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A CRITERION FOR RINGS WHICH ARE LOCALLY VALUATION RINGS

BY

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Abstract. Using the notion of cyclically pure injective modules, a characterization of rings which are locally valuation rings is established. As applications, new characterizations of Prüfer domains and pure semisimple rings are provided. Namely, we show that a domain R is Prüfer if and only if two of the three classes of pure injective, cyclically pure injective and RD-injective modules are equal. Also, we prove that a commutative ring R is pure semisimple if and only if every R-module is cyclically pure injective.

1. Introduction. Throughout this paper, R denotes a commutative ring with identity, and all modules are assumed to be left unitary. The notion of pure injective module has a substantial role in commutative algebra and model theory. Among various generalizations of this notion, the notion of cyclically pure injective module has been extensively studied by M. Hochster [9] and L. Melkersson [14]. Recall that an exact sequence $0 \to A \to B \to C \to 0$ of R-modules and R-homomorphisms is said to be *cyclically pure* if the induced map $R/\mathfrak{a}\otimes_R A \to R/\mathfrak{a}\otimes_R B$ is injective for all (finitely generated) ideals $\mathfrak{a}$ of R. Also, an R-module D is said to be *cyclically pure injective* if for any cyclically pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism $\operatorname{Hom}_R(B, D) \to \operatorname{Hom}_R(A, D)$ is surjective. In the following, we use the abbreviation CP for "cyclically pure".

More generally, let $\mathcal{S}$ be a class of R-modules. An exact sequence $0 \to A \to B \to C \to 0$ of R-modules and R-homomorphisms is said to be *$\mathcal{S}$-pure* if for all $M \in \mathcal{S}$, the induced homomorphism $\operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C)$ is surjective. An R-monomorphism $f : A \to B$ is said to be *$\mathcal{S}$-pure* if the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{\text{nat}} B/f(A) \to 0$ is $\mathcal{S}$-pure. An R-module D is said to be *$\mathcal{S}$-pure injective* if for any $\mathcal{S}$-pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism $\operatorname{Hom}_R(B, D) \to \operatorname{Hom}_R(A, D)$ is

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surjective (see [17]). When $\mathcal{S}$ is the class of finitely presented R-modules, $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules are simply called pure exact sequences and pure injective modules. If $\mathcal{S}$ denotes the class of all R-modules of the form R/Rr, $r \in R$, then $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules are called RD-exact sequences and RD-injective modules. For a survey on pure injective and RD-injective modules, we refer the reader to [6].

Let $\mathcal{S}$ be the class of all R-modules M for which there is a cyclic submodule G of R^n, for some $n \in \mathbb{N}$, such that M is isomorphic to R^n/G. In [3], we showed that CP-exact sequences and CP-injective modules coincide with $\mathcal{S}$-pure exact sequences and $\mathcal{S}$-pure injective modules, respectively. In the same paper we have systematically investigated the structure of CP-injective modules and presented several characterizations of this class of modules.

Our aims in this paper are twofold. First, we wish to classify the commutative rings over which the notions of "RD-injective" and "cyclically pure injective" coincide. Second, we wish to classify the commutative rings over which the notions of "pure injective" and "cyclically pure injective" coincide.

In Section 2, we show that $R_{\mathfrak{p}}$ is a valuation ring (not necessarily a domain) for all prime ideals $\mathfrak{p}$ of R if and only if every CP-injective R-module is RD-injective, if and only if every pure injective R-module is CP-injective. From this we obtain a characterization of semihereditary rings and also one for Prüfer domains. Namely, we show that a domain R is Prüfer if and only if every CP-injective R-module is RD-injective, if and only if every pure injective R-module is CP-injective. Also, we show that a domain R is Prüfer if and only if every absolutely CP-module is absolutely pure. Finally, a new characterization of pure semisimple rings is given. We show that a ring R is pure semisimple if and only if every R-module is CP-injective, if and only if every R-module is RD-pure injective.

The first example of a CP-exact sequence which is not pure was presented in [1]. Our first characterization of Prüfer domains mentioned above shows that over a non-Prüfer domain R, the class of CP-injective R-modules is strictly larger than that of RD-injective R-modules and strictly smaller than that of pure injective R-modules. However, these may be viewed as kind of implicit strict inclusions. In Section 3, we provide some examples for which we can explicitly show proper containments. In [3], we proved that in many aspects CP-injective modules behave similarly to pure injective and RD-injective modules. But Remark 2.2 and Example 3.5 below display some differences between the former class and the latter two.

2. A characterization of Prüfer rings. We say that an R-module M is *cyclically presented* if there are an integer $n \in \mathbb{N}$ and a cyclic submodule G of R^n such that M is isomorphic to R^n/G; we denote the class of all cyclically presented R-modules by $\mathcal{S}_{CP}$. Also, in what follows, the class of all cyclically presented cyclic R-modules will be denoted by $\mathcal{S}_{CPC}$, (so $\mathcal{S}_{CPC}$ is the class of all R-modules M which are isomorphic to R-modules of the form R/Rr, $r \in R$). Finally, the class of all finitely presented (respectively, finitely presented cyclic) R-modules will be denoted by $\mathcal{S}_{FP}$ (respectively, $\mathcal{S}_{FPC}$).

Definition 2.1. Let $\mathcal{S}$ be a class of R-modules. An exact sequence $0 \to A \to B \to C \to 0$ of R-modules and R-homomorphisms is called *$\mathcal{S}$-flat* if for all $M \in \mathcal{S}$ the induced map $A \otimes_R M \to B \otimes_R M$ is injective.

Remark 2.2. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules and R-homomorphisms.

(i) The above exact sequence is $\mathcal{S}_{CPC}$-pure if and only if it is $\mathcal{S}_{CPC}$-flat (see [17, Propositions 2 and 3]).
(ii) The above exact sequence is $\mathcal{S}_{FP}$-pure if and only if it is $\mathcal{S}_{FP}$-flat (see [17, Propositions 2 and 3]).
(iii) By [3, Proposition 2.2], the above exact sequence is $\mathcal{S}_{CP}$-pure if and only if it is $\mathcal{S}_{FPC}$-flat.

Example 3.5 in the next section shows that there exist $\mathcal{S}_{FPC}$-flat exact sequences which are not $\mathcal{S}_{FPC}$-pure.

Definition 2.3. Let $\mathcal{S}$ be a class of R-modules. An R-module P is said to be *$\mathcal{S}$-pure projective* if for any $\mathcal{S}$-pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is surjective.

Lemma 2.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be two classes of R-modules. The following are equivalent*:

(i) *Every $\mathcal{T}$-pure exact sequence is $\mathcal{S}$-pure exact.*
(ii) *Every $\mathcal{S}$-pure projective R-module is $\mathcal{T}$-pure projective.*
(iii) *Every element of $\mathcal{S}$ is a direct summand of a direct sum of modules in $\mathcal{T}$.*

Moreover, if $\mathcal{S}$ and $\mathcal{T}$ are both contained in $\mathcal{S}_{FP}$, then the above conditions are equivalent to the following:

(iv) *Every $\mathcal{S}$-pure injective R-module is $\mathcal{T}$-pure injective.*

Proof. Let $\mathcal{U}$ be a class of R-modules. By the definition every element of $\mathcal{U}$ is $\mathcal{U}$-pure projective. In general, by [17, Proposition 1], an R-module M is $\mathcal{U}$-pure projective if and only if M is a direct summand of a direct sum of modules in $\mathcal{U}$. Hence the equivalence of (i)–(iii) is immediate.

Next, assume that $\mathcal{S}$ and $\mathcal{T}$ are both contained in $\mathcal{S}_{FP}$. Let $\mathcal{U} \subseteq \mathcal{S}_{FP}$ be a class of R-modules and E an injective cogenerator of R. By [5, Lemma 1.2], there is a class $\mathcal{U}^*$ of R-modules such that an exact sequence $0 \to A \to B \to C \to 0$ of R-modules and R-homomorphisms is $\mathcal{U}$-pure if and only if

$$0 \to A \otimes_R M^* \to B \otimes_R M^* \to C \otimes_R M^* \to 0$$

is exact for all $M^* \in \mathcal{U}^*$. Thus by using the adjointness property, it follows that $\operatorname{Hom}_R(M^*, E)$ is a $\mathcal{U}$-pure injective R-module for all $M^* \in \mathcal{U}^*$.

(iv)⇒(i). Let

$$(*) \qquad 0 \to A \to B \to C \to 0$$

be a $\mathcal{T}$-pure exact sequence and $M^* \in \mathcal{S}^*$ an arbitrary element. Since $\operatorname{Hom}_R(M^*, E)$ is $\mathcal{S}$-pure injective, it is also $\mathcal{T}$-pure injective, by our assumption. Thus, by applying the functor $\operatorname{Hom}_R(-, \operatorname{Hom}_R(M^*, E))$ to $(*)$ and using the adjointness property, we deduce the exact sequence

$$0 \to \operatorname{Hom}_R(C \otimes_R M^*, E) \to \operatorname{Hom}_R(B \otimes_R M^*, E) \to \operatorname{Hom}_R(A \otimes_R M^*, E) \to 0.$$

Thus, it turns out that the sequence

$$0 \to A \otimes_R M^* \to B \otimes_R M^* \to C \otimes_R M^* \to 0$$

is exact. Therefore $(*)$ is $\mathcal{S}$-pure exact.

Now, since the implication (i)⇒(iv) clearly holds, the proof is finished. ■

LEMMA 2.5. *Assume that every pure injective R-module is CP-injective. Then an exact sequence $l : 0 \to A \to B \to C \to 0$ is $\mathcal{S}_{FPC}$-pure exact if and only if it is CP-exact.*

Proof. Assume that l is a CP-exact sequence. Then, by Lemma 2.4, it is pure exact. Hence it is clearly $\mathcal{S}_{FPC}$-pure, because $\mathcal{S}_{FPC} \subseteq \mathcal{S}_{FP}$.

Now, assume that l is $\mathcal{S}_{FPC}$-pure exact. Let E be an injective cogenerator of R and $(\cdot)^\vee$ denote the faithfully exact functor $\operatorname{Hom}_R(-, E)$. Let $l^\vee$ denote the induced exact sequence $0 \to C^\vee \to B^\vee \to A^\vee \to 0$. Let $\mathfrak{I}$ be a finitely generated ideal of R. Since $R/\mathfrak{I}$ is finitely presented, the two R-modules $R/\mathfrak{I} \otimes_R M^\vee$ and $\operatorname{Hom}_R(R/\mathfrak{I}, M)^\vee$ are naturally isomorphic for all R-modules M. So the exact sequence $l^\vee$ is CP-exact. Hence $l^\vee$ is pure exact, by Lemma 2.4. Let $N \in \mathcal{S}_{CP}$. Then by Remark 2.2(ii), the sequence $N \otimes_R l^\vee$ is exact. The exact sequences

$$0 \to N \otimes_R C^\vee \to N \otimes_R B^\vee \to N \otimes_R A^\vee \to 0$$

and

$$0 \to \operatorname{Hom}_R(N, C)^\vee \to \operatorname{Hom}_R(N, B)^\vee \to \operatorname{Hom}_R(N, A)^\vee \to 0$$

are naturally isomorphic. Thus the second sequence is also exact, and so

$$0 \to \operatorname{Hom}_R(N, A) \to \operatorname{Hom}_R(N, B) \to \operatorname{Hom}_R(N, C) \to 0$$

is an exact sequence, because $(\cdot)^\vee$ is a faithfully exact functor. Therefore l is a CP-exact sequence. ■

LEMMA 2.6. *Let $\mathfrak{a}$ be an ideal of R. Assume that every CP-injective R-module is RD-injective. Then every CP-injective $R/\mathfrak{a}$-module is an RD-injective $R/\mathfrak{a}$-module.*

Proof. Set $T = R/\mathfrak{a}$. Let $M = T^n/V$, where $n \in \mathbb{N}$ and V is a cyclic T-submodule of T^n. So, there are $b_1, \dots, b_n \in R$ such that

$$V = T(b_1 + \mathfrak{a}, \dots, b_n + \mathfrak{a}).$$

Let $N = R^n/U$, where $U = R(b_1, \dots, b_n)$. We show that M and $N \otimes_R T$ are naturally isomorphic as T-modules. To this end, let $\phi : M \to N \otimes_R T$ be the map defined by

$$(x_1 + \mathfrak{a}, \dots, x_n + \mathfrak{a}) + V \mapsto ((x_1, \dots, x_n) + U) \otimes (1 + \mathfrak{a})$$

for all $(x_1 + \mathfrak{a}, \dots, x_n + \mathfrak{a}) + V \in M$. Also, we define $\psi : N \otimes_R T \to M$ by

$$((x_1, \dots, x_n) + U) \otimes (r + \mathfrak{a}) \mapsto (rx_1 + \mathfrak{a}, \dots, rx_n + \mathfrak{a}) + V.$$

It is a routine check to see that ϕ and ψ are well defined T-homomorphisms and that $\psi\phi = \mathrm{id}_M$ and $\phi\psi = \mathrm{id}_{N \otimes_R T}$. Now, as $- \otimes_R T$ commutes with direct sums, the conclusion is immediate by Lemma 2.4(iii)⇔(iv). ∎

Recall that a *valuation ring* (not necessarily a domain) is a commutative ring whose ideals are linearly ordered by inclusion.

THEOREM 2.7. *The following are equivalent*:

(i) *$R_\mathfrak{p}$ is a valuation ring for all prime ideals $\mathfrak{p}$ of R.*
(ii) *Every pure injective R-module is RD-injective.*
(iii) *Every CP-injective R-module is RD-injective.*
(iv) *Every pure injective R-module is CP-injective.*
(v) *Every pure projective R-module is RD-projective.*
(vi) *Every CP-projective R-module is RD-projective.*
(vii) *Every pure projective R-module is CP-projective.*

Proof. By Lemma 2.4, the equivalences (ii)⇔(v), (iii)⇔(vi) and (iv)⇔(vii) are obvious. Also, the implications (ii)⇒(iii) and (ii)⇒(iv) are clear.

(i)⇒(v). As mentioned in the proof Lemma 2.4, for a given class $\mathcal{U}$ of R-modules, an R-module M is $\mathcal{U}$-pure projective if and only if M is a direct summand of a direct sum of modules in $\mathcal{U}$. So, to deduce (v), it is enough to show that every finitely presented R-module is RD-projective. By [6, VI, Lemma 12.3], a finitely presented R-module M is RD-projective if and only if $M_\mathfrak{m}$ is an RD-projective $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R. Hence (v) follows by [19, Theorem 1].

(v)⇒(i). This follows by [17, Proposition 1] and [19, Theorem 3].

(iii)⇒(i). Assume that there exists a prime ideal $\mathfrak{p}$ of R so that $R_\mathfrak{p}$ is not a valuation ring. Let $N = (R_\mathfrak{p})^n/G$, where $n \in \mathbb{N}$ and G is a cyclic $R_\mathfrak{p}$-submodule of $(R_\mathfrak{p})^n$. Clearly N is equal to the localization at $\mathfrak{p}$ of an element of $\mathcal{S}_{CP}$. Hence, as localization at $\mathfrak{p}$ commutes with direct sums, by Lemma 2.4, we may and do assume that R is a local ring which is not a valuation ring. Denote by $\mathfrak{m}$ the maximal ideal of R. Since R is not a valuation ring, there are two elements $a, b \in R$ such that $Ra \nsubseteq Rb$ and $Rb \nsubseteq Ra$. Set $\mathfrak{I} := \mathfrak{m}a + \mathfrak{m}b$. Lemma 2.6 implies that every CP-injective $R/\mathfrak{I}$-module is an RD-injective $R/\mathfrak{I}$-module. Replace R, a and b by $R/\mathfrak{I}$, $a+\mathfrak{I}$ and $b+\mathfrak{I}$, respectively. So we can assume that R is a local ring which is not a valuation ring and that there are two elements $a, b \in R$ such that $Ra \nsubseteq Rb$, $Rb \nsubseteq Ra$, $\mathfrak{m}a = \mathfrak{m}b = 0$ and $Ra \cap Rb = 0$. In view of the proof of [19, Theorem 2], it becomes clear that $M := (R \oplus R)/R(a, -b)$ is a noncyclic indecomposable R-module. Lemma 2.4 implies that M is a direct summand of a direct sum of cyclic modules. Now, by [18, Proposition 3], over a commutative local ring, any indecomposable direct summand of a direct sum of cyclic modules is cyclic. We have achieved a contradiction.

(iv)⇒(i). By Lemmas 2.4 and 2.5, it follows that every finitely presented R-module is a direct summand of a direct sum of cyclic modules. Now, we assume that (i) does not hold and search for a contradiction. Then there is a prime ideal $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ is not a valuation ring. Hence, by [19, Theorem 2], there exists an indecomposable finitely presented $R_{\mathfrak{p}}$-module M which is not cyclic. Since every finitely presented $R_{\mathfrak{p}}$-module is the localization at $\mathfrak{p}$ of a finitely presented R-module, we deduce that M is a direct summand of a direct sum of cyclic $R_{\mathfrak{p}}$-modules. But then by [18, Proposition 3], M should be a cyclic $R_{\mathfrak{p}}$-module. ■

Definition 2.8.

(i) A ring R is said to be a *projective principal ring* (P.P.R.) if every principal ideal of R is projective (see [4]).
(ii) A ring R is said to be *semihereditary* if every finitely generated ideal of R is projective.
(iii) An R-module M is said to be *absolutely pure* (resp. *absolutely cyclically pure*) if it is pure (resp. cyclically pure) as a submodule in every extension of M (see [11]).
(iv) An R-module M is said to be *divisible* if for every $r \in R$ and $x \in M$, $\operatorname{Ann}_R r \subseteq \operatorname{Ann}_R x$ implies that $x \in rM$ (see [5]). (This is equivalent to the usual definition when R is a domain.)

In the proof of the following lemma we use the methods of the proofs of [12, Proposition 1 and Corollary 2].

Lemma 2.9. *Let M be an R-module.*

(i) *M is absolutely cyclically pure if and only if* $\operatorname{Ext}^1_R(N, M) = 0$ *for all* $N \in \mathcal{S}_{CP}$.

(ii) *M is absolutely cyclically pure if and only if for any diagram*

$$\begin{array}{ccc} P' & \xrightarrow{\alpha} & P \\ \downarrow \beta & & \\ M & & \end{array}$$

with P' cyclic, α monic and P projective, there exists a homomorphism $\gamma : P \to M$ such that $\gamma\alpha = \beta$.

Proof. (i) Let L be an extension of M and $N \in \mathcal{S}_{CP}$. From the exact sequence $0 \to M \hookrightarrow L \to L/M \to 0$, we deduce the following exact sequence

$$(*) \quad 0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(N, L/M) \to \mathrm{Ext}^1_R(N, M) \to \mathrm{Ext}^1_R(N, L).$$

Assume that M is an absolutely CP-module and let L be an injective extension of M. Then by Remark 2.2(iii) and $(*)$, we conclude that $\mathrm{Ext}^1_R(N, M)=0$ for all $N \in \mathcal{S}_{CP}$.

Now, assume that $\mathrm{Ext}^1_R(N, M) = 0$ for all $N \in \mathcal{S}_{CP}$. Let L be an extension of M. Then Remark 2.2(iii) and $(*)$ imply that the exact sequence $0 \to M \hookrightarrow L \to L/M \to 0$ is CP-exact.

(ii) We may assume that P is a finitely generated free R-module. Then the result follows by using (i) and the exact sequence

$$\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P', M) \to \mathrm{Ext}^1_R(P/\alpha(P'), M) \to 0. \ \blacksquare$$

LEMMA 2.10. *The following are equivalent*:

(i) *R is a P.P.R.*
(ii) *Every cyclic submodule of a projective R-module is projective.*
(iii) *Every quotient of an absolutely CP-module is also an absolutely CP-module.*

Proof. (i)⇔(ii) follows by [4, Theorem 3.2].

(ii)⇔(iii). In view of Lemma 2.9, the proof is immediate by adapting the argument of [12, Theorem 2] and replacing the phrases "absolutely pure" and "finitely generated submodule" with "absolutely cyclically pure" and "cyclic submodule", respectively. ■

Now, it is time to present our main result which is a new characterizations of Prüfer domains. In the literature, there are several characterizations of Prüfer domains. In particular, by [6, Chapter XIII, Theorem 2.8], it is known that a domain R is Prüfer if and only if every pure injective R-module is RD-injective. Also, it is known by [6, Chapter IX, Proposition 3.4] that a domain R is Prüfer if and only if every divisible R-module is absolutely pure.

COROLLARY 2.11. *Assume that R is a P.P.R.* (*respectively*, *domain*). *The following are equivalent*:

(i) *R is a semihereditary ring (respectively, Prüfer domain).*
(ii) *Every pure injective R-module is RD-injective.*
(iii) *Every CP-injective R-module is RD-injective.*
(iv) *Every pure injective R-module is CP-injective.*
(v) *Every divisible R-module is absolutely pure.*
(vi) *Every absolutely CP-module is absolutely pure.*
(vii) *Every pure projective R-module is RD-projective.*
(viii) *Every CP-projective R-module is RD-projective.*
(ix) *Every pure projective R-module is CP-projective.*

Proof. Since a domain R is Prüfer if and only if it is semihereditary, it is enough to prove the claim only for P.P.R. rings. Note that every domain is a P.P.R. As mentioned in the proof of Theorem 2.7, by Lemma 2.4, the equivalences (ii)$\Leftrightarrow$(vii), (iii)$\Leftrightarrow$(viii) and (iv)$\Leftrightarrow$(ix) are obvious.

Now, assume that R is semihereditary. Let $\mathfrak{p}$ be a prime ideal of R. Then $R_{\mathfrak{p}}$ is also a semihereditary ring. Hence for each nonzero element a of $R_{\mathfrak{p}}$, the $R_{\mathfrak{p}}$-module $aR_{\mathfrak{p}}$ is a nonzero free $R_{\mathfrak{p}}$-module. Thus, we conclude that $R_{\mathfrak{p}}$ is a domain. But it is known that a domain is semihereditary if and only if it is Prüfer. So $R_{\mathfrak{p}}$ is a valuation domain for all prime ideals $\mathfrak{p}$ of R. Therefore the implication (i)$\Rightarrow$(ii) and the equivalences (ii)$\Leftrightarrow$(iii) and (iii)$\Leftrightarrow$(iv) are immediate by Theorem 2.7.

(ii)$\Rightarrow$(v). Let M be a divisible R-module and E denote the injective envelope of M. Then [5, Lemma 2.2] implies that the sequence $0 \to M \hookrightarrow E \to E/M \to 0$ is RD-exact. Hence, by Lemma 2.4, it is pure and so $\mathrm{Ext}^1_R(N,M) = 0$ for all $N \in \mathcal{S}_{FP}$. Thus, by [12, Proposition 1], M is absolutely pure.

(v)$\Rightarrow$(vi). Let M be an absolutely CP-module. Then, by Lemma 2.9(i), $\mathrm{Ext}^1_R(N,M) = 0$ for all $N \in \mathcal{S}_{CP}$. In particular, $\mathrm{Ext}^1_R(R/Rr,M) = 0$ for all $r \in R$, and so M is a divisible R-module by [5, Lemma 2.2]. Thus M is absolutely pure, as required.

Finally, we prove (vi)$\Rightarrow$(i). Since R is a P.P.R., Lemma 2.10 implies that every quotient of an absolutely CP-module is again an absolutely CP-module. So, if (vi) holds, then every quotient of an absolutely pure module is again absolutely pure. Thus (i) follows by [12, Theorem 2]. ∎

Let $\mathcal{C}_{RDR}$ denote the class of all RD-injective R-modules. Also, let $\mathcal{C}_{CPR}$ and $\mathcal{C}_{PR}$ denote the class of all CP-injective R-modules and of all pure injective R-modules, respectively. It follows from Theorem 2.7 that if two of the three classes $\mathcal{C}_{RDR}$, $\mathcal{C}_{CPR}$ and $\mathcal{C}_{PR}$ are equal, then all three classes are equal. The following result shows that if any of these three classes is equal to the class of all R-modules, then the other two are also equal to the class of all R-modules. First, we recall from [15] the following definition (see also [7] and [8]).

DEFINITION 2.12. A ring R is said to be *pure-semisimple* if every R-module is a direct sum of finitely generated R-modules.

There are many characterizations of pure-semisimple rings in the literature (see e.g. [7], [8], [10] and [16]). In particular, it is known that a ring R is pure-semisimple if and only if it is Artinian and of finite representation type, that is, has only a finite number of pairwise nonisomorphic indecomposable finitely generated R-modules.

THEOREM 2.13. *The following are equivalent*:

(i) *Every R-module is RD-pure injective.*
(ii) *Every R-module is CP-injective.*
(iii) *Every R-module is pure injective.*
(iv) *R is pure-semisimple.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are clear.

Assume that (iii) holds. Then every pure exact sequence of R-modules splits, and so it follows from [8] that every R-module is a direct sum of finitely generated R-modules. Thus (iii) implies (iv).

Now, we prove the implication (iv)⇒(i). By [7, Theorem 4.3], R is an Artinian principal ideal ring and every R-module is a direct sum of cyclic R-modules. Hence, since every ideal of R is principal, it follows that every R-module is a direct sum of modules of the form $R/Rr, r \in R$. From this we can conclude that every RD-exact sequence splits. Therefore, every R-module is RD-injective. ■

3. Some examples. Theorem 2.7 shows that if R is not Prüfer, then $\mathcal{C}_{RDR} \subsetneq \mathcal{C}_{CPR} \subsetneq \mathcal{C}_{PR}$. In this section, we present some explicit examples for these strict containments.

EXAMPLE 3.1. (i) Let $\mathbb{Z}$ be the ring of integers and p a prime integer. Since every ideal of $\mathbb{Z}$ is principal, the notions of RD-injectivity and of CP-injectivity coincide for $\mathbb{Z}$-modules. Hence by [3, Theorem 3.6], $D = \mathbb{Z}/p\mathbb{Z}$ is an RD-injective $\mathbb{Z}$-module, while it is not an injective $\mathbb{Z}$-module.

(ii) By [1, Example 1], there are an Artinian local ring R and an R-algebra S containing R such that the inclusion map $R \hookrightarrow S$ is cyclically pure, but not pure. It is known that every Artinian R-module is pure injective (see e.g. [13, Corollary 4.2]). Hence R is a pure injective R-module. But R is not CP-injective, because otherwise by [3, Theorem 3.4], the inclusion map $R \hookrightarrow S$ splits.

LEMMA 3.2. *Let R be a domain, B a torsion-free R-module and $0 \to K \hookrightarrow B \to M \to 0$ an exact sequence of R-modules. The following are equivalent*:

(i) *M is torsion-free.*
(ii) *The inclusion map $K \hookrightarrow B$ is RD-pure.*

Proof. It is easy to see that an R-module L is torsion-free if and only if

$$\operatorname{Tor}_1^R(R/Rr, L) = 0 \quad \text{for all } r \in R.$$

Since B is torsion-free for any $r \in R$, from the exact sequence $0 \to K \hookrightarrow B \to M \to 0$ we deduce the exact sequence

$$0 \to \operatorname{Tor}_1^R(R/Rr, M) \to (R/Rr) \otimes_R K \to (R/Rr) \otimes_R B \to (R/Rr) \otimes_R M \to 0.$$

Therefore, the assertion follows by Remark 2.2(i). ∎

Lemma 3.3. *Let R be a domain and D an RD-injective R-module. Then $\operatorname{Ext}_R^1(M, D) = 0$ for all torsion-free R-modules M.*

Proof. Let M be a torsion-free R-module. Consider an exact sequence $0 \to K \overset{i}{\hookrightarrow} F \to M \to 0$, in which F is a free R-module. Then, by Lemma 3.2, the inclusion map i is RD-pure. Now, from the exact sequence

$$0 \to \operatorname{Hom}_R(M, D) \to \operatorname{Hom}_R(F, D) \to \operatorname{Hom}_R(K, D) \to \operatorname{Ext}_R^1(M, D) \to 0,$$

we deduce that $\operatorname{Ext}_R^1(M, D) = 0$. Note that since D is RD-injective, the map $\operatorname{Hom}_R(i, \operatorname{id}_D)$ is surjective. ∎

Example 3.4. Let $(R, \mathfrak{m})$ be a local Noetherian domain with $\dim R > 1$. Since R is not a Prüfer domain, it turns out that R possesses an ideal $\mathfrak{a}$ which is not projective. Thus

$$\operatorname{Ext}_R^1(\mathfrak{a}, R/\mathfrak{m}) \neq 0,$$

by [2, Proposition 1.3.1]. Now, by [3, Theorem 3.6], $R/\mathfrak{m}$ is a CP-injective R-module, while by Lemma 3.3, $R/\mathfrak{m}$ is not RD-injective.

The following example shows that the notions of $\mathcal{S}_{FPC}$-flatness and $\mathcal{S}_{FPC}$-pureness are not the same.

Example 3.5. Assume that R is a Noetherian domain such that $\dim R > 1$. Hence R is not Prüfer, and so by Corollary 2.11, there exists an absolutely CP-module M which is not injective. So, there is an ideal $\mathfrak{a}$ such that

$$\operatorname{Ext}_R^1(R/\mathfrak{a}, M) \neq 0.$$

Let E denote the injective envelope of M. Then from the exact sequence

$$(*) \qquad 0 \to M \hookrightarrow E \xrightarrow{\pi} E/M \to 0,$$

we deduce the exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, M) \to \operatorname{Hom}_R(R/\mathfrak{a}, E) \to \operatorname{Hom}_R(R/\mathfrak{a}, E/M) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, M) \to 0.$$

Hence the map $\operatorname{Hom}_R(\mathrm{id}_{R/\mathfrak{a}}, \pi)$ is not surjective. Thus $(*)$ is an $\mathcal{S}_{FPC}$-flat sequence which is not $\mathcal{S}_{FPC}$-pure.

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