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A CATEGORICAL QUOTIENT IN THE CATEGORY OF DENSE CONSTRUCTIBLE SUBSETS

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Abstract. A. A'Campo-Neuen and J. Hausen gave an example of an algebraic torus action on an open subset of the affine four space that admits no quotient in the category of algebraic varieties. We show that this example admits a quotient in the category of dense constructible subsets and thereby answer a question of A. Białynicki-Birula.

The purpose of this note is to answer a question by A. Białynicki-Birula on categorical quotients in the category of dense constructible subsets (dc-subsets) introduced in [4]. The objects of this category are pairs $X \subseteq X'$, where X' is a complex algebraic variety and X is a dense constructible subset of X'. A morphism from $X \subseteq X'$ to $Y \subseteq Y'$ is a map $X \to Y$ that extends to a morphism $X'_1 \to Y'$ for some open neighbourhood $X'_1 \subseteq X'$ of X. Consider the $\mathbb{C}^*$-variety

$$X := \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \times \mathbb{C}^2, \qquad tx := (tx_1, tx_2, x_3, t^{-1}x_4).$$

A categorical quotient for this $\mathbb{C}^*$-variety is a $\mathbb{C}^*$-invariant morphism $\pi\colon X \to Y$, i.e. a morphism constant along the $\mathbb{C}^*$-orbits, such that any other $\mathbb{C}^*$-invariant morphism $\varphi\colon X \to Z$ has a unique factorization $\varphi = \psi \circ \pi$ with a morphism $\psi\colon Y \to Z$. This concept depends strongly on the category one works with. For example, in the category of complex algebraic varieties, there is no such quotient for X (see [2, Prop. 5.1(ii)]), whereas in the category of toric varieties, X admits a quotient, namely the toric morphism

$$\pi\colon X \to \mathbb{C}^3, \qquad (x_1, x_2, x_3, x_4) \mapsto (x_1x_4, x_2x_4, x_3)$$

(use [1, Thm. 2.3]). Now, set $Y := \pi(X)$. Then $Y \subseteq \mathbb{C}^3$ is a proper dc-subset, and π defines a morphism of dc-subsets from $X \subseteq X$ to $Y \subseteq \mathbb{C}^3$. A. Białynicki-Birula asks [4, p. 53] whether or not this is a categorical quotient in the category of dc-subsets.

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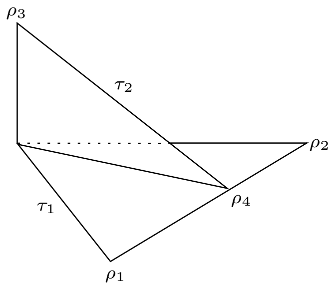
THEOREM. *The map $\pi\colon X \to Y$ is a categorical quotient for the $\mathbb{C}^*$-action on X in the category of dc-subsets.*

Proof. The most convenient way is to treat the problem in the setting of the toric varieties; we work in the notation of [5]. Note that X arises from the fan Δ in $\mathbb{Z}^4$ that has $\sigma_1 := \mathrm{cone}(e_1, e_2)$ and $\sigma_2 := \mathrm{cone}(e_3, e_4)$ as its maximal cones. Let $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ denote the homomorphism defined by

$$P(e_1) := e_1, \quad P(e_2) := e_2, \quad P(e_3) := e_3, \quad P(e_4) := e_1 + e_2.$$

Moreover, set $\tau_i := P(\sigma_i)$, $\rho_i := \mathrm{cone}(e_i)$, where $i = 1, 2, 3$, and $\rho_4 := \mathrm{cone}(e_1 + e_2)$. Then these cones are located in $\mathbb{R}^3$ as indicated in the figure below.



Note that $\pi\colon X \to \mathbb{C}^3$ is the toric morphism defined by the homomorphism $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$, and that the image $Y = \pi(X)$ is given by

$$\begin{aligned} Y &= \mathbb{C}^3 \setminus (\{0\} \times \mathbb{C}^* \times \{0\} \cup \mathbb{C}^* \times \{0\} \times \{0\}) \\ &= \mathbb{T}y_0 \cup \bigcup_{i=1}^{3} \mathbb{T}y_{\rho_i} \cup \mathbb{T}y_{\tau_1} \cup \mathbb{T}y_\delta \end{aligned}$$

where y_0, y_{ρ_i}, etc. in $\mathbb{C}^3$ denote the distinguished points corresponding to the faces 0, ρ_i etc. of the cone $\delta := \mathrm{cone}(e_1, e_2, e_3)$ describing the toric variety $\mathbb{C}^3$.

Moreover, let $\widetilde{Y}$ denote the toric prevariety obtained by gluing the affine toric varieties $\widetilde{Y}_{\tau_i} := \mathrm{Spec}(\mathbb{C}[\tau_i^\vee \cap \mathbb{Z}^3])$, where $i = 1, 2$, along the big torus $\mathbb{T} := (\mathbb{C}^*)^3 \subseteq \widetilde{Y}_{\tau_i}$. Then $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ defines a $\mathbb{C}^*$-invariant toric morphism $\widetilde{\pi}\colon X \to \widetilde{Y}$. Finally, the lattice homomorphism $\mathrm{id}\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ gives us a toric morphism $\kappa\colon \widetilde{Y} \to \mathbb{C}^3$ with $\kappa(\widetilde{Y}) = Y$ and $\pi = \kappa \circ \widetilde{\pi}$.

In order to verify the universal property of the morphism $\pi\colon X \to Y$, let $Z \subseteq Z'$ be a dc-subset and $\varphi\colon X \to Z$ a $\mathbb{C}^*$-invariant morphism. According to [2, Prop. 5.1(i)], the toric morphism $\widetilde{\pi}\colon X \to \widetilde{Y}$ is a categorical quotient for the $\mathbb{C}^*$-action on X in the category of complex prevarieties. Consequently, we have $\varphi = \widetilde{\psi} \circ \widetilde{\pi}$ with a morphism $\widetilde{\psi}\colon \widetilde{Y} \to Z'$. Our task is to show that there is a morphism $\psi\colon Y \to Z$ of dc-subsets making the following diagram commutative:

$$
\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[d, "\widetilde{\pi}"'] \arrow[dr, "\pi"] & & Z & \subseteq & Z' \\
\widetilde{Y} \arrow[r, "\kappa"'] \arrow[rrrru, bend right, "\widetilde{\psi}"'] & Y \arrow[ur, "\psi"'] & & &
\end{tikzcd}
$$

In a first step, we show that such a ψ exists as a set-theoretical map. Since $\widetilde{\psi}(\widetilde{Y}) = \varphi(X) \subseteq Z$, this amounts to verifying that $\widetilde{\psi}$ is constant along the fibres of κ. Let $\widetilde{y}_0, \widetilde{y}_{\rho_1}, \widetilde{y}_{\rho_2}, \widetilde{y}_{\tau_1} \in \widetilde{Y}_{\tau_1}$ and $\widetilde{y}_0, \widetilde{y}_{\rho_3}, \widetilde{y}_{\rho_4}, \widetilde{y}_{\tau_2} \in \widetilde{Y}_{\tau_2}$ denote the distinguished points corresponding to the cones 0, ρ_i etc. Further, denote by $\mathbb{T}_{y_{\tau_i}}$ the isotropy groups of y_{τ_i}. Then, using e.g. [3, Prop. 3.5], we find that the κ-fibres are

$$
\begin{gathered}
\kappa^{-1}(ty_0) = t\widetilde{y}_0, \qquad \kappa^{-1}(ty_{\rho_i}) = t\widetilde{y}_{\rho_i}, \qquad i = 1, 2, 3, \\
\kappa^{-1}(ty_{\tau_1}) = t\mathbb{T}_{y_{\tau_1}}\widetilde{y}_{\tau_1} \cup t\mathbb{T}_{y_{\tau_1}}\widetilde{y}_{\rho_4}, \qquad \kappa^{-1}(y_\delta) = \mathbb{T}\widetilde{y}_{\tau_2},
\end{gathered}
$$

where $t \in \mathbb{T}$. Thus, only for $\kappa^{-1}(ty_{\tau_1})$ and $\kappa^{-1}(y_\delta)$ is there something to show. Take $v \in \mathbb{Z}^3$ from the relative interior ρ_4 and let $\lambda_v \colon \mathbb{C}^* \to \mathbb{T}$ denote the corresponding one-parameter subgroup. Then, in $\widetilde{Y}$, we have

$$
\overline{\lambda_v(\mathbb{C}^*)t\widetilde{y}_0} \setminus \lambda_v(\mathbb{C}^*)t\widetilde{y}_0 = \{t\widetilde{y}_{\tau_1}, t\widetilde{y}_{\rho_4}\}.
$$

Note that $t\widetilde{y}_{\tau_1} \in \widetilde{Y}_{\tau_1}$ and $t\widetilde{y}_{\rho_4} \in \widetilde{Y}_{\tau_2}$. Since Z' is separated, $\widetilde{\psi}(t\widetilde{y}_{\tau_1}) = \widetilde{\psi}(t\widetilde{y}_{\rho_4})$. In particular, for $t \in \mathbb{T}_{y_{\tau_1}} = \mathbb{T}_{\widetilde{y}_{\tau_1}}$ we obtain the equation

$$
\widetilde{\psi}(t\widetilde{y}_{\rho_4}) = \widetilde{\psi}(t\widetilde{y}_{\tau_1}) = \widetilde{\psi}(\widetilde{y}_{\tau_1}).
$$

Consequently, for every $t \in \mathbb{T}$, the set $\widetilde{\psi}(\kappa^{-1}(ty_{\tau_1}))$ consists of a single point. Now we treat $\kappa^{-1}(ty_\delta) = \mathbb{T}\widetilde{y}_{\tau_2}$. First note that $\mathbb{T} = \mathbb{T}_{\widetilde{y}_{\tau_1}}\mathbb{T}_{\widetilde{y}_{\tau_2}}$ and hence any $t \in \mathbb{T}$ can be written as $t = t_{\tau_1}t_{\tau_2}$ with $t_{\tau_i} \in \mathbb{T}_{\widetilde{y}_{\tau_i}}$. Consider again the one-parameter subgroup $\lambda_v \colon \mathbb{C}^* \to \mathbb{T}$ corresponding to a lattice vector $v \in \mathbb{Z}^3$ in the relative interior of ρ_4. Then

$$
\begin{aligned}
\widetilde{\psi}(t\widetilde{y}_{\tau_2}) &= \lim_{s \to 0} \widetilde{\psi}(t_{\tau_1}t_{\tau_2}\lambda_v(s)\widetilde{y}_{\rho_4}) = \lim_{s \to 0} \widetilde{\psi}(t_{\tau_1}t_{\tau_2}\lambda_v(s)\widetilde{y}_{\tau_1}) \\
&= \lim_{s \to 0} \widetilde{\psi}(t_{\tau_2}\lambda_v(s)\widetilde{y}_{\tau_1}) = \widetilde{\psi}(t_{\tau_2}\widetilde{y}_{\tau_2}) = \widetilde{\psi}(\widetilde{y}_{\tau_2}).
\end{aligned}
$$

We now show that $\psi \colon Y \to Z$ is a morphism of dc-subsets. Consider the open affine toric subvarieties $Y_{\rho_3}, Y_{\tau_1} \subseteq \mathbb{C}^3$ and $\widetilde{Y}_{\rho_3} \subseteq \widetilde{Y}$ defined by the cones ρ_3 and τ_1. Then $Y_{\tau_1}, Y_{\rho_3} \subseteq Y$ and, for $U := \widetilde{Y}_{\tau_1} \cup \widetilde{Y}_{\rho_3}$ and $V := Y_{\tau_1} \cup Y_{\rho_3}$, the restriction $\kappa_{|U} \colon U \to V$ is an isomorphism of varieties. Thus, $\psi = \widetilde{\psi} \circ \kappa^{-1}$ on V, and hence ψ is a rational map.

We have to show that ψ is defined near the point y_δ. For this we choose an affine open neighbourhood $W_0 \subseteq Z'$ of $\psi(y_\delta)$ in Z' and set $U_0 := \widetilde{\psi}^{-1}(W_0)$

and $V_0 := \psi^{-1}(W_0)$. Then we have an open subset

$$V_0 \cap V = \kappa(U_0 \cap U) \subseteq \mathbb{C}^3,$$

on which ψ is a morphism. After realizing W_0 as a closed subset of some $\mathbb{K}^r$, we obtain functions $g_1, \dots, g_r \in \mathcal{O}(V_0 \cap V)$ such that the morphism ψ is given on $V_0 \cap V$ as

$$\psi_{|V_0 \cap V} \colon V_0 \cap V \to W_0, \qquad y \mapsto (g_1(y), \dots, g_r(y)).$$

Suppose for the moment that each g_i is defined at y_δ. Then there are an open set $V' \subseteq \mathbb{C}^3$ containing $V_0 \cap V$ and the point y_δ such that $g_1, \dots, g_r$ define a morphism $\psi' \colon V' \to W_0$ extending $\psi_{|V_0 \cap V}$. This morphism fits into the diagram

$$\begin{tikzcd}
U_0 \cap U \arrow[d, "\kappa"'] & \subseteq & \kappa^{-1}(V') \arrow[r, "\widetilde{\psi}"] \arrow[d, "\kappa"'] & W_0 \\
V_0 \cap V & \subseteq & V' \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

Since $\widetilde{\psi}$ and $\psi' \circ \kappa$ coincide on the dense open subset $U_0 \cap U$, they coincide on $\kappa^{-1}(V')$. Thus, by surjectivity of κ, we obtain $\psi' = \psi_{|V'}$ and see that ψ is a morphism on the neighbourhood $V' \subseteq \mathbb{C}^3$ of $y_\delta \in Y$. Hence, ψ is a morphism on the open set $V \cup V' \subseteq \mathbb{C}^3$. This shows that $\psi \colon Y \to Z$ is a mophism in the category of dc-subsets.

Thus, our task is to show that every g_i is defined at y_δ. For this, we regard g_i as a rational function on the normal ambient variety $\mathbb{C}^3$ of Y and show that $\operatorname{div}(g_i)$ is nonnegative at y_δ. For this in turn, it suffices to show that any prime divisor on $\mathbb{C}^3$ containing y_δ meets the open set $V \cap V_0$.

Let D be a prime divisor on $\mathbb{C}^3$ passing through y_δ. We show that D meets the open set $V \cap V_0$. Since $\mathbb{C}^3 \setminus V$ is of codimension 2 in $\mathbb{C}^3$, we obtain $D \cap V \neq \emptyset$. First consider the case that $\kappa^{-1}(D)$ contains the prime divisor $\overline{\mathbb{T}\widetilde{y}_{\rho_4}} = \mathbb{T}\widetilde{y}_{\rho_4} \cup \mathbb{T}\widetilde{y}_{\tau_2}$. Then $\kappa^{-1}(D) \cap U_0 \neq \emptyset$, hence

$$D \cap V_0 \neq \emptyset \;\Rightarrow\; D \cap V \cap V_0 \;\neq\; \emptyset.$$

Next suppose that $\kappa^{-1}(D)$ does not contain $\overline{\mathbb{T}\widetilde{y}_{\rho_4}} = \widetilde{Y} \setminus U$. Then any component of $\kappa^{-1}(D)$ meets U. Since $\mathbb{C}^3$ is factorial, D is principal, and thus $\kappa^{-1}(D)$ is of pure codimension one in $\widetilde{Y}$. Consequently, $\kappa^{-1}(D)$ equals the closure of $\kappa^{-1}(D) \cap U$ in $\widetilde{Y}$. Since $\kappa^{-1}(D)$ intersects $\kappa^{-1}(y_\delta)$, we can conclude

$$\kappa^{-1}(D) \cap U \cap U_0 \neq \emptyset \;\Rightarrow\; D \cap V \cap V_0 \neq \emptyset. \;\blacksquare$$

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