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THE RUELLE ROTATION OF KILLING VECTOR FIELDS

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Abstract. We present an explicit formula for the Ruelle rotation of a nonsingular Killing vector field of a closed, oriented, Riemannian 3-manifold, with respect to Riemannian volume.

Let M be a closed, oriented Riemannian 3-manifold and X be a nonsingular Killing vector field on M with trivial normal bundle. The plane bundle E orthogonal to X is then spanned by two globally defined orthogonal unit vector fields Y and Z such that $\{X(x), Y(x), Z(x)\}$ is a positively oriented basis of the tangent space at $x \in M$. Once we have chosen the unit vector field Z orthogonal to X, there is only one choice of a unit vetor field Y such that $\{X, Y, Z\}$ is a positively oriented orthogonal frame on M. The flow $(\phi_t)_{t\in\mathbb{R}}$ of X is a one-parameter group of isometries of M, and thus $\phi_{t*}(x)(E_x) = E_{\phi_t(x)}$ for every $t \in \mathbb{R}$ and $x \in M$. The matrix of $\phi_{t*}(x)|E_x$ with respect to the bases $\{Y(x), Z(x)\}$ and $\{Y(\phi_t(x)), Z(\phi_t(x))\}$ is a rotation, denoted by $f(t, x)$. The resulting function $f : \mathbb{R} \times M \to \mathrm{SO}(2, \mathbb{R})$ is a smooth cocycle of the flow, by the chain rule, and can be lifted to a smooth cocycle $\widetilde{f} : \mathbb{R} \times M \to \mathbb{R}$.

From the ergodic theorem for isometric systems (see [4]), the limit

$$F(x) = \lim_{t\to\infty} \frac{\widetilde{f}(t, x)}{t}$$

exists uniformly for every $x \in M$. If ω is the normalized Riemannian volume, the integral

$$\varrho(X) = \int_M F\omega = \int_M \widetilde{f}(1, \cdot)\omega$$

is the Ruelle rotation number of X with respect to the trivializaton $\{Y, Z\}$ of E. If $\{\overline{Y}, \overline{Z}\}$ is another trivialization of E as above and $\overline{\varrho}(X)$ is the corresponding Ruelle rotation number of X, it follows from Proposition 3.4

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in [2] that

$$\varrho(X) - \overline{\varrho}(X) = \int_M h^*\left(\frac{d\theta}{2\pi}\right) \wedge i_X\omega,$$

where $d\theta/2\pi$ is the natural representative of the standard generator of $H^1(\mathrm{SO}(2,\mathbb{R});\mathbb{Z})$ and $h : M \to \mathrm{SO}(2,\mathbb{R})$ is the smooth function such that the matrix $h(x)$ gives the change of basis from $\{Y(x), Z(x)\}$ to $\{\overline{Y}(x), \overline{Z}(x)\}$ in E_x. Since X preserves the Riemannian volume, $i_X\omega$ is closed. If it is exact, X is called *homologically trivial*, and in this case $\varrho(X) = \overline{\varrho}(X)$, that is, the Ruelle rotation number of X does not depend on the trivialization of E.

David Ruelle defined what we now call the Ruelle rotation in [5] for any nowhere vanishing smooth vector field with trivial normal bundle on a closed, oriented, smooth 3-manifold, with respect to a trivialization of the normal bundle and an invariant Borel probability measure. If the manifold is a homology 3-sphere, then the Ruelle rotation does not depend on the choice of the trivialization of the normal bundle [2].

In this note we present an explicit formula for $\varrho(X)$ and make some remarks. More precisely, we prove the following.

THEOREM. *Let X be a nonsingular Killing vector field with trivial normal bundle E on an oriented, Riemannian, closed 3-manifold M with normalized Riemannian volume element ω. Let $\{Y, Z\}$ be an orthonormal frame trivializing E such that $\{X, Y, Z\}$ is a positively oriented orthogonal frame on M. Then the Ruelle rotation number of X with respect to the given trivialization of E is given by the formula*

$$\varrho(X) = \frac{1}{2\pi}\int_M \langle [X, Z], Y\rangle\omega.$$

Proof. Let $(\phi_t)_{t\in\mathbb{R}}$ be the flow of X. For every $t \in \mathbb{R}$ we have

$$\begin{aligned}
\phi_{t*}(x)Y(x) &= \cos 2\pi\widetilde{f}(t,x)\, Y(\phi_t(x)) + \sin 2\pi\widetilde{f}(t,x)\, Z(\phi_t(x)),\\
\phi_{t*}(x)Z(x) &= -\sin 2\pi\widetilde{f}(t,x)\, Y(\phi_t(x)) + \cos 2\pi\widetilde{f}(t,x)\, Z(\phi_t(x)).
\end{aligned}$$

From [3, p. 235 and p. 245] we have

$$\tau_0^t \circ \phi_{t*}(x) = \exp(t(\nabla_{.}X)_x),$$

where τ_0^t is the parallel translation along the orbit of x from $\phi_t(x)$ to x. So,

$$(*)\qquad \begin{aligned}
\cos 2\pi\widetilde{f}(t,x) &= \langle \exp(t(\nabla_{.}X)_x)Y(x), \tau_0^t(Y(\phi_t(x)))\rangle,\\
\sin 2\pi\widetilde{f}(t,x) &= \langle \exp(t(\nabla_{.}X)_x)Y(x), \tau_0^t(Z(\phi_t(x)))\rangle.
\end{aligned}$$

Differentiating the second equation with respect to t we get

$$\widetilde{f}'(t,x) = \frac{1}{2\pi\langle\phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle}\cdot\frac{d}{dt}\langle \exp(t(\nabla_{.}X)_x)Y(x), \tau_0^t(Z(\phi_t(x)))\rangle$$

for $t \in \mathbb{R}$ with $\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle \neq 0$ and

$$\begin{aligned}
\frac{d}{dt}&\langle \exp(t(\nabla_{.}X)_x)Y(x), \tau_0^t(Z(\phi_t(x)))\rangle \\
&= \langle \exp(t(\nabla_{.}X)_x)(\nabla_{Y(x)}X), \tau_0^t(Z(\phi_t(x)))\rangle \\
&\quad + \left\langle \exp(t(\nabla_{.}X)_x)Y(x), \frac{d}{dt}(\tau_0^t(Z(\phi_t(x))))\right\rangle \\
&= \langle (\tau_0^t \circ \phi_{t*}(x))(\nabla_{Y(x)}X), \tau_0^t(Z(\phi_t(x)))\rangle \\
&\quad + \langle (\tau_0^t \circ \phi_{t*}(x))Y(x), \tau_0^t(\nabla_{X(\phi_t(x))}Z)\rangle \\
&= \langle \phi_{t*}(x)(\nabla_{Y(x)}X), Z(\phi_t(x))\rangle + \langle \phi_{t*}(x)Y(x), \nabla_{X(\phi_t(x))}Z\rangle.
\end{aligned}$$

So we have

$$\widetilde{f}'(t,x) = \frac{\langle \phi_{t*}(x)(\nabla_{Y(x)}X, Z(\phi_t(x))\rangle + \langle \phi_{t*}(x)Y(x), \nabla_{X(\phi_t(x))}Z\rangle}{2\pi\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle}.$$

Since Z has constant unit length, $2\langle \nabla_X Z, Z\rangle = X\langle Z, Z\rangle = 0$. Therefore

$$\nabla_X Z = \langle \nabla_X Z, Y\rangle Y + \frac{\langle \nabla_X Z, X\rangle}{\|X\|^2}X$$

and

$$\langle \phi_{t*}(x)Y(x), \nabla_{X(\phi_t(x))}Z\rangle = \langle \nabla_{X(\phi_t(x))}Z, Y(\phi_t(x))\rangle \cdot \langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle.$$

It follows that

$$\widetilde{f}'(t,x) = \frac{\langle \phi_{t*}(x)(\nabla_{Y(x)}X), Z(\phi_t(x))\rangle}{2\pi\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle} + \frac{1}{2\pi}\langle \nabla_{X(\phi_t(x))}Z, Y(\phi_t(x))\rangle$$

for $t \in \mathbb{R}$ with $\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle \neq 0$. If we differentiate the first equation in $(*)$ with respect to t and use the fact that $\langle \nabla_X Y, Z\rangle = -\langle \nabla_X Z, Y\rangle$, we get

$$\widetilde{f}'(t,x) = -\frac{\langle \phi_{t*}(x)(\nabla_{Y(x)}X), Y(\phi_t(x))\rangle}{2\pi\langle \phi_{t*}(x)Y(x), Z(\phi_t(x))\rangle} + \frac{1}{2\pi}\langle \nabla_{X(\phi_t(x))}Z, Y(\phi_t(x))\rangle$$

for $t \in \mathbb{R}$ with $\langle \phi_{t*}(x)Y(x), Z(\phi_t(x))\rangle \neq 0$. The last two formulas are the same for $t \in \mathbb{R}$ with $\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle \cdot \langle \phi_{t*}(x)Y(x), Z(\phi_t(x))\rangle \neq 0$, because

$$\begin{aligned}
&\langle \phi_{t*}(x)Y(x), Y(\phi_t(x))\rangle \cdot \langle \phi_{t*}(x)(\nabla_{Y(x)}X), Y(\phi_t(x))\rangle \\
&\qquad + \langle \phi_{t*}(x)Y(x), Z(\phi_t(x))\rangle \cdot \langle \phi_{t*}(x)(\nabla_{Y(x)}X), Z(\phi_t(x))\rangle \\
&\quad = \langle \phi_{t*}(x)(\nabla_{Y(x)}X), \phi_{t*}(x)Y(x)\rangle = \langle \nabla_{Y(x)}X, Y(x)\rangle = 0,
\end{aligned}$$

since X is Killing. Now $\phi_{t*}Y = \langle \phi_{t*}Y, Y\rangle Y + \langle \phi_{t*}Y, Z\rangle Z$ and so

$$\frac{1}{\langle \phi_{t*}Y, Y\rangle}\nabla_{\phi_{t*}Y}X = \nabla_Y X + \frac{\langle \phi_{t*}Y, Z\rangle}{\langle \phi_{t*}Y, Y\rangle}\nabla_Z X,$$

from which it follows that

$$\frac{\langle \nabla_{\phi_{t*}Y} X, Z\rangle}{\langle \phi_{t*}Y, Y\rangle} = \langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle = -\frac{\langle \nabla_{\phi_{t*}Y} X, Y\rangle}{\langle \phi_{t*}Y, Z\rangle}$$
$$= \langle [X,Z] - \nabla_X Z, Y\rangle,$$

since X is a Killing vector field. Consequently, for every $t \in \mathbb{R}$ we have

$$\widetilde{f}'(t,x) = \frac{1}{2\pi}\,\langle [X,Z](\phi_t(x)), Y(\phi_t(x))\rangle,$$

and so

$$\widetilde{f}(t,x) = \frac{1}{2\pi}\int_0^t \langle [X,Z](\phi_s(x)), Y(\phi_s(x))\rangle\, ds.$$

Hence

$$F(x) = \lim_{t\to\infty} \frac{1}{2\pi t}\int_0^t \langle [X,Z](\phi_s(x)), Y(\phi_s(x))\rangle\, ds.$$

By Fubini's theorem and the invariance of the Riemannian volume we get

$$\varrho(X) = \frac{1}{2\pi}\int_M \langle [X,Z], Y\rangle \omega,$$

as asserted.

Remark 1. Note that since X is a Killing vector field, we have

$$\langle [X,Z], X\rangle = \langle \nabla_X Z, X\rangle + \langle \nabla_X X, Z\rangle = X\langle Z, X\rangle = 0,$$
$$\langle [X,Z], Z\rangle = \langle \nabla_X Z, Z\rangle - \langle \nabla_Z X, Z\rangle = \frac{1}{2}X(\|Z\|^2) = 0.$$

So $[X,Z] = \langle [X,Z], Y\rangle Y$, and if for every $x \in M$ we let

$$\varepsilon(x) = \begin{cases} +1 & \text{if } \omega_x(X(x), [X,Z](x), Z(x)) > 0, \\ -1 & \text{if } \omega_x(X(x), [X,Z](x), Z(x)) < 0, \\ 0 & \text{if } [X,Z](x) = 0, \end{cases}$$

then

$$\varrho(X) = \frac{1}{2\pi}\int_M (\varepsilon \cdot \|[X,Z]\|)\omega.$$

If η is the dual 1-form of Z with respect to the Riemannian metric, then it is not hard to see that $\|X\| \cdot \eta \wedge d\eta = \mathrm{vol}(M)\langle [X,Z], Y\rangle \omega$. Therefore

$$\varrho(X) = \frac{1}{2\pi\,\mathrm{vol}(M)}\int_M \|X\| \cdot \eta \wedge d\eta.$$

Remark 2. If $H^1(M;\mathbb{Z}) = 0$, the function F does not depend on the trivialization $\{Y,Z\}$ of E. Indeed, let $\{Y_1, Z_1\}$ and $\{Y_2, Z_2\}$ be two trivializations of E as at the beginning. There exists a smooth function

$g : M \to \mathrm{SO}(2,\mathbb{R})$ such that $Y_2(x) = g(x)(Y_1(x))$ and $Z_2(x) = g(x)(Z_1(x))$ for every $x \in M$. Since $H^1(M;\mathbb{Z}) = 0$, there is a smooth function $\theta : M \to \mathbb{R}$ such that $g(x)$ is the rotation by the angle $\theta(x)$. Thus,

$$\begin{aligned} Y_2(x) &= \cos\theta(x)\cdot Y_1(x) + \sin\theta(x)\cdot Z_1(x), \\ Z_2(x) &= -\sin\theta(x)\cdot Y_1(x) + \cos\theta(x)\cdot Z_1(x), \end{aligned}$$

and

$$\begin{aligned} &\langle [X, Z_2], Y_2\rangle \\ &\quad = \langle -\sin\theta[X,Y_1] - X(\sin\theta)Y_1 + \cos\theta[X,Z_1] + X(\cos\theta)Z_1, \cos\theta Y_1 + \sin\theta Z_1\rangle \\ &\quad = \langle [X,Z_1],Y_1\rangle - X(\theta) = \langle [X,Z_1],Y_1\rangle - \frac{\partial(\theta\circ\phi)}{\partial t}. \end{aligned}$$

If f_1 and f_2 are the corresponding cocycles, we get

$$\widetilde{f}_1' - \widetilde{f}_2' = \frac{1}{2\pi}\cdot\frac{\partial(\theta\circ\phi)}{\partial t}$$

and

$$\widetilde{f}_1(t,x) - \widetilde{f}_2(t,x) = \frac{1}{2\pi}\,[\theta(\phi_t(x)) - \theta(x)],$$

that is, the two cocycles are cohomologous, and therefore $F_1 = F_2$.

According to the topological classification of nonsingular Killing vector fields on Riemannian 3-manifolds, given in [1], if M is a homology 3-sphere, the orbits of X are periodic and M is a Seifert manifold. If $T(x) > 0$ denotes the period of the orbit of x, then

$$F(x) = \frac{1}{2\pi T(x)}\int_0^{T(x)} \langle [X,Z](\phi_s(x)), Y(\phi_s(x))\rangle\, ds$$

and F is smooth except at a finite number of orbits, the exceptional fibers of the Seifert fibration.

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