_X(L) \in q_X(\Omega_2(X)) \setminus \{F_X\}$. Then $L \in \Omega_2(X) \setminus \mathcal{F}_1(X)$. Hence, by Lemmas 5.12, 5.15 and 5.14, we deduce that $L \cap (N(X) \cup R(X)) = \emptyset$. This implies that L is an arc (L is nondegenerate). By [2, (3.6)], there exists a maximal arc Y in X such that $L \subset Y$. Hence, since $L \cap (N(X) \cup R(X)) = \emptyset$, there exists $B \in \mathcal{A}(X)$ such that $L \subset B$. This proves that $q_X(\Omega_2(X)) \setminus \{F_X\} \subset \{\chi \in \mathrm{HS}(X) \setminus \{F_X\} \mid q_X^{-1}(\chi) \subset$

$B \setminus (N(X) \cup R(X))$ for some $B \in \mathcal{A}(X)\}$. The other inclusion follows from Lemma 5.13. ∎

5.20. LEMMA. *Let X be a dendroid such that* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous. If X is not locally connected, $\mathcal{A}(X) \neq \emptyset$ and $\chi_1 \in q_X(\Omega_2(X)) \setminus \{F_X\}$, then there exists $B \in \mathcal{A}(X)$ such that $\chi_1 \in i(\mathrm{HS}(B))$. Furthermore, if $\mathcal{K}$ is the arc component of $q_X(\Omega_2(X)) \setminus \{F_X\}$ that contains χ_1, then $\mathcal{K} = i(\mathrm{HS}(B))$.*

Proof. By Lemma 5.19, there exists $B \in \mathcal{A}(X)$ such that $q_X^{-1}(\chi_1) \subset B \setminus (N(X) \cup R(X))$. By Lemma 5.18, we have $E(B) \subset N(X) \cup R(X)$. Thus, $q_X^{-1}(\chi_1) \subset i(B)$. Then $\chi_1 \in i(\mathrm{HS}(B))$. Since B is an arc, we see that $i(\mathrm{HS}(B))$ is a 2-cell (Example 4.1), in particular, it is arcwise connected. Hence, by Lemma 5.19, we obtain $i(\mathrm{HS}(B)) \subset \mathcal{K}$. Next, suppose there exists $\chi \in \mathcal{K} \setminus i(\mathrm{HS}(B))$. Then there exists an arc $\mathcal{L} \subset \mathcal{K}$ whose end points are χ_1 and χ. By [29, (1.49)], $\bigcup q_X^{-1}(\mathcal{L})$ is a subdendroid of X that contains both $q_X^{-1}(\chi_1)$ and $q_X^{-1}(\chi)$. Since $q_X^{-1}(\chi_1) \subset i(B)$ and $q_X^{-1}(\chi) \setminus i(B) \neq \emptyset$, it follows that $E(B) \cap \bigcup q_X^{-1}(\mathcal{L}) \neq \emptyset$. Let $\chi_0 \in \mathcal{L}$ be such that $\emptyset \neq q_X^{-1}(\chi_0) \cap E(B) \subset q_X^{-1}(\chi_0) \cap (N(X) \cup R(X))$. Thus, by Lemma 5.19, we have $\chi_0 \in \mathrm{HS}(X) \setminus (q_X(\Omega_2(X)) \setminus \{F_X\}) \subset \mathrm{HS}(X) \setminus \mathcal{L}$. This contradiction proves that $i(\mathrm{HS}(B)) = \mathcal{K}$. ∎

5.21. THEOREM. *Let X be a dendroid and assume there exists an arc L such that $i(L) \cap R(X) = \emptyset$. Then* $\mathrm{HS}(X)$ *is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. Suppose $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous. By [2, (3.6)], there exists a maximal arc Y in X such that $L \subset Y$. Hence, there exists $B \in \mathcal{A}(X)$ such that $L \subset B$.

We prove that X is locally connected. Suppose that X is not locally connected. Then $R(X) \neq \emptyset$. By Remark 5.9, we have $B \cap (N(X) \cup R(X)) \neq \emptyset$. It follows from Lemmas 5.19 and 5.14 that $q_X(B) \in q_X(\Omega_1(X)) \cup \{F_X\}$. Since $X \in \Omega_1(X)$ (Remark 5.11), both T_X and $q_X(B)$ belong to the orbit $q_X(\Omega_1(X)) \cup \{F_X\}$ (Lemma 5.14). Note that $q_X(B) \in \mathrm{Cl}_{\mathrm{HS}(X)}(i(\mathrm{HS}(B)))$. Thus, by Lemma 5.20, there exists an element $B_1 \in \mathcal{A}(X)$ such that $T_X \in \mathrm{Cl}_{\mathrm{HS}(X)}(i(\mathrm{HS}(B_1))) \subset \mathrm{HS}(B_1)$. This implies that $X = B_1$, a contradiction to the assumption that X is not locally connected. Therefore, X is locally connected; i.e., X is a dendrite. Since dendrites are planar [31, 10.37], the result now follows from Corollary 5.5.

The reverse implication is a consequence of Example 4.1. ∎

Throughout this section we have given partial answers to Question 7.6 below.

6. A theorem for $\mathcal{C}(X)$. We present a result for $\mathcal{C}(X)$ analogous to Theorem 5.21. The proof of the following lemma is similar to the one given for Lemma 5.16.

6.1. Lemma. *Let X be a dendroid and let $B \in \mathcal{A}(X)$. Let $y \in i(B)$ and let $M \in \mathcal{C}(B) \setminus \mathcal{F}_1(B)$ be such that $M \cap E(B) = \emptyset$. Then $\{y\}$ does not belong to the manifold interior of any 2-cell in $\mathcal{C}(X)$, but M does.*

6.2. Theorem. *Let X be a dendroid and suppose there exists an arc L in X such that $i(L) \cap R(X) = \emptyset$. Then $\mathcal{C}(X)$ is $\frac{1}{2}$-homogeneous if and only if X is an arc.*

Proof. If X is an arc, then the result is clear by [32, Proposition 1.1].

Suppose $\mathcal{C}(X)$ is $\frac{1}{2}$-homogeneous. By [2, (3.6)], there exists a maximal arc Y in X such that $L \subset Y$. Hence, there exists $B \in \mathcal{A}(X)$ such that $L \subset B$.

Suppose X is not connected im kleinen at some point $y \in i(B)$. By Lemma 3.3, there exists $M \in \mathcal{C}(B) \setminus \mathcal{F}_1(B)$ such that $y \in M$, $M \cap E(B) = \emptyset$ and $\mathcal{C}(X)$ is not connected im kleinen at M.

Note that, by Lemma 6.1, $\mathcal{O}_{\mathcal{C}(X)}(\{y\}) \neq \mathcal{O}_{\mathcal{C}(X)}(M)$. Also note that, by Remark 5.9, $B \cap (N(X) \cup R(X)) \subset E(B)$. Hence, by Lemma 5.13, we see that both $\{y\}$ and M belong to $\Omega_2(X)$.

Since $X \in \Omega_1(X)$ (Remark 5.11), we obtain

$$\mathcal{O}_{\mathcal{C}(X)}(\{y\}) \neq \mathcal{O}_{\mathcal{C}(X)}(X) \neq \mathcal{O}_{\mathcal{C}(X)}(M).$$

This contradicts the $\frac{1}{2}$-homogeneity of $\mathcal{C}(X)$. Thus, X is connected im kleinen at some point of $i(B)$. Hence, X contains a free arc. The theorem now follows from [23, 3.14]. ∎

7. Questions. The following questions seem natural and interesting:

7.1. Question. Is there a decomposable, nonlocally connected continuum X such that $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous?

More generally:

7.2. Question. Is the arc the only decomposable continuum with $\frac{1}{2}$-homogeneous hyperspace suspension?

7.3. Remark. Note that, by [23, 6.13], if X is a decomposable homogeneous continuum and $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, then $\mathrm{HS}(X)$ is infinite-dimensional.

7.4. Question. Is there an infinite-dimensional continuum with $\frac{1}{2}$-homogeneous hyperspace suspension?

7.5. Question. Are solenoids the only indecomposable continua with $\frac{1}{2}$-homogeneous hyperspace suspension?

7.6. Question. If X is a dendroid such that $\mathrm{HS}(X)$ is $\frac{1}{2}$-homogeneous, then is X an arc?

REFERENCES

[1] C. E. Burgess, *Chainable continua and indecomposability*, Pacific J. Math. 9 (1959), 653–659.

[2] J. J. Charatonik, *On acyclic curves—a survey of results and problems*, Bol. Soc. Mat. Mexicana (3) 1 (1995), 1–39.

[3] W. D. Collins and E. J. Vought, *Continuum chainability and monotone decompositions in certain classes of unicoherent continua*, Houston J. Math. 10 (1984), 457–465.

[4] C. Eberhart, *Intervals of continua which are Hilbert cubes*, Proc. Amer. Math. Soc. 68 (1978), 220–224.

[5] R. Escobedo, M. de J. López and S. Macías, *On the hyperspace suspension of a continuum*, Topology Appl. 138 (2004), 109–124.

[6] J. T. Goodykoontz, *Connectedness im kleinen and local connectedness in* 2^X *and* $\mathcal{C}(X)$, Pacific J. Math. 53 (1974), 387–397.

[7] J. T. Goodykoontz, *More on connectedness im kleinen and local connectedness in* $\mathcal{C}(X)$, Proc. Amer. Math. Soc. 65 (1977), 357–364.

[8] C. L. Hagopian, *Arcwise connectivity of continuum-chainable plane continua*, Houston J. Math. 8 (1982), 69–74.

[9] C. L. Hagopian and L. E. Rogers, *Arcwise connectivity and continuum chainability*, Houston J. Math. 7 (1981), 249–259.

[10] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, NJ, 1948.

[11] K. Kuratowski, *Topology*, Vol. I, Academic Press and PWN, New York and Warszawa, 1966.

[12] K. Kuratowski, *Topology*, Vol. II, Academic Press and PWN, New York and Warszawa, 1968.

[13] M. de J. López, P. Pellicer-Covarrubias, A. Santiago-Santos, $\frac{1}{2}$*-Homogeneous suspensions*, Topology Appl. 157 (2010), 482–493.

[14] S. López, *Hyperspaces locally 2-cell at the top*, in: Continuum Theory: Proceedings of the Special Session in Honor of Professor Sam B. Nadler, Jr.'s 60th Birthday, A. Illanes et al. (eds.), Lecture Notes in Pure Appl. Math. 230, Dekker, New York, 2002, 9–31.

[15] S. Macías, *On symmetric products of continua*, Topology Appl. 92 (1999), 173–182.

[16] S. Macías, *On arcwise accessibility in hyperspaces*, Topology Proc. 26 (2001-2002), 247–254.

[17] S. Macías, *On the* n*-fold hyperspace suspension of a continuum*, Topology Appl. 138 (2004), 125–138.

[18] S. Macías, *Topics on Continua*, Pure Appl. Math. Ser. 275, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[19] S. Macías, *On the* n*-fold hyperspace suspension of continua, II*, Glasnik Mat. 41 (61) (2006), 335–343.

[20] S. Macías, *On* n*-fold hyperspaces of continua, II*, Topology Proc. 38 (2011), 137–147.

[21] S. Macías and S. B. Nadler, Jr., n*-fold hyperspaces cones and products*, Topology Proc. 26 (2001-2002), 255–270.

[22] S. Macías and S. B. Nadler, Jr., *Various types of local connectedness in* n*-fold hyperspaces*, Topology Appl. 154 (2007), 39–53.

[23] S. Macías and S. B. Nadler, Jr., $\frac{1}{2}$*-homogeneous hyperspaces*, preprint.

[24] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.) 158 (1979), 91 pp.

[25] J. van Mill, *Infinite-Dimensional Topology*, North-Holland, Amsterdam, 1989.
[26] P. Minc, *Bottlenecks in dendroids*, Topology Appl. 129 (2003), 187–209.
[27] J. R. Moon, K. Hur and C. J. Rhee, *Connectedness im kleinen and components in* $C(X)$, Bull. Korean Math. Soc. 34 (1997), 225–231.
[28] S. B. Nadler, Jr., *Continua whose cone and hyperspace are homeomorphic*, Trans. Amer. Math. Soc. 230 (1977), 321–345.
[29] S. B. Nadler, Jr., *Hyperspaces of Sets: a Text with Research Questions*, Monogr. Textbooks Pure Appl. Math. 49, Dekker, New York, 1978. Reprinted in: Aportaciones Matemáticas de la Sociedad Matemática Mexicana, Serie Textos 33, 2006.
[30] S. B. Nadler, Jr., *A fixed point theorem for hyperspace suspensions*, Houston J. Math. 5 (1979), 125–132.
[31] S. B. Nadler, Jr., *Continuum Theory: an Introduction*, Monogr. Textbooks Pure Appl. Math. 158, Dekker, New York, 1992.
[32] S. B. Nadler, Jr. and P. Pellicer-Covarrubias, *Hyperspaces with exactly two orbits*, Glasnik Mat. 41 (61) (2006), 141–157.
[33] L. G. Oversteegen, *Every contractible fan is locally connected at its vertex*, Trans. Amer. Math. Soc. 260 (1980), 379–402.
[34] A. Santiago-Santos, *Degree of homogeneity on suspensions*, Topology Appl. 158 (2011), 2125–2139.
[35] R. H. Sorgenfrey, *Concerning triodic continua*, Amer. J. Math. 66 (1944), 439–460.
[36] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. 28, Amer. Math. Soc., Providence, RI, 1942.

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