TOPOLOGICAL TRANSITIVITY OF SOLVABLE GROUP ACTIONS
ON THE LINE $\mathbb{R}$

BY

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Abstract. Let $\phi : G \to \text{Homeo}_+(\mathbb{R})$ be an orientation preserving action of a discrete solvable group $G$ on $\mathbb{R}$. In this paper, the topological transitivity of $\phi$ is investigated. In particular, the relations between the dynamical complexity of $G$ and the algebraic structure of $G$ are considered.

1. Introduction and preliminaries. Recently, there has been considerable progress in studying the dynamics of group actions on 1-manifolds (see e.g. [1], [3], [6], [7], [9], [10], [13]–[16], [18], [21], [22]). Topological transitivity is one of the most basic notions in dynamical systems. In this paper, we consider the topological transitivity of solvable group actions on the line $\mathbb{R}$.

We are mainly interested in the following two questions:

(1) Which solvable groups possess a faithful topologically transitive action on the line?

(2) What can one say about actions with higher transitivity?

Before stating the main results in this paper, let us recall some definitions. Let $X$ be a topological space and let Homeo($X$) be the homeomorphism group of $X$. Suppose that $G$ is a discrete group (i.e., $G$ is a topological group with discrete topology). Recall that a group homomorphism $\phi : G \to \text{Homeo}(X)$ is called an action of $G$ on $X$. The action $\phi$ is said to be faithful if it is injective. For convenience, we always use $gx$ or $g(x)$ to denote $\phi(g)(x)$.

Recall that the orbit of $x \in X$ is the set $Gx = \{gx : g \in G\}$. For a subset $A \subseteq X$, set $GA = \bigcup_{x \in A} Gx$. A subset $A \subseteq X$ is said to be $G$-invariant if $GA = A$. If $A$ is $G$-invariant, the symbol $G|_A$ denotes the restriction to $A$ of the action of $G$.

The action $\phi$ is said to be topologically transitive if for any two nonempty open subsets $U$ and $V$ of $X$, there is some $g \in G$ such that $g(U) \cap V \neq \emptyset$.

2000 Mathematics Subject Classification: Primary 37B05; Secondary 57S25.
Key words and phrases: topological transitivity, solvable group action, ping-pong game.

DOI: 10.4064/cm116-2-5 [203] © Instytut Matematyczny PAN, 2009
If there is some point \( x \in X \) such that the orbit \( Gx \) is dense in \( X \) then \( \phi \) is said to be point transitive, and such an \( x \) is called a transitive point. It is well known that when \( G \) is countable and \( X \) is a Polish space, these two notions are the same and in fact the collection of transitive points is a dense \( G_\delta \) set in \( X \). If for each \( x \in X \), \( Gx \) is dense in \( X \), then \( \phi \) is said to be minimal. For an integer \( k \geq 1 \), \( \phi \) is said to be topologically \( k \)-transitive if for any two \( \phi \)-families \( U_1, \ldots, U_k \) and \( V_1, \ldots, V_k \) of nonempty open subsets of \( X \), there is some \( g \in G \) such that \( g(U_i) \cap V_i \neq \emptyset \) for each \( i = 1, \ldots, k \). For commutative group actions, a well known theorem of H. Furstenberg says that topological \( 2 \)-transitivity implies topological \( k \)-transitivity for each \( k \geq 2 \) (see [8] and [11]). Topological \( k \)-transitivity of linear group actions is also studied in [4] and [5].

Denote by \( \text{Homeo}_+(\mathbb{R}) \) the group of all orientation preserving homeomorphisms on \( \mathbb{R} \). Let \( \phi : G \to \text{Homeo}_+(\mathbb{R}) \) be an orientation preserving group action on \( \mathbb{R} \). Since each element of \( G \) preserves the orientation of \( \mathbb{R} \), \( \phi \) cannot be topologically \( k \)-transitive for \( k \geq 2 \). However, we are mainly interested in orientation preserving group actions on \( \mathbb{R} \) in this paper, so we have to give the following definition of pseudo-\( k \)-transitivity.

Firstly we introduce an ordering \( \preceq \) in the collection of all open intervals contained in \( \mathbb{R} \). For any two open intervals \((a, b)\) and \((c, d)\) in \( \mathbb{R} \), we say that \((a, b) \preceq (c, d)\) if \( a \leq c \). We say \( \phi \) is pseudo-\( k \)-transitive if for any two families of open intervals \((a_1, b_1) \preceq \cdots \preceq (a_k, b_k)\) and \((c_1, d_1) \preceq \cdots \preceq (c_k, d_k)\) there is some \( g \in G \) such that \( g((a_i, b_i)) \cap (c_i, d_i) \neq \emptyset \) for all \( i = 1, \ldots, k \). It is easy to see that \( \phi \) is pseudo-\( k \)-transitive if and only if there are \( x_1 < \cdots < x_k \in \mathbb{R} \) such that for any nonempty open intervals \( U_1 \preceq \cdots \preceq U_k \) there is a \( g \in G \) such that \( g(x_i) \in U_i \) for each \( i = 1, \ldots, k \).

Now let us recall some definitions in group theory which will be used in the following. Suppose that \( G \) is a group with identity \( e \). Let \( a, b \in G \). The commutator \([a, b]\) is defined by \([a, b] = a^{-1}b^{-1}ab\). For any two subsets \( A \) and \( B \) of \( G \), define \([A, B] \) to be the subgroup generated by the set \([\{a, b : a \in A, b \in B\}]\). Let \( G_0 = G \) and \( G_{i+1} = [G_i, G] \) for \( i = 0, 1, \ldots \). Thus we get a sequence of normal subgroups of \( G \): \( G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \). If there is some natural number \( n \) such that \( G_n = \{e\} \), then \( G \) is called nilpotent. Also, we can define another sequence of normal subgroups \( G^0 = G \triangleright G^1 \triangleright G^2 \triangleright \cdots \) by letting \( G^0 = G \) and \( G^{i+1} = [G^i, G] \) for \( i = 0, 1, \ldots \). If there is some \( n \) such that \( G^n = \{e\} \) then \( G \) is called solvable. The minimal \( n \) such that \( G^n = \{e\} \) is called the derived length of \( G \). A solvable group with derived length at most 2 is called metabelian. A solvable group \( G \) is called polycyclic if for some \( k \), \( G \) has a sequence of normal subgroups \( G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\} \) such that each \( N_i/N_{i+1} \) is cyclic. When each of the quotients \( N_i/N_{i+1} \) is infinite cyclic, \( G \) is said to be poly-infinite-cyclic.
Before stating our results for solvable groups, let us put them in a broader context. First note that topologically transitive actions of nonsolvable groups on the line are quite abundant. For example, the group generated by the two elements $f(x) = x + 1$ and $g(x) = x^3$ is a free group [12, p. 37] and it is easy to verify that its action on the line is topologically transitive. The group of homeomorphisms of $\mathbb{R}$ that are piecewise linear with respect to a finite subdivision of $\mathbb{R}$ is not solvable but does not contain a nonabelian free subgroup [2], and its action is clearly pseudo-$k$-transitive for all $k$.

In Section 2, it is shown that minimal actions of commutative groups are building blocks of topologically transitive nilpotent group actions on $\mathbb{R}$. To illustrate this idea, for each finitely generated torsion free nilpotent group $G$, a topologically transitive $G \times \mathbb{Z}^2$ action on $\mathbb{R}$ is constructed. A more general result is also given, as we will now describe. First recall that a countable group $G$ has a faithful orientation preserving action on the real line if and only if $G$ is left orderable (see [10, Theorem 6.8]). Moreover, left orderable polycyclic groups are poly-infinite-cyclic [19]. Since the cyclic group $\mathbb{Z}$ obviously has no topologically transitive action on the line, it is therefore natural to look at noncyclic poly-infinite-cyclic groups. We have:

**Theorem 1.1.** Every noncyclic poly-infinite-cyclic group has a faithful topologically transitive orientation preserving action on the real line.

In particular, every finitely generated torsion free nilpotent group has a faithful topologically transitive orientation preserving action on the real line.

In Section 3, two examples are given: of pseudo-1-transitive but not pseudo-2-transitive and of pseudo-2-transitive but not pseudo-3-transitive metabelian group actions on $\mathbb{R}$. It is shown that each polycyclic solvable group action on $\mathbb{R}$ is at most pseudo-2-transitive, and if the derived length of a solvable group $G$ is $n$, then the action of $G$ is at most pseudo-$(4^n-1)$-transitive. At the end of that section, it is shown that no nilpotent group action on $\mathbb{R}$ is pseudo-2-transitive.

In the following, all group actions on $\mathbb{R}$ are assumed to be orientation preserving.

2. **The construction of topologically transitive actions.** In this section, we will construct some topologically transitive solvable group actions on $\mathbb{R}$. First we study minimal actions of nilpotent subgroups of $\text{Homeo}_+(\mathbb{R})$. The following proposition is in fact a special case of Corollary 4.6 in [18], but for completeness we reprove it here.

**Proposition 2.1.** Let $G$ be a finitely generated nilpotent subgroup of $\text{Homeo}_+(\mathbb{R})$. If the action of $G$ is minimal, then $G$ must be commutative.
and be topologically conjugate to a subgroup of \( \text{Homeo}_+ (\mathbb{R}) \) consisting of translations on \( \mathbb{R} \).

**Proof.** From [17], we know that there is a \( G \)-invariant Borel measure \( \mu \) on \( \mathbb{R} \) which is finite on compact sets. Since \( G \) is minimal, we have \( \text{supp}(\mu) = \mathbb{R} \) and \( \mu \) has no atoms. Let \( h : \mathbb{R} \to \mathbb{R} \) be the map defined by

\[
h(x) = \begin{cases} 
\mu([0, x]) & \text{if } x \geq 0, \\
-\mu([x, 0]) & \text{if } x < 0.
\end{cases}
\]

Then it is easy to see that \( h \) is a homeomorphism. Now let \( \tilde{G} = \{ hgh^{-1} : g \in G \} \). It is not difficult to check that each element of \( \tilde{G} \) is an isometry. Since \( \tilde{G} \) preserves the orientation of \( \mathbb{R} \), \( \tilde{G} \) consists of translations on \( \mathbb{R} \). Thus \( \tilde{G} \) is commutative and is conjugate to \( G \) by \( h \).

**Remark 2.2.** We can easily construct a minimal \( \mathbb{Z}^2 \) action on \( \mathbb{R} \). Indeed, let \( L_a \) and \( L_b \) be two translations on \( \mathbb{R} \) defined by \( L_a(x) = x + a \) and \( L_b(x) = x + b \) for all \( x \in \mathbb{R} \), where \( a \) and \( b \) are rationally independent. Then the subgroup \( \langle L_a, L_b \rangle \) of \( \text{Homeo}_+ (\mathbb{R}) \) generated by \( L_a \) and \( L_b \) is minimal.

Now we consider the structure of topologically transitive nilpotent group actions on \( \mathbb{R} \). The following proposition indicates that, for nilpotent group actions on \( \mathbb{R} \), minimal systems are building blocks of topologically transitive systems.

**Proposition 2.3.** Let \( G \) be a finitely generated nilpotent subgroup of \( \text{Homeo}_+ (\mathbb{R}) \) which is topologically transitive. Then there exists an open interval \( (\alpha, \beta) \) (\( \alpha \) may be \( -\infty \) and \( \beta \) may be \( \infty \)) such that the restriction of the action of the group \( F = \{ g \in G : g((\alpha, \beta)) = (\alpha, \beta) \} \) to \( (\alpha, \beta) \) is minimal.

**Proof.** If \( G \) is minimal then we need only let \( \alpha = -\infty \) and \( \beta = \infty \). Otherwise, there is some \( x_1 \in \mathbb{R} \) such that \( Gx_1 \) is not dense in \( \mathbb{R} \). Let \( a = \inf \{ Gx_1 \} \) and \( b = \sup \{ Gx_1 \} \).

**Claim 1.** \( a = -\infty \) and \( b = +\infty \).

Indeed, if \( a \in \mathbb{R} \) then it is not difficult to see that \( a \) is a fixed point of \( G \). Since \( G \) preserves the orientation of \( \mathbb{R} \), we have \( g((-\infty, a)) = (-\infty, a) \) and \( g((a, +\infty)) = (a, +\infty) \) for each \( g \in G \). This contradicts the topological transitivity of \( G \). So \( a = -\infty \). Similarly, we have \( b = +\infty \).

**Claim 2.** \( Gx_1 \) is nowhere dense in \( \mathbb{R} \).

Otherwise, there is some nonempty open set \( U \subset Gx_1 \). For any nonempty open set \( V \subset \mathbb{R} \), since \( G \) is topologically transitive, there is some \( g \in G \) such that \( g(U) \cap V \neq \emptyset \). Thus there is some \( g' \in G \) such that \( g'(x_1) \in U \) and \( gg'(x_1) \in V \). By the arbitrariness of \( V \), we deduce that \( x_1 \) is a transitive point. This contradicts our original assumption.
It follows from Claims 1 and 2 that $\mathbb{R} \setminus Gx_1 = \bigcup_{i=-\infty}^{\infty} (a_i, b_i)$, where
\( \{(a_i, b_i) : i \in \mathbb{Z}\} \) is a sequence of pairwise disjoint open intervals in $\mathbb{R}$. Let
\[ F_1 = \{g \in G : g((a_0, b_0)) = (a_0, b_0)\} \]
and let $G/F_1 = \bigcup_{i=-\infty}^{+\infty} g_i F_1$ be the coset decomposition of $G$ with respect to $F_1$, where $g_i((a_0, b_0)) = (a_i, b_i)$ for all $i \in \mathbb{Z}$. This implies that $[G : F_1] = \infty$ and the restrictive action of $F_1$ on $(a_0, b_0)$ is transitive. Let $(\alpha_1, \beta_1) = (a_0, b_0)$. If $F_1|_{(\alpha_1, \beta_1)}$ is not minimal, then, similarly to the above discussions, we can get a subgroup $F_2$ of $F_1$ and an open interval $(\alpha_2, \beta_2) \subset (\alpha_1, \beta_1)$ such that $F_2|_{(\alpha_2, \beta_2)}$ is topologically transitive and $[F_1 : F_2] = \infty$. Going on in this way, if for every $i \geq 1$, $F_i|_{(\alpha_i, \beta_i)}$ is topologically transitive but is not minimal, then we obtain a sequence of open intervals $(\alpha_1, \beta_1) \supset (\alpha_2, \beta_2) \supset \cdots$ and a sequence of subgroups of $G$: $F_1 \supset F_2 \supset \cdots$ such that

(i) each $(\alpha_i, \beta_i)$ is $F_i$-invariant,

(ii) $F_i|_{(\alpha_i, \beta_i)}$ is topologically transitive but is not minimal, and

(iii) $[F_i : F_{i+1}] = \infty$ for $i = 1, 2, \ldots$.

But (iii) contradicts the fact that $G$ is a finitely generated nilpotent group. So there must be some $n \in \mathbb{N}$ such that $F_n|_{(\alpha_n, \beta_n)}$ is minimal. This completes the proof. \[\blacksquare\]

To illustrate the ideas in Proposition 2.3, we construct some topologically transitive actions of finitely generated nilpotent groups on $\mathbb{R}$. In the proof of the following proposition, we follow some ideas in [7].

**Proposition 2.4.** Suppose $G$ is a finitely generated torsion free nilpotent group. Then $G \times \mathbb{Z}^2$ acts on $\mathbb{R}$ faithfully and topologically transitively.

**Proof.** We consider the group $\mathbb{Z}^n$ of $n$-tuples of integers and provide it with a linear order $\prec$ which is the lexicographic ordering, i.e. $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$ if and only if $x_i = y_i$ for $1 \leq i < k$ and $x_k < y_k$ for some $0 \leq k \leq n$.

It is well known that each finitely generated torsion free nilpotent group $G$ admits a linear order $\prec$ which is invariant under left translations, and there exists an order preserving bijection $\iota : G \rightarrow \mathbb{Z}^n$, i.e. $g_1 \prec g_2$ if and only if $\iota(g_1) \prec \iota(g_2)$. Thus $\iota$ naturally induces an action of $G$ on $\mathbb{Z}^n$ by letting
\[ g(p_1, \ldots, p_n) = jgj^{-1}(p_1, \ldots, p_n) \]
for all $(p_1, \ldots, p_n) \in \mathbb{Z}^n$.

Let $B : \mathbb{Z}^n \rightarrow \mathbb{R}$ be defined by
\[ B(q_1, \ldots, q_n) = \sum_{j=1}^{n} q_j^{2n-2j+2}, \]
and let
\[ s = \sum_{(q_1, \ldots, q_n) \in \mathbb{Z}^n} \frac{1}{B(q_1, \ldots, q_n)}. \]
For \((p_1, \ldots, p_n)\) we define \(\iota : \mathbb{Z}^n \to \mathbb{R}\) by
\[
\iota(p_1, \ldots, p_n) = \sum_{(q_1, \ldots, q_n) < (p_1, \ldots, p_n)} \frac{1}{B(q_1, \ldots, q_n)}.
\]
Then it is easy to see that \(\iota\) is an order preserving injection from \(\mathbb{Z}^n\) to \((0, s)\), i.e. \((p_1, \ldots, p_n) < (p'_1, \ldots, p'_n)\) if and only if \(\iota(p_1, \ldots, p_n) < \iota(p'_1, \ldots, p'_n)\). Thus \(\iota\) induces naturally an action of \(G\) on \(\iota(\mathbb{Z}^n)\). We extend this action to the closure \(\overline{\iota(\mathbb{Z}^n)}\) in \((0, s)\) by using continuity, and then extend it to \((0, s)\) by using affine extensions on the complementary intervals of this closure. Thus we get an orientation preserving action of \(G\) on \((0, s)\).

For each \((p_1, \ldots, p_n) \in \mathbb{Z}^n\), let \(U_{(p_1, \ldots, p_n)} = (\iota(p_1, \ldots, p_n), \iota(p_1, \ldots, p_n) + 1)\). Then it is easy to see that \((0, s) \setminus \iota(\mathbb{Z}^n) = \bigcup_{(p_1, \ldots, p_n) \in \mathbb{Z}^n} U_{(p_1, \ldots, p_n)}\). Now let \(f_{(0, \ldots, 0)}\) and \(h_{(0, \ldots, 0)}\) be two homeomorphisms on \(U_{(0, \ldots, 0)}\) such that the action of \(\langle f_{(0, \ldots, 0)}, h_{(0, \ldots, 0)} \rangle\) on \(U_{(0, \ldots, 0)}\) is minimal (for the existence of such homeomorphisms, see Remark 2.2). Evidently, for each \((p_1, \ldots, p_n) \in \mathbb{Z}^n\), there is a unique \(g \in G\) such that \(g(U_{(0, \ldots, 0)}) = U_{(p_1, \ldots, p_n)}\). Now define \(f_{(p_1, \ldots, p_n)}, h_{(p_1, \ldots, p_n)} : U_{(p_1, \ldots, p_n)} \to U_{(p_1, \ldots, p_n)}\) by letting \(f_{(p_1, \ldots, p_n)} = gf_{(0, \ldots, 0)}g^{-1}\), \(h_{(p_1, \ldots, p_n)} = gh_{(0, \ldots, 0)}g^{-1}\). Then we define two homeomorphisms \(f, h : (0, s) \to (0, s)\) by
\[
f(x) = \begin{cases} f_{(p_1, \ldots, p_n)}(x) & \text{for } x \in U_{(p_1, \ldots, p_n)}, \\ x & \text{for } x \in \overline{\iota(\mathbb{Z}^n)} , \end{cases}
\]
\[
h(x) = \begin{cases} h_{(p_1, \ldots, p_n)}(x) & \text{for } x \in U_{(p_1, \ldots, p_n)}, \\ x & \text{for } x \in \overline{\iota(\mathbb{Z}^n)} . \end{cases}
\]
From the above definitions, we see that \(fh = hf\) and \(fg = gf\), \(hg = gh\) for all \(g \in G\). Thus \(f\), \(h\) and \(G\) generate a \(G \times \mathbb{Z}^2\) action on \((0, s)\). It is not difficult to check that this action is topologically transitive. Since \((0, s)\) and \(\mathbb{R}\) are homeomorphic, we also obtain a topologically transitive and faithful action of \(G \times \mathbb{Z}^2\) on \(\mathbb{R}\). \(\blacksquare\)

Theorem 1.1, announced in Section 1, is more general than Proposition 2.4 and its proof relies on different ideas.

Proof of Theorem 1.1. Suppose that \(G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}\) where \(k > 1\) and each \(N_i/N_{i+1}\) is infinite cyclic. The proof is by induction on \(k\). The assertion holds when \(k = 2\), since the abelian case is given by Proposition 2.1, and in the nonabelian case, the group \(\langle a, b \mid aba^{-1} = b^n \rangle\) has a faithful minimal orientation preserving action on the line (see Proposition 3.1 below). It remains to deduce the result for \(G_k\) assuming it true for \(G_{k-1}\).

Let \(a \in G_k \setminus G_{k-1}\). We let \(a\) act on the line by the unit translation \(\phi(a) : x \mapsto x + 1\). By hypothesis, \(G_{k-1}\) has a faithful topologically transitive orientation preserving action on the open interval \((0, 1)\), which we extend to an action \(\phi\) on the closed interval \([0, 1]\) by fixing the endpoints. Then extend...
this action of $G_{k-1}$ to the line by setting, for each $i \in \mathbb{Z}$,
\[
\phi(b)(x) = \phi(a^{-i}ba^i)(x - i) + i
\]
for all $x \in [i, i+1]$ and $b \in G_{k-1}$. We then define $\phi$ on $G_k$ by setting
\[
\phi(a^ib) = (\phi(a))^i \phi(b)
\]
for all $i \in \mathbb{Z}$ and $b \in G_{k-1}$. It is easy to verify that $\phi$ is a continuous group action and that it is faithful, topologically transitive and orientation preserving. Indeed, these claims are all obvious except possibly the fact that $\phi$ is an action, which is a calculation. Consider two arbitrary elements $g_1 = a^{l_1}b_1$, $g_2 = a^{l_2}b_2$ in $G_k$, where $b_1, b_2 \in G_{k-1}$. We must show that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$. For all $x \in [i, i+1]$ we have
\[
\phi(g_1g_2)(x) = \phi(a^{l_1}b_1a^{l_2}b_2)(x) = \phi(a^{l_1+l_2}b_1a^{l_2}b_2)(x)
\]
\[
= \phi(a^{-i}a^{-l_2}b_1a^{l_2}b_2^i)(x - i) + i + l_1 + l_2,
\]
while
\[
\phi(g_1)\phi(g_2)(x) = \phi(a^{l_1}b_1)\phi(a^{l_2}b_2)(x) = \phi(a^{l_1}b_1)(\phi(a^{-i}b_2a^i)(x - i) + i + l_2)
\]
\[
= \phi(a^{-i-l_2}b_1a^i)\phi(a^{-i}b_2a^i)(x - i) + i + l_1 + l_2
\]
\[
= \phi(a^{-i-l_2}b_1a^{l_2}b_2a^i)(x - i) + i + l_1 + l_2,
\]
as required. \end{proof}

3. Higher transitivity. In this section, we first give two examples: of minimal but not pseudo-2-transitive and of pseudo-2-transitive but not pseudo-3-transitive solvable subgroups of $\text{Homeo}_+ (\mathbb{R})$. We use $\mathbb{R}_+$ and $\mathbb{R}_-$ to denote the sets of positive and of negative numbers respectively.

**Proposition 3.1.** Let $T, S : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x + 1$ and $S(x) = \alpha x$ for some $\alpha > 1$ and all $x \in \mathbb{R}$. Then the solvable group $G = \langle T, S \rangle$ is minimal but is not pseudo-2-transitive.

**Proof.** An easy computation shows that $S^{-m}T^n(x) = x + \alpha^{-n}$ for all $n \in \mathbb{N}$. Define $L_{\alpha^{-n}} : \mathbb{R} \to \mathbb{R}$ by $L_{\alpha^{-n}}(x) = x + \alpha^{-n}$ for all $x \in \mathbb{R}$. Then these $L_{\alpha^{-n}}$ belong to $G$. For any nonempty open interval $U \subset \mathbb{R}$, choose an $n' \in \mathbb{N}$ such that $\alpha^{-n'} < \text{diam}(U)$. So for any $x \in \mathbb{R}$, there must exist some $m \in \mathbb{N}$ such that $x + ma^{-n'} \in U$, that is, $L_{\alpha^{-n'}}(x) \in U$. Thus $G$ is minimal.

Now we show that $G$ is not pseudo-2-transitive. In fact, for any two different points $x, y \in \mathbb{R}$, let $d = |x - y|$. Then for any $g \in G$, from the definitions of $T$ and $S$, we see that there are some $n \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ such that $g(x) = \alpha^n x + \beta$ and $g(y) = \alpha^n y + \beta$. Thus $|g(x) - g(y)| = |\alpha^n x - \alpha^n y| = \alpha^n d$. It follows that the set $\{|g(x) - g(y)| : g \in G\}$ is not dense in the set of positive real numbers $\mathbb{R}_+$. This implies that $G$ is not pseudo-2-transitive. \end{proof}

**Lemma 3.2.** Given two numbers $\alpha > 0$, $\beta > 0$, let $K$ denote $\mathbb{R}_+$ or $\mathbb{R}_-$ and let $M_\alpha, M_\beta : K \to K$ be defined by $M_\alpha(x) = \alpha x$ and $M_\beta(x) = \beta x$ for
all \(x \in K\). If \(\log(\alpha)\) and \(\log(\beta)\) are rationally independent, then the action of the group \(G = \langle M_\alpha, M_\beta \rangle\) on \(K\) is minimal.

Proof. Suppose that \(K = \mathbb{R}_+\). Let \(a = \log(\alpha)\) and \(b = \log(\beta)\). It is easy to see that the action of \(G\) on \(\mathbb{R}_+\) is topologically conjugate to the action of \(G' = \langle L_\alpha, L_\beta \rangle\) via the homeomorphism \(h : \mathbb{R}_+ \to \mathbb{R}, x \mapsto \log(x)\). Since \(a\) and \(b\) are rationally independent, from Remark 2.2, we find that \(G\) is minimal. Similarly, the conclusion also holds when \(K = \mathbb{R}_-\). \(\blacksquare\)

Proposition 3.3. Let \(T, M_\alpha, M_\beta\) be defined by \(T(x) = x + 1, M_\alpha(x) = \alpha x\) and \(M_\beta(x) = \beta x\) for all \(x \in \mathbb{R}\). If \(\alpha > 1, \beta > 1\), and \(\log(\alpha)\) and \(\log(\beta)\) are rationally independent, then the solvable group \(G = \langle T, M_\alpha, M_\beta \rangle\) is pseudo-2-transitive but is not pseudo-3-transitive.

Proof. For any two different points \(x < y \in \mathbb{R}\) and any two nonempty open intervals \(U \leq V\) contained in \(\mathbb{R}\), we will show that there is some \(g \in G\) such that \(g(x) \in U\) and \(g(y) \in V\).

Since \(U \leq V\), there are two intervals \([a_1, b_1] \subseteq U\) and \([a_2, b_2] \subseteq V\) such that \(b_1 < a_2\) and \(|a_1 - b_1| = |a_2 - b_2|\). Let \(c = |a_1 - b_1| = |a_2 - b_2|\) and \(d = |x - y|\). From the proof of Proposition 3.1, we see that the set of translations \(\{L_{\alpha^{-n}} : n \in \mathbb{N}\}\) lies in \(G\). Since \(\alpha > 1\), there is some \(n' \in \mathbb{N}\) such that \(\alpha^{-n'} < \min\{c/6, a_2 - b_1\}\). For the translation \(L_{\alpha^{-n'}}\), there is some \(p \in \mathbb{Z}\) such that

\[
L^{p}_{\alpha^{-n'}}(b_1) < 0 < L^{p}_{\alpha^{-n'}}(a_2)
\]

and

\[
||L^{p}_{\alpha^{-n'}}(b_1)| - |L^{p}_{\alpha^{-n'}}(a_2)|| = ||b_1 + p\alpha^{-n'}| - |a_2 + p\alpha^{-n'}|| \leq \alpha^{-n'}.
\]

Define \(a'_i = L^{p}_{\alpha^{-n'}}(a_i)\) and \(b'_i = L^{p}_{\alpha^{-n'}}(b_i)\) for \(i = 1, 2\). From the inequalities above, we have

\[
|b'_2 + a'_1| = |b'_1 + a'_2| \leq \alpha^{-n'} < c/6.
\]

Next choose \(n'' \in \mathbb{N}\) such that

\[
\alpha^{-n''} < \min \left\{ \frac{d}{4}, \frac{c}{4}, \frac{4}{d} \left( \frac{c}{6} - a'_1 \right) \right\}^{-1}
\]

For \(L_{\alpha^{-n''}}\), there exists some \(q \in \mathbb{Z}\) such that

\[
L^{q}_{\alpha^{-n''}}(x) < 0 < L^{q}_{\alpha^{-n''}}(y)
\]

and

\[
||L^{q}_{\alpha^{-n''}}(x)| - |L^{q}_{\alpha^{-n''}}(y)|| = ||x + q\alpha^{-n''}| - |y + q\alpha^{-n''}|| \leq \alpha^{-n''}.
\]

Let \(x' = L^{q}_{\alpha^{-n''}}(x)\) and \(y' = L^{q}_{\alpha^{-n''}}(y)\). Then it is not hard to see that

\[
|x' + y'| \leq \alpha^{-n''} < \frac{c}{6} \left[ \frac{4}{d} \left( \frac{c}{6} - a'_1 \right) \right]^{-1} \quad \text{and} \quad |x'| > \frac{d}{4}.
\]
Since the action of $\langle M_\alpha, M_\beta \rangle$ on $\mathbb{R}_-$ is minimal from Lemma 3.2, there are $s, t \in \mathbb{Z}$ such that

\[(3.3) \quad \left| M_\alpha^s M_\beta^t(x') - \frac{a_1' + b_1'}{2} \right| = \left| \alpha^s \beta^t x' - \frac{a_1' + b_1'}{2} \right| < \frac{c}{6},\]

which implies that

\[(3.4) \quad \alpha^s \beta^t < \frac{1}{|x'|} \left( \frac{c}{6} + \left| \frac{a_1' + b_1'}{2} \right| \right) < \frac{4}{d} \left( \frac{c}{6} - a_1' \right).\]

Then by the conditions (3.1)–(3.4) we have

\[
\begin{align*}
\left| M_\alpha^s M_\beta^t(y') - \frac{a_2' + b_2'}{2} \right| &= \left| \alpha^s \beta^t (-y') + \frac{a_2' + b_2'}{2} \right| \\
&\leq \left| \alpha^s \beta^t (-y') - \alpha^s \beta^t x' \right| + \left| \alpha^s \beta^t x' - \frac{a_1' + b_1'}{2} \right| + \left| \frac{a_1' + b_1'}{2} + \frac{a_2' + b_2'}{2} \right| \\
&\leq \alpha^s \beta^t |x' + y'| + \left| \alpha^s \beta^t x' - \frac{a_1' + b_1'}{2} \right| + \left| \frac{a_1' + b_2'}{2} + \frac{b_1' + a_2'}{2} \right| \\
&\leq \frac{4}{d} \left( \frac{c}{6} - a_1' \right) \cdot \alpha^{-n''} + \frac{c}{6} + |b_1' + a_2'| \\
&< \frac{4}{d} \left( \frac{c}{6} - a_1' \right) \cdot \frac{c}{6} \cdot \left[ \frac{4}{d} \left( \frac{c}{6} - a_1' \right) \right]^{-1} + \frac{c}{6} + c = \frac{c}{2}.
\end{align*}
\]

So $M_\alpha^s M_\beta^t(x') \in [a_1', b_1']$ and $M_\alpha^s M_\beta^t(y') \in [a_2', b_2']$, thus $L_{\alpha^{-n''}, M_\alpha^s M_\beta^t L_q^{\alpha^{-n''}}(x)} \in [a_1, b_1] \subseteq U$ and $L_{\alpha^{-n''}, M_\alpha^s M_\beta^t L_q^{\alpha^{-n''}}(y)} \in [a_2, b_2] \subseteq V$. This implies that $G$ is pseudo-2-transitive.

Now, we show that $G$ is not pseudo-3-transitive. In fact, for any $g \in G$, there are some $n, m \in \mathbb{Z}$ and $r \in \mathbb{R}$ such that $g(x) = \alpha^n \beta^m x + r$ for all $x \in \mathbb{R}$. Thus for any three points $x < y < z \in \mathbb{R}$, we have

\[(3.5) \quad \frac{|g(x) - g(y)|}{|g(y) - g(z)|} = \frac{|\alpha^n \beta^m x - \alpha^n \beta^m y|}{|\alpha^n \beta^m y - \alpha^n \beta^m z|} = \frac{|x - y|}{|y - z|} \quad \text{for any } g \in G.
\]

Now choose points $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ in $\mathbb{R}$ such that

\[
\frac{a_2 - b_1}{b_3 - a_2} > 100 \frac{|x - y|}{|y - z|}.
\]

Then from (3.5) we see that for all $g \in G$, the containments $g(x) \in (a_1, b_1)$, $g(y) \in (a_2, b_2)$ and $g(z) \in (a_3, b_3)$ cannot occur simultaneously. This shows that the $G$-action is not pseudo-3-transitive. \(\blacksquare\)

Recall that a measure $\mu$ on $\mathbb{R}$ is called quasi-invariant for a group $G \subset \text{Homeo}_+(\mathbb{R})$ if for each $g \in G$ there is a positive constant $\alpha(g)$ such that $g_* \mu = \alpha(g) \mu$ (see [18]).
Proposition 3.4. Suppose that $G \subseteq \text{Homeo}_+(\mathbb{R})$ is a group and has a quasi-invariant measure $\mu$ on $\mathbb{R}$ which is finite on compact sets. Then $G$ is not pseudo-3-transitive.

Proof. Assume to the contrary that $G$ is pseudo-3-transitive and $\mu$ is a $G$-quasi-invariant measure on $\mathbb{R}$. First we claim that $\text{supp}(\mu) = \mathbb{R}$. In fact, fix an interval $[a, b] \subset \mathbb{R}$ such that $\mu([a, b]) > 0$. For any nonempty open interval $U \subset \mathbb{R}$, since $G$ is pseudo-3-transitive, there is some $g \in G$ such that $g(U) \cap (-\infty, a - 1) \neq \emptyset$ and $g(U) \cap (b + 1, \infty) \neq \emptyset$. Thus $g(U) \supseteq [a, b]$. Since $\mu$ is quasi-invariant for $G$, there exists a number $\alpha(g) > 0$ such that

$$\mu(U) = \alpha(g)^{-1}\mu(g(U)) \geq \alpha(g)^{-1}\mu([a, b]) > 0.$$ 

By the arbitrariness of $U$, we see that $\text{supp}(\mu) = \mathbb{R}$.

Thus for any three points $x < y < z \in \mathbb{R}$, we have $\mu([x, y]) > 0$ and $\mu([y, z]) > 0$. Then for any $g \in G$,

$$\frac{\mu([gx, gy])}{\mu([gy, gz])} = \frac{\alpha(g)\mu([x, y])}{\alpha(g)\mu([y, z])} = \frac{\mu([x, y])}{\mu([y, z])},$$

which contradicts the assumption that $G$ is pseudo-3-transitive. 

From Theorem 4.4 in [18], we see that each polycyclic solvable subgroup of $\text{Homeo}_+(\mathbb{R})$ must have a quasi-invariant measure $\mu$ on $\mathbb{R}$. Thus we get the following

Corollary 3.5. No polycyclic solvable subgroup of $\text{Homeo}_+(\mathbb{R})$ is pseudo-3-transitive.

Lemma 3.6. Let $G$ be a solvable subgroup of $\text{Homeo}_+(\mathbb{R})$. If $G$ is pseudo-$4k$-transitive on $\mathbb{R}$ for some $k \in \mathbb{N}$, then the commutator subgroup $[G, G]$ is at least pseudo-$k$-transitive.

Proof. Suppose $U_1 \subseteq \cdots \subseteq U_k$ and $V_1 \subseteq \cdots \subseteq V_k$ are arbitrary nonempty open intervals in $\mathbb{R}$. To see that $[G, G]$ is pseudo-$k$-transitive, we will show that there is some $f^{-1}g^{-1}fg \in [G, G]$ such that $f^{-1}g^{-1}fg(U_i) \cap V_i \neq \emptyset$ for all $i = 1, \ldots, k$. Without loss of generality, we can suppose that the length of each $U_i$ and $V_i$ is finite. Then there exists an $x \in \mathbb{R}$ such that $(\bigcup_{i=1}^k (U_i \cup V_i)) \cap [x, \infty) = \emptyset$. In the following, for an open interval $J = (a, b) \subset \mathbb{R}$, define $J^+(\varepsilon) = (b, b + \varepsilon)$ and $J^-(\varepsilon) = (a - \varepsilon, a)$ for some small positive number $\varepsilon$. For brevity, we always use $J^+$ and $J^-$ instead of $J^+(\varepsilon)$ and $J^-(\varepsilon)$ respectively.

Now take $3k$ pairwise disjoint open intervals of $(x + 1, \infty)$, $A_1 \subseteq \cdots \subseteq A_k \subseteq B_1 \subseteq \cdots \subseteq B_k \subseteq C_1 \subseteq \cdots \subseteq C_k$, such that the distance between any two consecutive intervals equals $1$. Then there is some sufficiently small $\varepsilon$ such that $A_i^+ \cap A_{i+1}^- = \emptyset$, $B_i^+ \cap B_{i+1}^- = \emptyset$ and $C_i^+ \cap C_{i+1}^- = \emptyset$ for all $i = 1, \ldots, k - 1$, and $A_k^+ \cap B_1^- = \emptyset$ and $B_k^+ \cap C_1^- = \emptyset$. For the following two
sequences of open intervals:

\[ U_1 \leq U_1 \leq U_2 \leq \cdots \leq U_k \leq B_1 \leq B_1 \leq B_2 \leq \cdots \leq B_k \leq B_k, \]

\[ A_i^{-} \leq A_i^{+} \leq A_2^{-} \leq A_2^{+} \leq \cdots \leq A_k^{-} \leq A_k^{+} \leq C_1^{-} \leq C_1^{+} \leq C_2^{-} \leq C_2^{+} \leq \cdots \leq C_k^{-} \leq C_k^{+}, \]

since \( G \) is pseudo-4\( k \)-transitive, there is some \( g \in G \) such that

\[ g(U_i) \cap A_i^{-} \neq \emptyset, \quad g(U_i) \cap A_i^{+} \neq \emptyset \quad \text{for all} \quad i = 1, \ldots, k, \]

\[ g(B_i) \cap C_i^{-} \neq \emptyset, \quad g(B_i) \cap C_i^{+} \neq \emptyset \quad \text{for all} \quad i = 1, \ldots, k. \]

From (3.6) and (3.7), it is not hard to see that

\[ A_i \subseteq g(U_i) \quad \text{and} \quad C_i \subseteq g(B_i) \quad \text{for each} \quad i = 1, \ldots, k. \]

Similarly, for the sequences

\[ V_1 \leq V_1 \leq V_2 \leq \cdots \leq V_k \leq A_1 \leq A_1 \leq A_2 \leq \cdots \leq A_k \leq A_k, \]

\[ B_1^{-} \leq B_1^{+} \leq B_2^{-} \leq B_2^{+} \leq \cdots \leq B_k^{-} \leq B_k^{+} \leq C_1^{-} \leq C_1^{+} \leq C_2^{-} \leq C_2^{+} \leq \cdots \leq C_k^{-} \leq C_k^{+}, \]

there is some \( f \in G \) such that

\[ f(V_i) \cap B_i^{-} \neq \emptyset, \quad f(V_i) \cap B_i^{+} \neq \emptyset \quad \text{for all} \quad i = 1, \ldots, k, \]

\[ f(A_i) \cap C_i^{-} \neq \emptyset, \quad f(A_i) \cap C_i^{+} \neq \emptyset \quad \text{for all} \quad i = 1, \ldots, k. \]

Thus we have

\[ B_i \subseteq f(V_i) \quad \text{and} \quad C_i \subseteq f(A_i) \quad \text{for all} \quad i = 1, \ldots, k. \quad (3.9) \]

From (3.8) and (3.9), we see that for all \( i = 1, \ldots, k, \)

\[ fg(U_i) \supseteq f(A_i) \supseteq C_i \quad \text{and} \quad gf(V_i) \supseteq g(B_i) \supseteq C_i. \]

Hence \( fg(U_i) \cap g(V_i) \neq \emptyset, \) that is, \( f^{-1}g^{-1}fg(U_i) \cap V_i \neq \emptyset \) for each \( i \in \{1, \ldots, k\}. \) This completes the proof. \( \blacksquare \)

**Proposition 3.7.** Suppose \( G \) is a solvable subgroup of \( \text{Homeo}_+(\mathbb{R}) \) and its derived length is \( n. \) Then \( G \) is at most pseudo-(4\( n \)-1)-transitive.

**Proof.** Assume to the contrary that \( G \) is pseudo-4\( n \)-transitive. From Lemma 3.6, the commutator subgroup \( G^{1} = [G, G] \) is pseudo-4\( n \)-1-transitive and \( G^{2} = [G^{1}, G^{1}] \) is pseudo-4\( n \)-2-transitive. Continuing this process, we see that \( G^{n} = [G^{n-1}, G^{n-1}] \) is topologically transitive on \( \mathbb{R}. \) However, as the derived length of \( G \) is \( n, \) \( G^{n} = \{e\}. \) This is a contradiction. \( \blacksquare \)

Ping-pong game is an important technique in determining the existence of free subgroups or free subsemigroups of some groups (see for example [13]). Let \( G \) be a group acting on \( \mathbb{R}. \) Suppose \( I, J \) and \( K \) are three closed intervals in \( \mathbb{R} \) such that \( I \) and \( J \) are disjoint, \( I \subset K \) and \( J \subset K. \) If there are two elements \( g_1, g_2 \in G \) such that \( g_1(K) \subset I \) and \( g_2(K) \subset J, \) then the pair \((g_1 : K \to I; g_2 : K \to J)\) is called a ping-pong game. It is not difficult to check that the semigroup \( S \) generated by such \( g_1 \) and \( g_2 \) is free. It is
well known that a finitely generated nilpotent group does not contain a free subsemigroup.

**Proposition 3.8.** No finitely generated nilpotent subgroup of the group $\text{Homeo}_+ (\mathbb{R})$ is pseudo-2-transitive.

**Proof.** Assume to the contrary that there is a finitely generated nilpotent subgroup $G$ of $\text{Homeo}_+ (\mathbb{R})$ which is pseudo-2-transitive. Let $I = [0,1]$ and $J = [2,3]$ since $G$ is pseudo-2-transitive, there are $g_1, g_2 \in G$ such that $g_1(I) \cap (-\infty, -1) \neq \emptyset$, $g_1(I) \cap (4, \infty) \neq \emptyset$, $g_2(J) \cap (-\infty, -1) \neq \emptyset$ and $g_2(J) \cap (4, \infty) \neq \emptyset$. Let $I' = g_1^{-1}([-1, 4])$ and $J' = g_2^{-1}([-1, 4])$. Then $(g_1^{-1} : [-1, 4] \to I'; g_2^{-1} : [-1, 4] \to J')$ is a ping-pong game, and thus $G$ contains a free semigroup $S$ which is generated by $g_1^{-1}$ and $g_2^{-1}$. This contradicts the fact that $G$ is a finitely generated nilpotent group. ■

**Acknowledgements.** E. H. Shi was supported by the National Natural Science Foundation of China (No. 10801103) and by the Natural Sciences Fund for Colleges and Universities in Jiangsu Province (No. 08KJB110010).

L. Z. Zhou was supported in part by the National Natural Science Foundation of China (No. 10871142).

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Received 18 August 2008;
revised 20 December 2008 (5081)