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A CLASSIFICATION OF SYMMETRIC ALGEBRAS OF STRICTLY CANONICAL TYPE

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Abstract. In continuation of our article in Colloq. Math. 116.1, we give a complete description of the symmetric algebras of strictly canonical type by quivers and relations, using Brauer quivers.

Introduction and the main result. Throughout the article, K will denote a fixed algebraically closed field. By an *algebra* is meant an associative, finite-dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$.

In [21] we describe the structure and homological properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type, that is, the orbit algebras $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra of a branch extension (equivalently, branch coextension) B of a canonical algebra (in the sense of Ringel [31]) and G is an infinite cyclic group generated by a strictly positive automorphism of $\widehat{B}$. Here, we are concerned with the classification of the symmetric algebras of strictly canonical type. Recall that an algebra A is called *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules. By a classical result due to Nakayama (see [23], [24]), an algebra A is symmetric if and only if there exists a nondegenerate, symmetric, associative, K-bilinear form $(-,-) : A \times A \to K$.

The classical examples of symmetric algebras are provided by the group algebras KG of finite groups G, or more generally their blocks. In the representation theory of blocks of group algebras, a prominent role is played by the Brauer tree algebras. Namely, by the deep theorem due to Dade, Janusz and Kupisch (see [1], [13], [18], [19], [20]), every block of finite representation type of a group algebra is Morita equivalent to a Brauer tree algebra. More-

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over, by results of Gabriel and Riedtmann [15] and Rickard [30], the Brauer tree algebras are exactly the symmetric algebras which are stably equivalent and derived equivalent to the symmetric Nakayama algebras.

In the representation theory of symmetric algebras an important role is played by the trivial extension algebras. The *trivial extension*

$$T(B) = B \ltimes D(B)$$

of an algebra B by its minimal injective cogenerator B-B-bimodule $D(B) = \mathrm{Hom}_K(B, K)$ is the K-algebra whose K-linear space is that of $B \oplus D(B)$, and whose multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in B$, $f, g \in D(B)$. Then $T(B)$ is a symmetric algebra. Namely, the K-bilinear form $(-,-) : T(B) \times T(B) \to K$ defined by

$$((a, f), (b, g)) = f(b) + g(a)$$

for $a, b \in B$ and $f, g \in D(B)$ is nondegenerate, symmetric and associative. The trivial extension algebras $T(B)$ are given by splittable Hochschild extensions

$$0 \to D(B) \to T(B) \to B \to 0,$$

corresponding to the trivial elements of the Hochschild cohomology spaces $H^2(B, D(B))$ [12, Chapter IX]. We also note that the trivial extension algebras $T(B)$ are the orbit algebras $\widehat{B}/(\nu_{\widehat{B}})$ of the repetitive algebras $\widehat{B}$ of B with respect to the action of the infinite cyclic group generated by the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$. The representation theory of the trivial extension algebras has been extensively developed (see [2], [3], [5]–[8], [14], [16], [17], [22], [25], [26], [28]–[30], [34]–[36], [38]–[40] for some research in this direction).

We will introduce families $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$, $\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, $\Gamma^{(k)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, $\Gamma^{(k,1)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}, \mu)$, $\Gamma^{(k,2)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}, \mu, \xi)$, $\Gamma^{(1,1)}(\boldsymbol{p}, \boldsymbol{T_p})$, $\Gamma^{(1,2)}(\boldsymbol{p}, \boldsymbol{T_p})$ of self-injective algebras, depending on weight sequences $\boldsymbol{p}$ and $\boldsymbol{q}$ of positive integers, parameter sequences $\boldsymbol{\lambda}$, μ, ξ of pairwise different elements of the projective line $\mathbb{P}_1(K)$, collections $\boldsymbol{T_p}$ and $\boldsymbol{T^*_q}$ of Brauer trees with distinguished extreme edges, and positive integers k; we prove the following theorem.

Main Theorem. *Let A be a basic, indecomposable, finite-dimensional algebra over an algebraically closed field K. The following statements are equivalent*:

(i) *A is a symmetric algebra of strictly canonical type.*

(ii) *A is isomorphic to a bound quiver algebra of one of the forms*

(1) $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$;

(2) $\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ *for* $\mathrm{char}\, K = 2$;

(3) $\Gamma^{(k)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, $\Gamma^{(k,1)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}, \mu)$, $\Gamma^{(1,1)}(\boldsymbol{p}, \boldsymbol{T_p})$;
(4) $\Gamma^{(k,2)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}, \mu, \xi)$, $\Gamma^{(1,2)}(\boldsymbol{p}, \boldsymbol{T_p})$ *for* $\operatorname{char} K \neq 2$.

We refer to [21] for needed facts on selfinjective algebras of strictly canonical type, and to the books [4], [32], [33] for background on the representation theory of algebras.

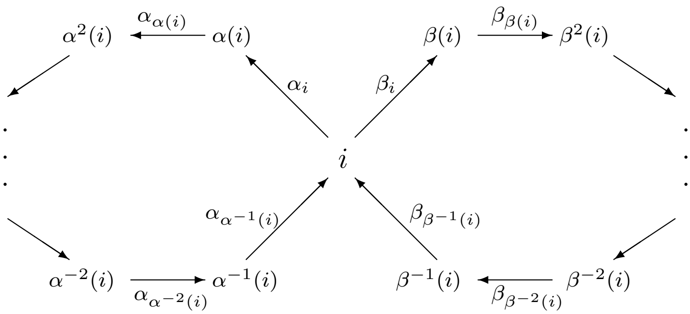
1. Brauer tree algebras. In this section we introduce the Brauer tree algebras of multiplicity one (see [1] and [34] for more general classes of Brauer tree algebras). In this paper, by a *Brauer tree* T we mean a finite connected tree, with at least one edge, where for each vertex there is a fixed circular order on the edges adjacent to it. We draw T in a plane and assume that the edges adjacent to a given vertex are clockwise ordered. A vertex of T having only one neighbour in T is said to be an *extreme vertex* of T. Moreover, the unique edge of T containing an extreme vertex is said to be an *extreme edge* of T.

We associate to a Brauer tree T a *Brauer quiver* Q_T defined as follows. The vertices of Q_T are the edges of T and there is an arrow $i \to j$ in Q_T if and only if j is the direct successor of i in the order around some vertex of T (to which i and j are both adjacent). Hence, the quiver Q_T is a union of oriented cycles corresponding to the vertices of T, and every vertex of Q_T belongs to exactly two cycles. Note that, for every extreme vertex of T, there is a loop in Q_T. The cycles of Q_T may be divided into two camps: α-camps and β-camps such that two cycles of Q_T having nontrivial intersection belong to different camps. In particular, for each vertex i of Q_T, we have

- $i \xrightarrow{\alpha_i} \alpha(i)$, the arrow in the α-camp of Q_T starting at i,
- $i \xrightarrow{\beta_i} \beta(i)$, the arrow in the β-camp of Q_T starting at i,

and the cycles

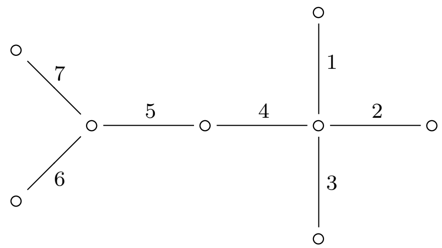
$$A_i = \alpha_i \alpha_{\alpha(i)} \ldots \alpha_{\alpha^{-1}(i)}, \qquad B_i = \beta_i \beta_{\beta(i)} \ldots \beta_{\beta^{-1}(i)},$$



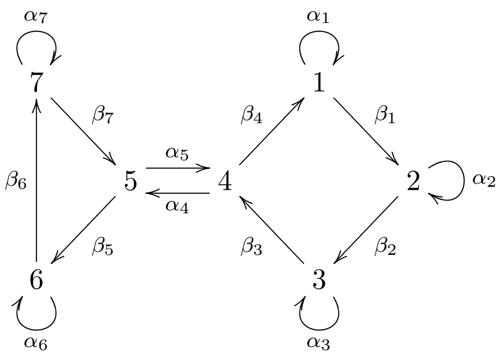
We associate to a Brauer tree T the *Brauer tree algebra* $A(T) = KQ_T/I_T$, where I_T is the ideal of the path algebra KQ_T of Q_T generated by the elements

$$A_i - B_i,\ \beta_{\beta^{-1}(i)}\alpha_i,\ \alpha_{\alpha^{-1}(i)}\beta_i \qquad \text{for all vertices } i \text{ of } Q_T.$$

EXAMPLE 1.1. Let T be a Brauer tree of the form



Then the Brauer quiver Q_T of T is of the form



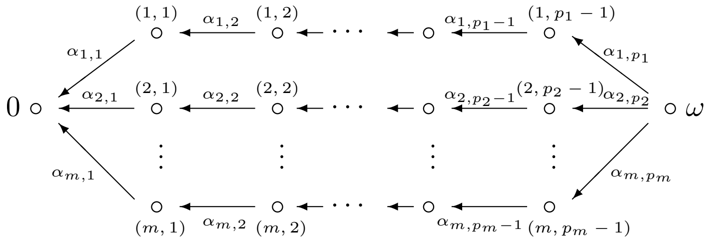
and $A(T) = KQ_T/I_T$, where I_T is the ideal of the path algebra KQ_T of Q_T generated by the elements $\alpha_1\beta_1$, $\beta_1\alpha_2$, $\alpha_2\beta_2$, $\beta_2\alpha_3$, $\alpha_3\beta_3$, $\beta_3\alpha_4$, $\alpha_5\beta_4$, $\beta_4\alpha_1$, $\alpha_1 - \beta_1\beta_2\beta_3\beta_4$, $\alpha_2 - \beta_2\beta_3\beta_4\beta_1$, $\alpha_3 - \beta_3\beta_4\beta_1\beta_2$, $\alpha_4\alpha_5 - \beta_4\beta_1\beta_2\beta_3$, $\alpha_4\beta_5$, $\beta_5\alpha_6$, $\alpha_6\beta_6$, $\beta_6\alpha_7$, $\alpha_7\beta_7$, $\beta_7\alpha_5$, $\alpha_5\alpha_4 - \beta_5\beta_6\beta_7$, $\alpha_6 - \beta_6\beta_7\beta_5$, $\alpha_7 - \beta_7\beta_5\beta_6$.

We recall the following result (see [17], [34, Section 3]).

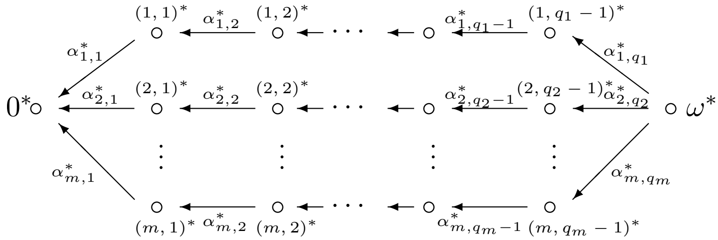
PROPOSITION 1.2. *Let A be an algebra. The following statements are equivalent*:

(i) *A is isomorphic to a Brauer tree algebra $A(T)$.*
(ii) *A is isomorphic to the trivial extension algebra $T(B)$ of a tilted algebra B of Dynkin type $\mathbb{A}_n$, $n \geq 1$.*

2. Trivial extension algebras of strictly canonical type. The aim of this section is to describe the trivial extension algebras of strictly canonical type. We introduce a family of symmetric algebras $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$. Let $m \geq 3$ be a natural number, $\boldsymbol{p} = (p_1, \ldots, p_m)$ and $\boldsymbol{q} = (q_1, \ldots, q_m)$ sequences of positive natural numbers, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ a sequence of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$, normalized so that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. Consider the quiver $Q(\boldsymbol{p}, \boldsymbol{q})$ obtained from the quivers



and



by identifying the vertices ω with 0^*, and 0 with ω^*. Further, for each pair (i, r_i) with $i \in \{1, \dots, m\}$, $r_i \in \{1, \dots, p_i - 1\}$, we define the cycle $A_{(i,r_i)}$ in $Q(\boldsymbol{p}, \boldsymbol{q})$ around (i, r_i) as follows:

$$A_{(1,r_1)} = \alpha_{1,r_1} \dots \alpha_{1,1}\alpha^*_{2,q_2} \dots \alpha^*_{2,1}\alpha_{1,p_1} \dots \alpha_{1,r_1+1},$$
$$A_{(i,r_i)} = \alpha_{i,r_i} \dots \alpha_{i,1}\alpha^*_{1,q_1} \dots \alpha^*_{1,1}\alpha_{i,p_i} \dots \alpha_{i,r_i+1} \qquad \text{for } i \geq 2.$$

Similarly, for each pair (i, s_i) with $i \in \{1, \dots, m\}$, $s_i \in \{1, \dots, q_i - 1\}$, we define the cycle $A^*_{(i,s_i)}$ in $Q(\boldsymbol{p}, \boldsymbol{q})$ around $(i, s_i)^*$ as follows:

$$A^*_{(1,s_1)} = \alpha^*_{1,s_1} \dots \alpha^*_{1,1}\alpha_{2,p_2} \dots \alpha_{2,1}\alpha^*_{1,q_1} \dots \alpha^*_{1,s_1+1},$$
$$A^*_{(i,s_i)} = \alpha^*_{i,s_i} \dots \alpha^*_{i,1}\alpha_{1,p_1} \dots \alpha_{1,1}\alpha^*_{i,q_i} \dots \alpha^*_{i,s_i+1} \qquad \text{for } i \geq 2.$$

Then we define the bound quiver algebra $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\lambda}) = KQ(\boldsymbol{p}, \boldsymbol{q})/I(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\lambda})$, where $I(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\lambda})$ is the ideal of the path algebra $KQ(\boldsymbol{p}, \boldsymbol{q})$ of $Q(\boldsymbol{p}, \boldsymbol{q})$ generated by the elements

- $\alpha_{i,1}\alpha^*_{i,q_i}$, $\alpha^*_{i,1}\alpha_{i,p_i}$, $i \in \{1, \dots, m\}$,
- $\alpha_{1,r_1+1}A_{(1,r_1)}$, $r_1 \in \{1, \dots, p_1 - 1\}$,
- $\alpha_{i,r_i+1}A_{(i,r_i)}$, $i \in \{2, \dots, m\}$, $r_i \in \{1, \dots, p_i - 1\}$,
- $\alpha^*_{1,s_1+1}A^*_{(1,s_1)}$, $s_1 \in \{1, \dots, q_1 - 1\}$,
- $\alpha^*_{i,s_i+1}A^*_{(i,s_i)}$, $i \in \{2, \dots, m\}$, $s_i \in \{1, \dots, q_i - 1\}$,
- $\alpha_{j,p_j} \dots \alpha_{j,1} + \alpha_{1,p_1} \dots \alpha_{1,1} + \lambda_j\alpha_{2,p_2} \dots \alpha_{2,1}$, $j \in \{3, \dots, m\}$,
- $\alpha^*_{j,q_j} \dots \alpha^*_{j,1} + \alpha^*_{1,q_1} \dots \alpha^*_{1,1} + \lambda_j\alpha^*_{2,q_2} \dots \alpha^*_{2,1}$, $j \in \{3, \dots, m\}$.

Observe that the canonical algebras $C(\boldsymbol{p},\boldsymbol{\lambda})$ and $C(\boldsymbol{q},\boldsymbol{\lambda})$ are quotient algebras of $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\lambda})$.

For $\boldsymbol{p}=(p_1,\dots,p_m)$ and $\boldsymbol{q}=(q_1,\dots,q_m)$ as above, we also consider collections $\boldsymbol{T_p}$ and $\boldsymbol{T_q^*}$ of Brauer trees defined as follows. We define $\boldsymbol{T_p}$ to be the collection of Brauer trees $T_{(i,r_i)}$, $i\in\{1,\dots,m\}$, $r_i\in\{1,\dots,p_i-1\}$, each of the trees $T_{(i,r_i)}$ with one fixed extreme edge, denoted by (i,r_i). We assume that the loop of the Brauer quiver $Q_{T_{(i,r_i)}}$ of $T_{(i,r_i)}$ at the vertex (i,r_i), corresponding to the chosen extreme edge of $T_{(i,r_i)}$, is an α-loop $\alpha_{(i,r_i)}$. Then we have in $Q_{T_{(i,r_i)}}$ two β-arrows

$$\beta(i,r_i)\xleftarrow{\beta_{(i,r_i)}}(i,r_i)\xleftarrow{\beta_{\beta^{-1}(i,r_i)}}\beta^{-1}(i,r_i).$$

Moreover, we have the β-cycle $B_{(i,r_i)}=\beta_{(i,r_i)}\dots\beta_{\beta^{-1}(i,r_i)}$ of $Q_{T_{(i,r_i)}}$ around (i,r_i). Similarly, $\boldsymbol{T_q^*}$ is a collection of Brauer trees $T^*_{(i,s_i)}$, where $i\in\{1,\dots,m\}$, $s_i\in\{1,\dots,q_i-1\}$, each of the trees $T^*_{(i,s_i)}$ with one fixed extreme edge, denoted by $(i,s_i)^*$. We assume also that the loop of the Brauer quiver $Q_{T^*_{(i,s_i)}}$ of $T^*_{(i,s_i)}$ at the vertex $(i,s_i)^*$, corresponding to the chosen extreme edge of $T^*_{(i,s_i)}$, is an α-loop $\alpha^*_{(i,s_i)}=\alpha_{(i,s_i)^*}$. Then we have in $Q_{T^*_{(i,s_i)}}$ two β-arrows

$$\beta((i,s_i)^*)\xleftarrow{\beta^*_{(i,s_i)}}(i,s_i)^*\xleftarrow{\beta^*_{\beta^{-1}((i,s_i)^*)}}\beta^{-1}((i,s_i)^*).$$

Moreover, we have the β-cycle $B^*_{(i,s_i)}=\beta^*_{(i,r_i)}\dots\beta^*_{\beta^{-1}((i,s_i)^*)}$ of $Q_{T_{(i,s_i)^*}}$ around $(i,s_i)^*$. Denote by $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*})$ the quiver obtained from the quiver $Q(\boldsymbol{p},\boldsymbol{q})$ and the Brauer quivers $Q_{T_{(i,r_i)}}$ and $Q_{T^*_{(i,s_i)}}$ of all Brauer trees of $\boldsymbol{T_p}$ and $\boldsymbol{T_q^*}$, respectively, by

- removing the α-loop $\alpha_{(i,r_i)}$ from $Q_{T_{(i,r_i)}}$, and glueing the remaining subquiver of $Q_{T_{(i,r_i)}}$ to the quiver $Q(\boldsymbol{p},\boldsymbol{q})$ at the vertex (i,r_i), for each pair (i,r_i) with $i\in\{1,\dots,m\}$, $r_i\in\{1,\dots,p_i-1\}$;
- removing the α-loop $\alpha^*_{(i,s_i)}$ from $Q_{T^*_{(i,s_i)}}$, and glueing the remaining subquiver of $Q_{T^*_{(i,s_i)}}$ to the quiver $Q(\boldsymbol{p},\boldsymbol{q})$ at the vertex $(i,s_i)^*$, for each pair (i,s_i) with $i\in\{1,\dots,m\}$, $s_i\in\{1,\dots,q_i-1\}$.

Then we define the bound quiver algebra

$$\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*},\boldsymbol{\lambda})=KQ(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*})/I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*},\boldsymbol{\lambda}),$$

where $I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*},\boldsymbol{\lambda})$ is the ideal of the path algebra $KQ(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*})$ of $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q^*})$ generated by the elements

- $\alpha_{i,1}\alpha^*_{i,q_i}$, $\alpha^*_{i,1}\alpha_{i,p_i}$, $i\in\{1,\dots,m\}$,
- A_v-B_v, $\beta_{\beta^{-1}(v)}\alpha_v$, $\alpha_{\alpha^{-1}(v)}\beta_v$ for all vertices v of $Q_{T_{(i,r_i)}}$ except the vertex (i,r_i), and the elements $\beta_{\beta^{-1}(i,r_i)}\alpha_{i,r_i}$, $\alpha_{i,r_i+1}\beta_{(i,r_i)}$, $A_{(i,r_i)}-B_{(i,r_i)}$ for all (i,r_i) with $i\in\{1,\dots,m\}$, $r_i\in\{1,\dots,p_i-1\}$,

- $A^*_w - B^*_w$, $\beta^*_{\beta^{-1}(w)}\alpha^*_w$, $\alpha^*_{\alpha^{-1}(w)}\beta^*_w$ for all vertices w of $Q^*_{T_{(i,s_i)}}$ except the vertex $(i,s_i)^*$, and the elements $\beta^*_{\beta^{-1}((i,s_i)^*)}\alpha^*_{i,s_i}$, $\alpha^*_{i,s_i+1}\beta^*_{(i,s_i)}$, $A^*_{(i,s_i)} - B^*_{(i,s_i)}$ for all (i,s_i) with $i \in \{1,\dots,m\}$, $s_i \in \{1,\dots,q_i-1\}$,
- $\alpha_{j,p_j}\dots\alpha_{j,1} + \alpha_{1,p_1}\dots\alpha_{1,1} + \lambda_j\alpha_{2,p_2}\dots\alpha_{2,1}$, $j \in \{3,\dots,m\}$,
- $\alpha^*_{j,q_j}\dots\alpha^*_{j,1} + \alpha^*_{1,q_1}\dots\alpha^*_{1,1} + \lambda_j\alpha^*_{2,q_2}\dots\alpha^*_{2,1}$, $j \in \{3,\dots,m\}$.

We note that $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda}) = \Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\lambda})$ if all Brauer trees of $\boldsymbol{T_p}$ and $\boldsymbol{T^*_q}$ consist only of one (extreme) edge.

PROPOSITION 2.1. *The algebra* $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})$ *is symmetric.*

Proof. Let $A = \Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda}) = KQ(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q})/I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})$. By general theory, it is enough to show that there exists a K-linear form $\varphi : A \to K$ such that $\varphi(ab) = \varphi(ba)$ for all elements $a,b \in A$ and the kernel of φ does not contain nonzero one-sided ideals of A (see [37, Theorem 2]). It follows from the definition that, for every indecomposable projective A-module $P_A(x)$ with x a vertex of $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})$, $P_A(x)$ has a one-dimensional socle and top, which are isomorphic to the simple module $S_A(x)$ at the vertex x. Hence, A is a weakly symmetric algebra, and consequently a selfinjective algebra, by the classical result due to Nakayama [24] (see also [37, Theorem 4.4]). For each vertex x of $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q})$, we fix a maximal cycle w_x around x in $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q})$ with $w_x \notin I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})$. Observe that the cosets $u_x = w_x + I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})$, with x a vertex of $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q})$, form a K-linear basis of the socle of A. Then the K-linear form $\varphi : A \to K$ with $\varphi(u_x) = 1$ and $\varphi(v + I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda})) = 0$ for any path v of $Q(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q})$ with $v + I(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T^*_q},\boldsymbol{\lambda}) \notin \operatorname{soc} A \setminus \{0\}$ satisfies the conditions required for A to be symmetric. ∎

EXAMPLE 2.2. Let $B = KQ/I$ be the bound quiver algebra from Example 3.3 in [21], and $T(B)$ the associated trivial extension algebra. Since $T(B)$ is the orbit algebra $\widehat{B}/(\nu_{\widehat{B}})$, it follows from the considerations in [21, Example 3.3] that $T(B)$ is the bound quiver algebra $T(B) = Q_{T(B)}/I_{T(B)}$, where $Q_{T(B)}$ is the quiver of Figure 2.1 and $I_{T(B)}$ is the ideal of the path algebra $KQ_{T(B)}$ of $Q_{T(B)}$ generated by the elements

- $\alpha_{3,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \alpha_{2,2}\alpha_{2,1}$,
  $\alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}\alpha_{4,1} + \alpha_{1,3}\alpha_{1,2}\alpha_{1,1} + \mu\alpha_{2,2}\alpha_{2,1}$,
- $\beta_{3,2}\beta_{3,1} + \beta_{1,1}\beta_{2,1}$, $\beta_{4,1} + \beta_{1,1} + \mu\beta_{2,1}$, $\mu \in K \setminus \{0,1\}$,
- $\beta_{1,1}\alpha_{1,3}$, $\beta_{2,1}\alpha_{2,2}$, $\beta_{3,1}\alpha_{3,1}$, $\beta_{4,1}\alpha_{4,5}$,
- $\alpha_{1,1}\beta_{1,1}$, $\alpha_{2,1}\beta_{2,1}$, $\alpha_{3,1}\beta_{3,2}$, $\alpha_{4,1}\beta_{4,1}$,
- $\alpha_{1,1}\beta_{2,1}\alpha_{1,3}\alpha_{1,2}\alpha_{1,2}$, $\alpha_{1,2}\alpha_{1,2}\beta_{2,1}\alpha_{1,3}\alpha_{1,2}$,
- $\beta_{3,2}\gamma_1$, $\alpha_2\beta_{3,1}$, $\beta_{3,1}\alpha_{1,3}\alpha_{1,2}\alpha_{1,1}\beta_{3,2} - \gamma_1\alpha_3\alpha_2$, $\alpha_3\alpha_2\gamma_1\alpha_3$,
- $\alpha_{4,2}\beta_4$, $\gamma_5\alpha_{4,1}$, $\alpha_{4,1}\beta_{1,1}\alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2} - \beta_4\beta_5\gamma_5$, $\beta_5\gamma_5\beta_4\beta_5$,
- $\alpha_{4,2}\alpha_{4,1}\beta_{1,1}\alpha_{4,5}\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}$, $\alpha_{4,4}\alpha_{4,3}\alpha_{4,2}\alpha_{4,1}\beta_{1,1}\alpha_{4,5}\alpha_{4,4}$,

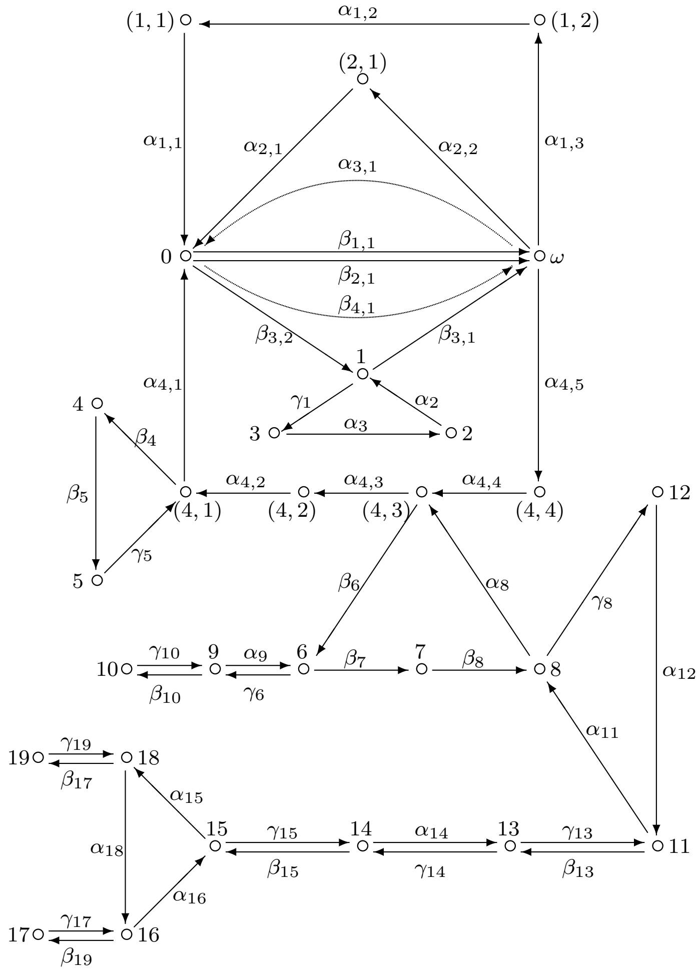
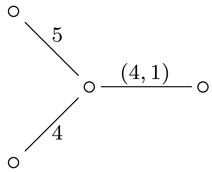


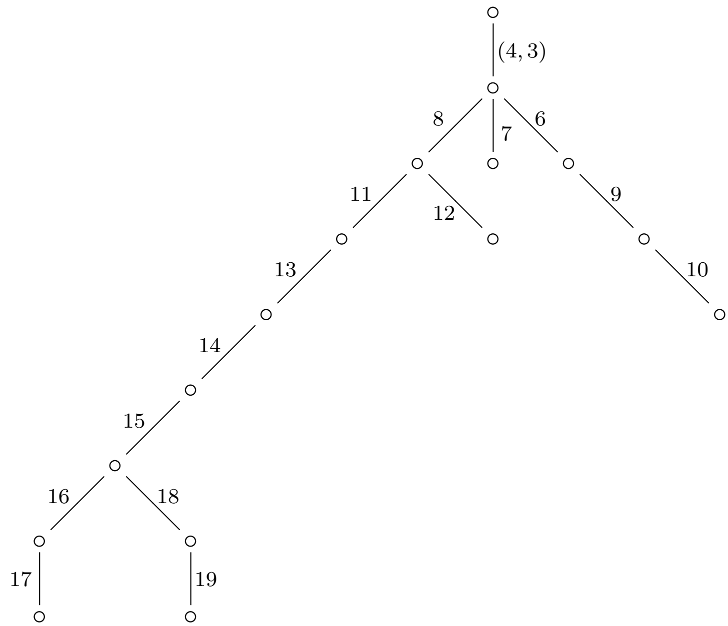
Fig. 2.1

- $\alpha_{4,4}\beta_6$, $\alpha_8\alpha_{4,3}$, $\alpha_{4,3}\alpha_{4,2}\alpha_{4,1}\beta_{1,1}\alpha_{4,5}\alpha_{4,4} - \beta_6\beta_7\beta_8\alpha_8$,
- $\beta_8\alpha_8\beta_6\beta_7\beta_8$, $\beta_6\gamma_6, \alpha_9\beta_7$, $\beta_7\beta_8\alpha_8\beta_6 - \gamma_6\alpha_9$,
- $\beta_8\gamma_8, \alpha_{1,1}\alpha_8$, $\alpha_8\beta_6\beta_7\beta_8 - \gamma_8\alpha_{12}\alpha_{11}$,
- $\gamma_{10}\alpha_9$, $\gamma_6\beta_{10}$, $\alpha_9\gamma_6 - \beta_{10}\gamma_{10}$, $\gamma_{13}\alpha_{11}$, $\alpha_{12}\beta_{13}$, $\alpha_{11}\gamma_8\alpha_{12} - \beta_{13}\gamma_{13}$,
- $\alpha_{14}\gamma_{13}$, $\beta_{13}\gamma_{14}$, $\gamma_{13}\beta_{13} - \gamma_{14}\alpha_{14}$, $\gamma_{15}\alpha_{14}$, $\gamma_{14}\beta_{15}$, $\alpha_{14}\gamma_{14} - \beta_{15}\gamma_{15}$,
- $\beta_{15}\alpha_{15}$, $\alpha_{16}\gamma_{15}$, $\gamma_{15}\beta_{15} - \alpha_{15}\alpha_{18}\alpha_{16}$, $\alpha_{18}\beta_{17}$, $\gamma_{17}\alpha_{16}$,
- $\alpha_{16}\alpha_{15}\alpha_{18} - \beta_{17}\gamma_{17}$, $\alpha_{15}\beta_{19}$, $\gamma_{19}\alpha_{18}$, $\alpha_{18}\alpha_{16}\alpha_{15} - \beta_{19}\gamma_{19}$.

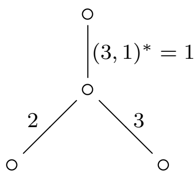
A simple analysis of the relations in $T(B) = KQ_{T(B)}/I_{T(B)}$ given by the above generators of $I_{T(B)}$ shows that $T(B)$ is isomorphic to the algebra $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$, where $\boldsymbol{p} = (3, 2, 1, 5)$, $\boldsymbol{q} = (1, 1, 2, 1)$, $\boldsymbol{\lambda} = (\infty, 0, 1, \mu)$, $\boldsymbol{T_p}$ is the collection of Brauer trees with $T_{(1,1)}$, $T_{(1,2)}$, $T_{(2,1)}$, $T_{(4,2)}$, $T_{(4,4)}$ consisting only of one edge, $T_{(4,1)}$ of the form



$T_{(4,3)}$ of the form



and $\boldsymbol{T}^*_{\boldsymbol{q}}$ consists of one tree $T^*_{(3,1)}$ of the form



The following theorem gives a complete description of the trivial extension algebras of strictly canonical type.

THEOREM 2.3. *An algebra A is a trivial extension algebra $T(B)$ of strictly canonical type if and only if A is isomorphic to an algebra of the form $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$.*

Proof. By general theory (see [21]) we know that the class of trivial extension algebras of strictly canonical type coincides with the class of trivial extensions $T(B) = B \ltimes D(B)$ of branch coextensions B of canonical algebras.

Let B be a branch coextension $[E_1, \mathcal{L}_1, \dots, E_s, \mathcal{L}_s]C$ of a canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ of type $(\boldsymbol{p}, \boldsymbol{\lambda})$. If $B = C$ then it follows from [21, Example 5.4] that $T(B) = T(C)$ is isomorphic to the algebra $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T_q^*}, \boldsymbol{\lambda})$ with $\boldsymbol{q} = (1, \dots, 1)$, $\boldsymbol{T_q^*}$ an empty collection of Brauer trees, $\boldsymbol{T_p}$ formed by Brauer trees with one (extreme) edge, or empty if $\boldsymbol{p} = (1, \dots, 1)$. Therefore assume that $B \neq C$. We will show that $T(B)$ is isomorphic to some algebra $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T_q^*}, \boldsymbol{\lambda})$.

Let $\boldsymbol{p} = (p_1, \dots, p_m)$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, $m \geq 2$, with $\lambda_1 = \infty$, $\lambda_2 = 0$. We first define the weight sequence $\boldsymbol{q} = (q_1, \dots, q_m)$. Fix $i \in \{1, \dots, m\}$. If the unique nonsimple module $E^{(\lambda_i)}$ of the stable tube $\mathcal{T}^C_{\lambda_i}$ of Γ_C is not one of the modules $E_1, \dots, E_s$, we set $q_i = 1$. Assume $E^{(\lambda_i)} = E_l$ for some $l \in \{1, \dots, s\}$, and let $0_l^* = b_1^* \to b_2^* \to \dots \to b_n^*$ be the maximal path of the branch $\mathcal{L}_i$ starting at its germ 0_l^*. Then we take $q_i = n + 1$. We note that the vertices $b_1^*, \dots, b_n^*$ of the above path yield the vertices

$$(i,1)^* = i_1 = b_n^*, \quad (i,2)^* = i_2 = b_{n-1}^*, \quad \dots, \quad (i, q_i - 1)^* = i_k = b_1^*$$

of the quiver $Q(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T_q^*})$ of the required algebra $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T_q^*}, \boldsymbol{\lambda})$ (see the proof of [21, Theorem 3.1]).

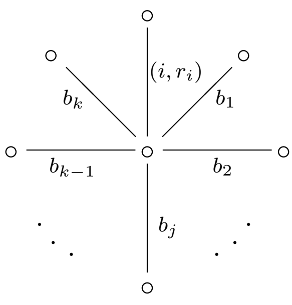
Assume now that $p_i \geq 2$ for some $i \in \{1, \dots, m\}$. Take a vertex (i, r_i), with $r_i \in \{1, \dots, p_i - 1\}$, lying on the ith path of Q_C from the source ω to the sink 0. Our aim is to define a Brauer tree $T_{(i,r_i)}$ with a fixed extreme edge. Let $S(i, r_i)$ be the simple C-module at the vertex (i, r_i) of Q_C. If $S(i, r_i)$ is not one of the modules $E_1, \dots, E_s$, we define $T_{(i,r_i)}$ to be the Brauer tree consisting of one edge

$$\circ \overset{(i,r_i)}{\frac{}{}} \circ$$

Assume that $S(i, r_i) = E_t$ for some $t \in \{1, \dots, s\}$. Let

$$0_t^* = b_1 \to b_2 \to \dots \to b_k$$

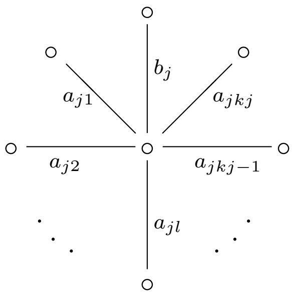
be the maximal path of the branch $\mathcal{L}_t$ starting at its germ 0_t^*. Observe that such a path consists of β-arrows of $\mathcal{L}_t$. The set of edges of the Brauer tree $T_{(i,r_i)}$ adjacent to the ends of the extreme edge (i, r_i) is defined as the Brauer tree $T^{(1)}_{(i,r_i)}$



with the clockwise order of edges. If a vertex b_j is not the sink of an α-arrow of $Q_{\mathcal{L}_t}$, then the edge b_j will be an extreme edge of $T_{(i,r_i)}$. If b_j is such a sink, let

$$b_j \leftarrow a_{j1} \leftarrow a_{j2} \leftarrow \cdots \leftarrow a_{jkj}$$

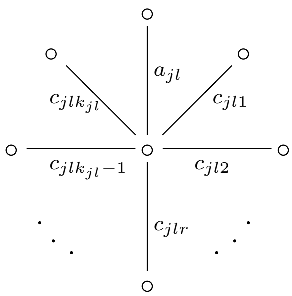
be the maximal α-path of $Q_{\mathcal{L}_t}$ with sink at the vertex b_j. Then we enlarge the above Brauer tree by glueing in the Brauer tree (with the clockwise order of edges)



at the edge b_j. Applying this procedure to all edges $b_1, \dots, b_k$ of $T^{(1)}_{(i,r_i)}$, we obtain a Brauer tree $T^{(2)}_{(i,r_i)}$. In the next step, take a vertex a_{jl} of $Q_{\mathcal{L}_t}$. If a_{jl} is not the source of a β-arrow of $Q_{\mathcal{L}_t}$, then the edge a_{jl} will be an extreme edge of $T_{(i,r_i)}$. If a_{jl} is such a source, let

$$a_{jl} \to c_{jl1} \to c_{jl2} \to \cdots \to c_{jlk_{jl}}$$

be the maximal β-path of $Q_{\mathcal{L}_t}$ with source at a_{jl}. Then we enlarge the above-constructed Brauer tree $T^{(2)}_{(i,r_i)}$ by glueing in the Brauer tree (with the clockwise order of edges)



at the edge a_{jl}. Applying this procedure to all edges a_{jl}, for all $j \in \{1, \dots, k\}$, $l \in \{1, \dots, k_j\}$, we obtain a Brauer tree $T^{(3)}_{(i,r_i)}$. Repeating this procedure, we obtain (in finitely many steps) a Brauer tree $T_{(i,r_i)} = T^{(k_i)}_{(i,r_i)}$ all of whose edges except the chosen extreme edge (i, r_i) correspond to the vertices of the branch $\mathcal{L}_t$, involved in the branch coextension of C using the module $E_t = S(i, r_i)$. In such a way, we define the required collection $\boldsymbol{T_p}$ of Brauer trees $T_{(i,r_i)}$ with $i \in \{1, \dots, m\}$, $r_i \in \{1, \dots, p_i - 1\}$. Observe that the nontrivial (with at least two edges) Brauer trees $T_{(i,r_i)}$ correspond to the simple modules $S(i,r_i)$ involved in the branch coextension B of C.

The Brauer trees $T^*_{(i,s_i)}$ of the collection $\boldsymbol{T}^*_{\boldsymbol{q}}$ correspond to the branches $\mathcal{L}_l$ with E_l being nonsimple modules lying on the mouth of stable tubes of the $\mathbb{P}_1(K)$-family $\mathcal{T}^C$. Namely, assume that $q_i \geq 2$ for some $i \in \{1,\dots,m\}$. We know that the vertices $(i,1)^* = b^*_n, \dots, (i, q_i - 1)^* = b^*_1$ of the quiver $\Delta(q)^*$ correspond to the maximal β-path

$$0^* = b^*_1 \to b^*_1 \to \dots \to b^*_n$$

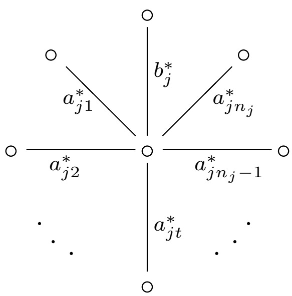
of the branch $\mathcal{L}_l$ corresponding to the coextension module $E_l = E^{(\lambda_i)}$. Fix $(i,s_i)^*$ for some $s_i \in \{1,\dots,q_i - 1\}$. We indicate how to construct the Brauer tree $T^*_{(i,s_i)}$. Let $(i,s_i)^* = b^*_j$. If the vertex b^*_j is not the sink of an α-arrow of the branch $\mathcal{L}_l$, then $T^*_{(i,s_i)}$ is defined as the Brauer tree consisting of one edge

$$\circ \overset{b^*_j}{\text{———}} \circ$$

If b^*_j is such a sink, let

$$b^*_j \leftarrow a^*_{j1} \leftarrow a^*_{j2} \leftarrow \dots \leftarrow a^*_{jn_j}$$

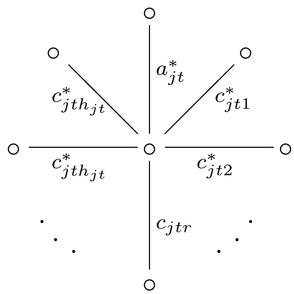
be the maximal α-path of $Q_{\mathcal{L}_l}$ with sink at the vertex b^*_j. Then the set of edges of the Brauer tree $T^*_{(i,s_i)}$ incident to the extreme edge $b^*_j = (i,s_i)^*$ is the Brauer tree $(T^*_{(i,s_i)})^{(1)}$



with the clockwise order of edges. If a vertex a^*_{jt} is not source of a β-path of $Q_{\mathcal{L}_l}$, then a^*_{jt} will be an extreme edge of $T^*_{(i,s_i)}$. If a^*_{jt} is such a source, let

$$a^*_{jt} \to c^*_{jt1} \to c^*_{jt2} \to \ldots \to c^*_{jtn_{jt}}$$

be the maximal β-path of $Q_{\mathcal{L}_l}$ with source a^*_{jt}. Then we enlarge the Brauer tree $(T^*_{(i,s_i)})^{(1)}$ by glueing in the Brauer tree (with the clockwise order of edges)

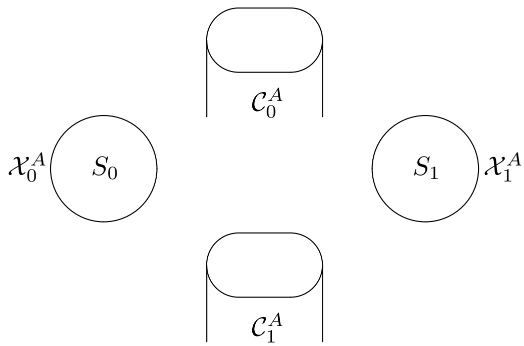


at the edge a^*_{jt}. Applying this procedure to all vertices a^*_{jt}, for all $j \in \{1,\dots,n\}$, $t \in \{1,\dots,n_j\}$, we obtain a Brauer tree $(T^*_{(i,s_i)})^{(2)}$. Repeating this procedure, we obtain (in finitely many steps) a Brauer tree $T^*_{(i,s_i)} = (T^*_{(i,s_i)})^{(r_i)}$, whose edges correspond to the vertices of the connected subquiver, containing the vertex a^*_{j1}, of the quiver obtained from $Q_{\mathcal{L}_l}$ by removing the path $b^*_1 \to b^*_2 \to \cdots \to b^*_n$.

It follows from the proofs of Theorems 3.1 and 5.1 in [21] that indeed $T(B)$ is isomorphic to the algebra $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q},\boldsymbol{\lambda})$ with $\boldsymbol{q}$, $\boldsymbol{T_p}$, $\boldsymbol{T_q}$ defined earlier.

Conversely, the above discussion also shows that every algebra of the form $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q},\boldsymbol{\lambda})$ is isomorphic to the trivial extension algebra $T(B)$ for a unique branch coextension $B = B(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q},\boldsymbol{\lambda})$ of a canonical algebra $C = C(\boldsymbol{p},\boldsymbol{\lambda})$ (see also [21]). We also note that $\Lambda(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q},\boldsymbol{\lambda})$ is isomorphic to the trivial extension algebra $T(B^*)$ of the unique branch coextension $B^* = B^*(\boldsymbol{p},\boldsymbol{q},\boldsymbol{T_p},\boldsymbol{T_q},\boldsymbol{\lambda})$ of the canonical algebra $C^* = C(\boldsymbol{q},\boldsymbol{\lambda})$. ■

Furthermore, we note that, by the main result of [21], the Auslander–Reiten quiver Γ_A of a trivial extension algebra A of strictly canonical type has the following shape:



where, for $q \in \{0,1\}$, $\mathcal{C}_q^A = (\mathcal{C}_q^A(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes with $s(\mathcal{C}_q^A(\lambda)) + p(\mathcal{C}_q^A(\lambda)) = r(\mathcal{C}_q^A(\lambda)) - 1$ for each $\lambda \in \mathbb{P}_1(K)$, and $\mathcal{X}_q^A$ is a family of components, containing exactly one simple module S_q.

Here, $s(\mathcal{C}_q^A(\lambda))$ denotes the number of simple modules in $\mathcal{C}_q^A(\lambda)$, $p(\mathcal{C}_q^A(\lambda))$ the number of projective modules in $\mathcal{C}_q^A(\lambda)$, and $r(\mathcal{C}_q^A(\lambda))$ the rank of the stable tube of $\mathcal{C}_q^A(\lambda)$.

3. Weakly symmetric algebras. The aim of this section is to introduce some families of weakly symmetric algebras of strictly canonical type and discuss their symmetry.

Let $m \geq 3$ be a natural number, $\boldsymbol{p} = (p_1, \ldots, p_m)$ a sequence of positive natural numbers, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ a sequence of pairwise different elements of $\mathbb{P}_1(K) = K \cup \{\infty\}$ normalized so that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. Consider the quiver $Q(\boldsymbol{p})$ of Figure 3.1.

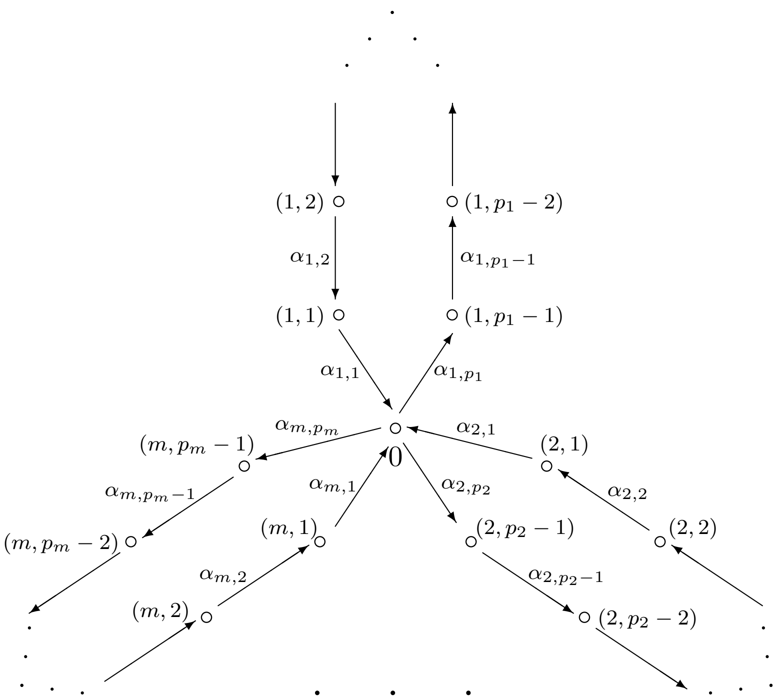


Fig. 3.1

Moreover, let $\boldsymbol{T_p}$ be a collection of Brauer trees $T_{(i,r_i)}$, $i \in \{1, \ldots, m\}$, $r_i \in \{1, \ldots, p_i - 1\}$, each with one fixed extreme edge, denoted by (i, r_i). We assume that the loop of the Brauer quiver $Q_{T_{(i,r_i)}}$ of $T_{(i,r_i)}$ at the vertex (i, r_i), corresponding to the chosen extreme edge of $T_{(i,r_i)}$, is an α-loop $\alpha_{(i,r_i)}$. Hence we have in $Q_{T_{(i,r_i)}}$ two β-arrows

$$\beta(i, r_i) \xleftarrow{\beta_{(i,r_i)}} (i, r_i) \xleftarrow{\beta_{\beta^{-1}(i,r_i)}} \beta^{-1}(i, r_i).$$

Denote by $Q(\boldsymbol{p}, \boldsymbol{T_p})$ the quiver obtained from the quiver $Q(\boldsymbol{p})$ and the Brauer quivers $Q_{T_{(i,r_i)}}$ of the Brauer trees $T_{(i,r_i)}$ of $\boldsymbol{T_p}$ by

- removing the α-loop $\alpha_{(i,r_i)}$ from $Q_{T_{(i,r_i)}}$, and glueing the remaining subquiver of $Q_{T_{(i,r_i)}}$ to the quiver $Q(\boldsymbol{p})$ at the vertex (i, r_i) for each pair (i, r_i) with $i \in \{1, \dots, m\}$, $r_i \in \{1, \dots, p_i - 1\}$.

For each pair (i, r_i) with $i \in \{1, \dots, m\}$ and $r_i \in \{1, \dots, p_i - 1\}$, we define the cycles $A_{(i,r_i)}$ and $\bar{A}_{(i,r_i)}$ of $Q(\boldsymbol{p}, \boldsymbol{T_p})$ around (i, r_i) as follows:

$$\begin{aligned}
A_{(1,r_1)} &= \alpha_{1,r_1} \dots \alpha_{1,1}\alpha_{2,p_2} \dots \alpha_{2,1}\alpha_{1,p_1} \dots \alpha_{1,r_1+1}, && \\
A_{(i,r_i)} &= \alpha_{i,r_i} \dots \alpha_{i,1}\alpha_{1,p_1} \dots \alpha_{1,1}\alpha_{i,p_i} \dots \alpha_{i,r_i+1} && \text{for } i \geq 2, \\
\bar{A}_{(i,r_i)} &= \alpha_{i,r_i} \dots \alpha_{i,1}\alpha_{i,p_i} \dots \alpha_{i,1}\alpha_{i,p_i} \dots \alpha_{i,r_i+1} && \text{for } i \geq 1.
\end{aligned}$$

We define the bound quiver algebra

$$\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}) = KQ(\boldsymbol{p}, \boldsymbol{T_p})/I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}),$$

where $I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is the ideal of the path algebra $KQ(\boldsymbol{p}, \boldsymbol{T_p})$ of $Q(\boldsymbol{p}, \boldsymbol{T_p})$ generated by the elements

- $\alpha_{i,1}\alpha_{i,p_i}$, $i \in \{1, \dots, m\}$,
- $A_v - B_v$, $\beta_{\beta^{-1}(v)}\alpha_v$, $\alpha_{\alpha^{-1}(v)}\beta_v$ for all vertices v of $Q_{T_{(i,r_i)}}$ except the vertex (i, r_i), and the elements $\beta_{\beta^{-1}(i,r_i)}\alpha_{(i,r_i)}$, $\alpha_{(i,r_i+1)}\beta_{(i,r_i)}$, $A_{(i,r_i)} - B_{(i,r_i)}$ for all (i, r_i) with $i \in \{1, \dots, m\}$, $r_i \in \{1, \dots, p_i - 1\}$,
- $\alpha_{j,p_j} \dots \alpha_{j,1} + \alpha_{1,p_1} \dots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \dots \alpha_{2,1}$, $j \in \{3, \dots, m\}$.

Proposition 3.1.

(i) *$\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is a weakly symmetric algebra of strictly canonical type.*
(ii) *$\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is a symmetric algebra if and only if* $\operatorname{char} K = 2$.

Proof. (i) Let A be an algebra $\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$. We set $\boldsymbol{q} = \boldsymbol{p}$, $\boldsymbol{T^*_q} = \boldsymbol{T_p}$, and let $B = B(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}, \boldsymbol{\lambda})$ be the unique branch coextension of the canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ such that $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T^*_q}\boldsymbol{\lambda}) \cong T(B)$. Then $A \cong \widehat{B}/(\varphi_{\widehat{B}})$ for a canonical automorphism $\varphi_{\widehat{B}}$ of $\widehat{B}$ with $\varphi^2_{\widehat{B}} = \nu_{\widehat{B}}$, and consequently A is a weakly symmetric algebra of strictly canonical type.

(ii) Assume $A = \Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is a symmetric algebra. Then there exists a K-linear form $\varphi : A \to K$ such that $\varphi(ab) = \varphi(ba)$ for all elements $a, b \in A$ and the kernel of φ does not contain nonzero one-sided ideals of A. For each vertex x of $Q(\boldsymbol{p}, \boldsymbol{T_p})$, the indecomposable projective A-module $P_A(x)$ at x has a one-dimensional socle $\operatorname{soc} P_A(x)$, isomorphic to $\operatorname{top} P_A(x)$, since A is weakly symmetric. Hence every nonzero element $a \in \operatorname{soc} P_A(x)$ generates the two-sided ideal $AaA = Ka = aK$, and then $\varphi(a) \neq 0$.

We now claim that A being symmetric implies that $\operatorname{char} K = 2$. In order to simplify the calculations, we identify an element w of the path algebra

$KQ(\boldsymbol{p}, \boldsymbol{T_p})$ with the coset $w + I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ of the bound quiver algebra $A = KQ(\boldsymbol{p}, \boldsymbol{T_p})/I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$. Then we have in A the relations

- $\alpha_{i,1}\alpha_{i,p_i} = 0,\ i \in \{1, \ldots, m\}$,
- $\alpha_{j,p_j} \ldots \alpha_{j,1} + \alpha_{1,p_1} \ldots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \ldots \alpha_{2,1} = 0,\ j \in \{3, \ldots, m\}$.

Then, for each $j \in \{3, \ldots, m\}$, we obtain the relations

- $\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{1,p_1} \ldots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \ldots \alpha_{2,1}\alpha_{1,p_1} \ldots \alpha_{1,1} = 0$,
- $\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{2,p_2} \ldots \alpha_{2,1} + \alpha_{1,p_1} \ldots \alpha_{1,1}\alpha_{2,p_2} \ldots \alpha_{2,1} = 0$,
- $\alpha_{1,p_1} \ldots \alpha_{1,1}\alpha_{j,p_j} \ldots \alpha_{j,1} + \lambda_j \alpha_{2,p_2} \ldots \alpha_{2,1}\alpha_{j,p_j} \ldots \alpha_{j,1} = 0$,

for $j \in \{3, \ldots, m\}$. Applying the symmetry property $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$, we obtain the equalities

$$\begin{aligned}
\varphi(\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{1,p_1} \ldots \alpha_{1,1}) &= -\lambda_j \varphi(\alpha_{2,p_2} \ldots \alpha_{2,1}\alpha_{1,p_1} \ldots \alpha_{1,1}) \\
&= -\lambda_j \varphi(\alpha_{1,p_1} \ldots \alpha_{1,1}\alpha_{2,p_2} \ldots \alpha_{2,1}) \\
&= \lambda_j \varphi(\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{2,p_2} \ldots \alpha_{2,1}) \\
&= \lambda_j \varphi(\alpha_{2,p_2} \ldots \alpha_{2,1}\alpha_{j,p_j} \ldots \alpha_{j,1}) \\
&= -\varphi(\alpha_{1,p_1} \ldots \alpha_{1,1}\alpha_{j,p_j} \ldots \alpha_{j,1}) \\
&= -\varphi(\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{1,p_1} \ldots \alpha_{1,1}).
\end{aligned}$$

Moreover, $\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{1,p_1} \ldots \alpha_{1,1}$ is a nonzero element of the socle of the indecomposable projective A-module $P_A(0)$ at the central vertex 0 of $Q(\boldsymbol{p}, \boldsymbol{T_p})$, and so $\varphi(\alpha_{j,p_j} \ldots \alpha_{j,1}\alpha_{1,p_1} \ldots \alpha_{1,1}) \neq 0$. Therefore, we conclude that $1 = -1$, and so $\operatorname{char} K = 2$.

Conversely, assume $\operatorname{char} K \neq 2$. For each vertex x of $Q(\boldsymbol{p}, \boldsymbol{T_p})$ fix a maximal cycle w_x in $Q(\boldsymbol{p}, \boldsymbol{T_p})$ with $w_x \notin I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$. Then the coset $u_x = w_x + I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ generates the one-dimensional socle of the indecomposable projective A-module $P_A(x)$ at x. Then the K-linear form $\varphi : A \to K$ with $\varphi(u_x + I(\boldsymbol{p})) = 1$ and $\varphi(v + I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})) = 0$ for any path v of $Q(\boldsymbol{p}, \boldsymbol{T_p})$ with $v + I(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}) \notin \operatorname{soc} A \setminus \{0\}$ satisfies the conditions required for A to be symmetric, and consequently $A = \Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is a symmetric algebra. ■

Given two integers $k \geq 1$ and $s \geq 0$ with $2k + s = m$, we define the bound quiver algebra

$$\Lambda^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}) = KQ(\boldsymbol{p}, \boldsymbol{T_p})/I^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}),$$

where $I^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is the ideal of the path algebra $KQ(\boldsymbol{p}, \boldsymbol{T_p})$ of $Q(\boldsymbol{p}, \boldsymbol{T_p})$ generated by the elements

- $\alpha_{2i-1,1}\alpha_{2i,p_{2i}},\ \alpha_{2i,1}\alpha_{2i-1,p_{2i-1}},\ i \in \{1, \ldots, k\}$,
- $\alpha_{j,1}\alpha_{j,p_j},\ j \in \{2k+1, \ldots, m\}$,
- $A_v - B_v,\ \beta_{\beta^{-1}(v)}\alpha_v,\ \alpha_{\alpha^{-1}(v)}\beta_v$ for all vertices v of $Q_{T_{(i,r_i)}}$ except the vertex (i, r_i), for all (i, r_i) with $i \in \{1, \ldots, m\}$, $r_i \in \{1, \ldots, p_i - 1\}$,

- $\bar{A}_{(l,r_l)} - B_{(l,r_l)}$, $\beta_{\beta^{-1}(l,r_l)}\alpha_{(l,r_l)}$, $\alpha_{(l,r_l+1)}\beta_{(l,r_l)}$ for all (l,r_l) with $l \in \{1,\ldots,2k\}$, $r_l \in \{1,\ldots,p_l-1\}$,
- $A_{(l,r_l)} - B_{(l,r_l)}$, $\beta_{\beta^{-1}(l,r_l)}\alpha_{(l,r_l)}$, $\alpha_{(l,r_l+1)}\beta_{(l,r_l)}$ for all (l,r_l) with $l \in \{2k+1,\ldots,m\}$, $r_l \in \{1,\ldots,p_l-1\}$,
- $\alpha_{j,p_j}\ldots\alpha_{j,1} + \alpha_{1,p_1}\ldots\alpha_{1,1} + \lambda_j\alpha_{2,p_2}\ldots\alpha_{2,1}$, $j \in \{3,\ldots,m\}$.

PROPOSITION 3.2. *Let $m = 2k+s$ with $k \geq 1$, $s \geq 0$. Then*

(i) $\Lambda^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ *is a weakly symmetric algebra of strictly canonical type.*

(ii) $\Lambda^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ *is a symmetric algebra if and only if the following conditions hold*:

(1) $\lambda_{2i-1}\lambda_{2i} = \lambda_4$ *for* $k \geq 3$ *and* $i \in \{3,\ldots,k\}$,

(2) $s \in \{0,1,2\}$,

(a) $\lambda_4 = -1$ *for* $k = 1$, $s = 2$,

(b) $\lambda_{2k+1}^2 = \lambda_4$ *for* $k \geq 2$, $s = 1$,

(c) $\lambda_{2k+1}^2 = \lambda_4 = \lambda_{2k+2}^2$ *for* $k \geq 2$, $s = 2$.

Proof. (i) Let $m = 2k+s$ with $k \geq 1$, $s \geq 0$, and $A = \Lambda^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$. Let $\boldsymbol{q} = (q_1,\ldots,q_m)$ be the weight sequence obtained from the weight sequence $\boldsymbol{p} = (p_1,\ldots,p_m)$ by the exchange $q_{2i-1} = p_{2i}$ and $q_{2i} = p_{2i-1}$ for $i \in \{1,\ldots,k\}$, and taking $q_j = p_j$ for $j \in \{2k+1,\ldots,m\}$. Further, let $\boldsymbol{T}^*_{\boldsymbol{q}}$ be the collection of Brauer trees obtained from $\boldsymbol{T_p}$ by taking $T^*_{2i-1,r_{2i-1}} = T_{2i,r_{2i-1}}$ and $T^*_{2i,r_{2i}} = T_{2i-1,r_{2i}}$ for $i \in \{1,\ldots,k\}$, $r_{2i-1} \in \{1,\ldots,q_{2i-1}-1\} = \{1,\ldots,p_{2i}-1\}$, $r_{2i} \in \{1,\ldots,q_{2i}-1\} = \{1,\ldots,p_{2i-1}-1\}$, and $T^*_{j,r_j} = T_{j,r_j}$ for $j \in \{2k+1,\ldots,m\}$ and $r_j \in \{1,\ldots,q_j-1\} = \{1,\ldots,p_j-1\}$. Consider the branch coextension $B = B(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T}^*_{\boldsymbol{q}}, \boldsymbol{\lambda})$ of the canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ such that $\Lambda(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{T_p}, \boldsymbol{T}^*_{\boldsymbol{q}}, \boldsymbol{\lambda}) \cong T(B)$. Then $A \cong \widehat{B}/(\psi_{\widehat{B}})$ for a canonical automorphism $\psi_{\widehat{B}}$ of $\widehat{B}$ with $\psi^2_{\widehat{B}} = \nu_{\widehat{B}}$, and consequently A is a weakly symmetric algebra of strictly canonical type.

(ii) Assume that $A = \Lambda^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ is a symmetric algebra. Let $\varphi : A \to K$ be a K-linear form such that $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$ and the kernel of φ does not contain nonzero one-sided ideals of A. For each vertex x of the quiver $Q(\boldsymbol{p}, \boldsymbol{T_p})$ of the bound quiver algebra $A = KQ(\boldsymbol{p}, \boldsymbol{T_p})/I^{(k,s)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, the indecomposable projective A-module $P_A(x)$ at x has one-dimensional socle $\operatorname{soc} P_A(x)$, isomorphic to $\operatorname{top} P_A(x)$, since A is weakly symmetric. Hence every nonzero element $a \in \operatorname{soc} P_A(x)$ generates the two-sided ideal $AaA = Ka = aK$, and then $\varphi(a) \neq 0$. In particular, this is the case for nonzero elements of the socle $\operatorname{soc} P_A(0)$ of the indecomposable projective A-module $P_A(0)$ at the center 0 of the quiver $Q(\boldsymbol{p}, \boldsymbol{T_p})$. We claim that the required conditions (1) and (2) hold.

In order to simplify the calculations, we identify an element w of the path algebra $KQ(\boldsymbol{p},\boldsymbol{T_p})$ with the coset $w+I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})$ of the bound quiver algebra $A=KQ(\boldsymbol{p},\boldsymbol{T_p})/I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})$. Then we have in A the relations

- $\alpha_{2i-1,1}\alpha_{2i,p_{2i}}=0,\ \alpha_{2i,1}\alpha_{2i-1,p_{2i-1}}=0,\ i\in\{1,\dots,k\}$;
- $\alpha_{l,1}\alpha_{l,p_l}=0,\ l\in\{2k+1,\dots,m\}$,
- $\alpha_{j,p_j}\dots\alpha_{j,1}+\alpha_{1,p_1}\dots\alpha_{1,1}+\lambda_j\alpha_{2,p_2}\dots\alpha_{2,1}=0,\ j\in\{3,\dots,m\}$.

Thus we arrive at the relations

- $\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{1,p_1}\dots\alpha_{1,1}+\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}=0$,
- $\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{2,p_2}\dots\alpha_{2,1}+\lambda_j\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1}=0$,
- $\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{2i,p_{2i}}\dots\alpha_{2i,1}+\lambda_{2i-1}\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2i,p_{2i}}\dots\alpha_{2i,1}=0$,
- $\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{2i-1,p_{2i-1}}\dots\alpha_{2i-1,1}+\lambda_{2i}\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2i-1,p_{2i-1}}\dots\alpha_{2i-1,1}=0$,
- $\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{s,p_s}\dots\alpha_{s,1}+\lambda_s\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{s,p_s}\dots\alpha_{s,1}=0$,

for $j\in\{3,\dots,m\}$, $i\in\{2,\dots,k\}$ and $s\in\{2k+1,\dots,m\}$.

Applying the property $\varphi(ab)=\varphi(ba)$ for all $a,b\in A$, we obtain the equalities

$$\begin{aligned}\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1})&=-\varphi(\alpha_{2i,p_{2i}}\dots\alpha_{2i,1}\alpha_{1,p_1}\dots\alpha_{1,1})\\&=-\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{2i,p_{2i}}\dots\alpha_{2i,1})\\&=\lambda_{2i-1}\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2i,p_{2i}}\dots\alpha_{2i,1})\\&=\lambda_{2i-1}\varphi(\alpha_{2i,p_{2i}}\dots\alpha_{2i,1}\alpha_{2,p_2}\dots\alpha_{2,1})\\&=-\lambda_{2i-1}\lambda_{2i}\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})\end{aligned}$$

for $i\in\{2,\dots,k\}$, and the equalities

$$\begin{aligned}\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1})&=-\varphi(\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{1,p_1}\dots\alpha_{1,1})\\&=-\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{j,p_j}\dots\alpha_{j,1})\\&=\lambda_j\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{j,p_j}\dots\alpha_{j,1})\\&=\lambda_j\varphi(\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{2,p_2}\dots\alpha_{2,1})\\&=-\lambda_j^2\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})\end{aligned}$$

for $j\in\{2k+1,\dots,m\}$. Moreover, $\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}$ is a nonzero element of the socle of the indecomposable projective A-module $P_A(0)$ at the central vertex 0 of $Q(\boldsymbol{p},\boldsymbol{T_p})$, and so $\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1})\neq 0$. Hence, we conclude that $\lambda_{2i-1}\lambda_{2i}=\lambda_{2j-1}\lambda_{2j}$ for $i,j\in\{2,\dots,k\}$. In particular, for $k\geq 3$ and $j=2$, we obtain $\lambda_{2i-1}\lambda_{2i}=\lambda_3\lambda_4=\lambda_4$ for $i\in\{3,\dots,k\}$, and so (1) holds.

Assume that $k\geq 2$. Then we have the equalities

- $\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1})=-\lambda_{2k+1}^2\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})$ if $s\geq 1$,

- $\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}) = -\lambda_{2k+2}^2\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})$ if $s\geq 2$,
- $\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}) = -\lambda_{2k+3}^2\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})$ if $s\geq 3$,

and

- $\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}) = -\lambda_{2i-1}\lambda_{2i}\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})$

if $i\in\{2,\dots,k\}$. Hence, we obtain

- $\lambda_4=\lambda_{2k+1}^2$ for $s\geq 1$,
- $\lambda_4=\lambda_{2k+1}^2=\lambda_{2k+2}^2$ for $s\geq 2$,
- $\lambda_4=\lambda_{2k+1}^2=\lambda_{2k+2}^2=\lambda_{2k+3}^2$ for $s\geq 3$.

Because λ_{2k+1}, λ_{2k+2}, λ_{2k+3} are pairwise different, we conclude that $s\in\{0,1,2\}$. Therefore, conditions (2b) and (2c) hold.

Assume now $k=1$ and $s=2$. Then we have in A the relations

- $\alpha_{1,1}\alpha_{2,p_2}=0,\ \alpha_{2,1}\alpha_{1,p_1}=0,\ \alpha_{3,1}\alpha_{3,p_3}=0,\ \alpha_{4,1}\alpha_{4,p_4}=0$,
- $\alpha_{j,p_j}\dots\alpha_{j,1}+\alpha_{1,p_1}\dots\alpha_{1,1}+\lambda_j\alpha_{2,p_2}\dots\alpha_{2,1}=0,\ j\in\{3,4\}$.

Hence, we get the relations

- $\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{1,p_1}\dots\alpha_{1,1}+\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}=0$,
- $\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{2,p_2}\dots\alpha_{2,1}+\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1}=0$,
- $\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{j,p_j}\dots\alpha_{j,1}+\lambda_j\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{j,p_j}\dots\alpha_{j,1}=0$ for $j\in\{3,4\}$.

Applying the property $\varphi(ab)=\varphi(ba)$ for all $a,b\in A$, we obtain the equalities

$$\begin{aligned}\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{1,p_1}\dots\alpha_{1,1}) &= -\varphi(\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{1,p_1}\dots\alpha_{1,1})\\ &= -\varphi(\alpha_{1,p_1}\dots\alpha_{1,1}\alpha_{j,p_j}\dots\alpha_{j,1})\\ &= \lambda_j\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{j,p_j}\dots\alpha_{j,1})\\ &= \lambda_j\varphi(\alpha_{j,p_j}\dots\alpha_{j,1}\alpha_{2,p_2}\dots\alpha_{2,1})\\ &= -\lambda_j^2\varphi(\alpha_{2,p_2}\dots\alpha_{2,1}\alpha_{2,p_2}\dots\alpha_{2,1})\end{aligned}$$

for $j\in\{3,4\}$. Therefore, we conclude that $1=\lambda_3^2=\lambda_4^2$, and hence $\lambda_4=-1$ (because $\lambda_3\neq\lambda_4$).

Conversely, assume that conditions (1) and (2) of (ii) are satisfied. For each vertex x of $Q(\boldsymbol{p},\boldsymbol{T_p})$ fix a maximal path w_x in $Q(\boldsymbol{p},\boldsymbol{T_p})$ with $w_x\notin I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})$. Then the coset $u_x=w_x+I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})$ generates the one-dimensional socle of the indecomposable projective A-module $P_A(x)$ at x. Hence the K-linear form $\varphi: A\to K$ with $\varphi(u_x+I^{(k,s)}(\boldsymbol{p}))=1$ and $\varphi(v+I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda}))=0$ for any path v of $Q(\boldsymbol{p},\boldsymbol{T_p})$ with $v+I^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})\notin\operatorname{soc}A\setminus\{0\}$ satisfies the conditions required for A to be symmetric, and consequently $A=\Lambda^{(k,s)}(\boldsymbol{p},\boldsymbol{T_p},\boldsymbol{\lambda})$ is a symmetric algebra. ∎

We note that the elements $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4, \dots, \lambda_m$ of the projective line $\mathbb{P}_1(K) = K \cup \{\infty\}$ are pairwise different. Hence, the algebra $\Lambda^{(k,2)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda})$ being symmetric forces $\operatorname{char} K \neq 2$.

For an integer $k \geq 2$ and a sequence $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4, \dots, \lambda_{2k}$ of pairwise different elements of $\mathbb{P}_1(K)$ satisfying the equations $\lambda_{2i-1}\lambda_{2i} = \lambda_4$ for $i \in \{3, \dots, k\}$ (in case $k \geq 3$), we have the following symmetric algebras of strictly canonical type:

- $\Gamma^{(k)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda}) = \Lambda^{(k,0)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda})$ for $m = 2k$;
- $\Gamma^{(k,1)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda}, \mu) = \Lambda^{(k,1)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda}^{(1)})$ for $m = 2k+1$, with $\boldsymbol{\lambda}^{(1)} = (\lambda_1, \dots, \lambda_{2k}, \mu)$ for a square root μ of λ_4;
- $\Gamma^{(k,2)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda}, \mu, \xi) = \Lambda^{(k,2)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda}^{(2)})$ for $m = 2k+2$, $\operatorname{char} K \neq 2$, with $\boldsymbol{\lambda}^{(2)} = (\lambda_1, \dots, \lambda_{2k}, \mu, \xi)$ for two different square roots μ, ξ of λ_4.

Moreover, for $k = 1$, we have the following symmetric algebras of strictly canonical type:

- $\Gamma^{(1,1)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}) = \Lambda^{(1,1)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda})$ for $m = 3$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (\infty, 0, 1)$,
- $\Gamma^{(1,2)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}) = \Lambda^{(1,2)}(\boldsymbol{p}, \boldsymbol{T}_{\boldsymbol{p}}, \boldsymbol{\lambda})$ for $m = 4$, $\operatorname{char} K \neq 2$, and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\infty, 0, 1, -1)$.

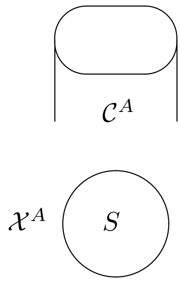
4. Symmetric algebras of strictly canonical type. In this section we complete the proof of the Main Theorem of the paper.

First we recall a general result proved by Ohnuki, Takeda and Yamagata in [27].

THEOREM 4.1. *Let B be a basic, indecomposable algebra, φ a positive automorphism of $\widehat{B}$, and $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$. Then A is a symmetric algebra if and only if $A \cong B$.*

The trivial extension algebras of strictly canonical type have been described in Theorem 2.3. Therefore, in order to complete the description of the symmetric algebras of strictly canonical type, we have to describe the symmetric algebras of the form $A = \widehat{B}/(\psi)$, where B is a branch extension (equivalently, coextension) of a canonical algebra C and ψ is a strictly positive automorphism of $\widehat{B}$ of the form $f\varphi_{\widehat{B}}$, with f a rigid automorphism of $\widehat{B}$ and $\varphi_{\widehat{B}}$ the canonical automorphism of $\widehat{B}$ such that $\varphi_{\widehat{B}}^2 = \nu_{\widehat{B}}$ (see [21, Proposition 5.2]). Moreover, since every symmetric algebra is weakly symmetric, for such $\psi = f\varphi_{\widehat{B}}$ we have $\psi^2(x) = x$ for any object $x = e_{m,i}$ of $\widehat{B}$. In fact, as $A = \widehat{B}/(\psi)$ is symmetric, by Theorem 4.1 we may take ψ with $\psi^2 = \nu_{\widehat{B}}$.

We also note that, by the main result of [21], the Auslander–Reiten quiver Γ_A of such a symmetric algebra $A = \widehat{B}/(\psi)$, $\psi^2 = \nu_{\widehat{B}}$, has the following shape:



where $\mathcal{C}^A = (\mathcal{C}^A(\lambda))_{\lambda\in\mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes with $s(\mathcal{C}^A(\lambda))+p(\mathcal{C}^A(\lambda)) = r(\mathcal{C}^A(\lambda)) - 1$ for each $\lambda \in \mathbb{P}_1(K)$, and $\mathcal{X}^A$ is a family of components, containing exactly one simple module S and exactly one projective module $P(S)$ (the projective cover of S).

The following theorem is a direct consequence of Theorem 4.1 and Propositions 3.1 and 3.2, and completes the proof of the Main Theorem.

THEOREM 4.2. *Let $A = \widehat{B}/(\psi)$ be a selfinjective algebra of strictly canonical type with $\psi^2 = \nu_{\widehat{B}}$. Then A is a symmetric algebra if and only if A is isomorphic to the bound quiver algebra of one of the forms*

(1) $\Lambda(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$ *for* $\operatorname{char} K = 2$;
(2) $\Gamma^{(k)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, $\Gamma^{(k,1,\mu)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda})$, $\Gamma^{(1,1)}(\boldsymbol{p}, \boldsymbol{T_p})$;
(3) $\Gamma^{(k,2)}(\boldsymbol{p}, \boldsymbol{T_p}, \boldsymbol{\lambda}, \mu, \xi)$, $\Gamma^{(1,2)}(\boldsymbol{p}, \boldsymbol{T_p})$ *for* $\operatorname{char} K \neq 2$.

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