COLLOQUIUM MATHEMATICUM

VOL. 100 2004 NO. 1

BESOV SPACES AND 2-SUMMING OPERATORS

BY

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Abstract. Let Π_2 be the operator ideal of all absolutely 2-summing operators and let I_m be the identity map of the m-dimensional linear space. We first establish upper estimates for some mixing norms of I_m. Employing these estimates, we study the embedding operators between Besov function spaces as mixing operators. The result obtained is applied to give sufficient conditions under which certain kinds of integral operators, acting on a Besov function space, belong to Π_2; in this context, we also consider the case of the square $\Pi_2 \circ \Pi_2$.

1. Introduction. Let us start with some preliminaries. For the general theory of operator ideals we refer the reader to the monograph [14].

The class of all (bounded linear) operators between arbitrary Banach spaces is denoted by $\mathcal{L}$, while $\mathcal{L}(E,F)$ stands for the space of those operators acting from E into F, equipped with the usual *operator norm*

$$\|S\| = \|S : E \to F\| := \sup\{\|Sx\| : \|x\| \le 1\}.$$

The set $\mathcal{F}_n(E,F)$ consists of all $S \in \mathcal{L}(E,F)$ such that $S(E) := \{Sx : x \in E\}$ is at most n-dimensional. The dual of E is denoted by E', the value of $a \in E'$ at $x \in E$ by $\langle x, a\rangle$, and the identity map of the m-dimensional linear space by I_m.

In the following, by $[\mathcal{M}_{s,r}, \mu_{s,r}]$ and $[\Pi_{q,p}, \pi_{q,p}]$, with $1 \le r \le s \le \infty$ and $1 \le p \le q \le \infty$, we denote the normed operator ideals of (s,r)-mixing and absolutely (q,p)-summing operators, respectively. For $p = q$ we have the normed operator ideal $[\Pi_p, \pi_p]$ of absolutely p-summing operators. The basic facts related to them are established in [14, Chapters 17 and 20]. Further information is also given in [6] and [7].

For $0 < p, u \le \infty$ the *Lorentz sequence space* $l_{p,u}$ consists of all bounded sequences $x = (\xi_k)$ having a finite quasi-norm

$$\lambda_{p,u}(x) := \begin{cases} \Big(\sum_{n=1}^{\infty} [n^{1/p-1/u} s_n(x)]^u\Big)^{1/u} & \text{if } 0 < u < \infty, \\ \sup_n\, [n^{1/p} s_n(x)] & \text{if } u = \infty, \end{cases}$$

2000 *Mathematics Subject Classification*: 47B10, 47L20, 46E35, 47G10.

where $(s_n(x))$ is the non-increasing rearrangement of x. For $p = u$ we get the classical space of p-summable sequences, denoted by l_p.

If $T \in \mathcal{L}(E,F)$ and $n = 1, 2, \ldots$, then the nth *approximation number* and *Weyl number* are defined by

$$a_n(T) := \inf\{\|T - L\| : L \in \mathcal{F}_{n-1}(E,F)\},$$
$$x_n(T) := \sup\{a_n(TX) : X \in \mathcal{L}(l_2, E),\ \|X\| \le 1\}$$

respectively. We write $T \in \mathcal{L}_{p,u}^{(x)}(E,F)$ if $(x_n(T)) \in l_{p,u}$, and we define

$$L_{p,u}^{(x)}(T) := \lambda_{p,u}(x_n(T)).$$

Then $[\mathcal{L}_{p,u}^{(x)}, L_{p,u}^{(x)}]$ is a quasi-normed operator ideal, introduced by A. Pietsch in [12] (see also [15, Chapter 2]).

If $1 \le p \le \infty$, then the dual exponent p' is determined by $1/p + 1/p' = 1$.

By $c, c_1, c_2, \ldots$ we always denote positive constants, possibly depending on certain exponents or operators, but not on other quantities like natural numbers.

2. Inequalities for mixing norms. First, we have

LEMMA 2.1. *Let $2 < q, s \le \infty$ with $1/2 - 1/s > 1/q$. Let E and F be Banach spaces and let $T \in \mathcal{F}_n(E,F)$ for $n = 1, 2, \ldots$. Then*

$$\mu_{s,2}(T) \le c n^{1/2 - 1/s - 1/q} \pi_{q,2}(T).$$

Proof. If $1/t + 1/s = 1/2$, from [3] we have $\mathcal{L}_{t,1}^{(x)} \subseteq \mathcal{M}_{s,2}$. Combining the above inclusion with well-known inequalities of Lewis type related to Weyl numbers (see [12]), we arrive at

$$\mu_{s,2}(T) \le c_1 L_{t,1}^{(x)}(T) \le c_2 n^{1/t - 1/q} L_{q,\infty}^{(x)}(T) \le c_2 n^{1/t - 1/q} \pi_{q,2}(T)$$

since $0 < t < q \le \infty$ and $L_{q,\infty}^{(x)}(T) \le \pi_{q,2}(T)$, which also follows by [12]. ■

We are now in a position to give

PROPOSITION 2.2. *Let*

$$1/r := \begin{cases} 1/p - 1/q & \text{if } 1 \le p \le q \le 2, \\ 1/p - 1/2 & \text{if } 1 \le p \le 2 \le q \le \infty, \\ 0 & \text{if } 2 \le p \le q \le \infty. \end{cases}$$

(i) *If $2 < s \le \infty$ and $1/2 - 1/s > 1/r$, then*

$$\mu_{s,2}(I_n : l_p^n \to l_q^n) \le c_1 n^{1/2 - 1/s - 1/r}$$

for $n = 1, 2, \ldots$, whenever $2 < r \le \infty$.

(ii) *If $2 \le s \le \infty$ and $1/2 - 1/s \le 1/r$, then*

$$\mu_{s,2}(I_n : l_p^n \to l_q^n) \le c_2$$

for $n = 1, 2, \ldots$.

Proof. (i) By [1] and [2] (see also [15, (1.6.7)]), the embedding operator I from l_p into l_q satisfies $I \in \Pi_{r,2}(l_p, l_q)$. Hence

$$\pi_{r,2}(I_n : l_p^n \to l_q^n) \le c_3 := \pi_{r,2}(I : l_p \to l_q)$$

for $n = 1, 2, \ldots$. Since $2 < r \le \infty$, in view of Lemma 2.1 we have

$$\mu_{s,2}(I_n : l_p^n \to l_q^n) \le c n^{1/2-1/s-1/r} \pi_{r,2}(I_n : l_p^n \to l_q^n) \le c c_3 n^{1/2-1/s-1/r},$$

which is the desired estimate with $c_1 := c c_3$.

(ii) If $1 \le p \le q \le 2$ and $1 \le p \le 2 \le q \le \infty$, from [3] we know that the above embedding I satisfies $I \in \mathcal{M}_{s,2}(l_p, l_q)$, whenever $1/2 - 1/s \le 1/r$. Consequently,

$$\mu_{s,2}(I_n : l_p^n \to l_q^n) \le c_2 := \mu_{s,2}(I : l_p \to l_q)$$

for $n = 1, 2, \ldots$. In the case $2 \le p \le q \le \infty$ we have $0 \le 1/2 - 1/s \le 1/r = 0$, hence $s = 2$ and

$$[\mathcal{M}_{s,2}, \mu_{s,2}] = [\mathcal{L}, \| \ \|],$$

and the inequality follows with $c_2 := 1$. ∎

Remark. By [14, (22.3.7)] we know that the operator ideals $\mathcal{M}_{s,2}$ and $\Pi_{t,2}$, with $1/s + 1/t = 1/2$, have the same limit order. Using [4] (see also [14, (22.6.8)]) for the limit order of $\Pi_{t,2}$, one sees that the estimates given in Proposition 2.2 are the best possible.

3. Besov spaces and mixing operators. Let $-\infty < \sigma < \infty$ and $1 \le p, u \le \infty$. The *Besov sequence space* $b^\sigma_{p,u}$ consists of all scalar sequences $x = (\xi_{m,n})$, with the index set

$$\{(m,n) : m = 0, 1, \ldots;\ n = 1, \ldots, 2^m\}$$

lexicographically ordered, such that the norm

$$\|x\|_{b^\sigma_{p,u}} := \Big(\sum_{m=0}^{\infty} \Big[2^{m\sigma} \Big(\sum_{n=1}^{2^m} |\xi_{m,n}|^p \Big)^{1/p} \Big]^u \Big)^{1/u}$$

is finite; see [13] and [15, (5.4.1)]. In the cases when $p = \infty$ or $u = \infty$ the usual modifications are required.

According to [15, (5.4.1)] we have $b^\sigma_{p,u} := [l_u, 2^{m\sigma} l_p^{2^m}]$, and using [14, (C.4.2)] we obtain $(b^\sigma_{p,u})' = b^{-\sigma}_{p',u'}$ if $-\infty < \sigma < \infty$ and $1 \le p, u < \infty$.

In order to prove the next proposition, an auxiliary result is required.

Lemma 3.1. *Let $-\infty < \sigma, \tau < \infty$, $1 \le p, q, u, v \le \infty$ and $\sigma - \tau > \max(1/q - 1/p, 0)$. Let $[\mathcal{A}, A]$ be a normed operator ideal. Assume there exist constants $c, \alpha \ge 0$ such that $\sigma - \tau > \alpha$ and*

$$A(I_{2^m} : l_p^{2^m} \to l_q^{2^m}) \le c 2^{m\alpha}$$

for $m = 0, 1, 2, \ldots$. *Then* $I \in \mathcal{A}(b^{\sigma}_{p,u}, b^{\tau}_{q,v})$, *where* I *is the natural embedding from* $b^{\sigma}_{p,u}$ *into* $b^{\tau}_{q,v}$.

Proof. We consider the canonical operators $J_{2^m} \in \mathcal{L}(l_q^{2^m}, b^{\tau}_{q,v})$ and $Q_{2^m} \in \mathcal{L}(b^{\sigma}_{p,u}, l_p^{2^m})$ defined by

$$J_{2^m}(\xi_1, \ldots, \xi_{2^m}) := (0; \ldots; 0, \ldots, 0; \xi_1, \ldots, \xi_{2^m}; 0, \ldots, 0; \ldots),$$
$$Q_{2^m}(\xi_{0,1}; \ldots; \xi_{m,1}, \ldots, \xi_{m,2^m}; \ldots) := (\xi_{m,1}, \ldots, \xi_{m,2^m}).$$

Then $\|J_{2^m}\| = 2^{m\tau}$ and $\|Q_{2^m}\| = 2^{-m\sigma}$. Hence

$$\sum_{m=0}^{\infty} A(J_{2^m} I_{2^m} Q_{2^m}) \le \sum_{m=0}^{\infty} \|J_{2^m}\|\, A(I_{2^m})\, \|Q_{2^m}\| \le c \sum_{m=0}^{\infty} 2^{m(\alpha+\tau-\sigma)} < \infty.$$

Therefore $\sum_{m=0}^{\infty} J_{2^m} I_{2^m} Q_{2^m}$ is convergent in the Banach space $\mathcal{A}(b^{\sigma}_{p,u}, b^{\tau}_{q,v})$ and since $I = \sum_{m=0}^{\infty} J_{2^m} I_{2^m} Q_{2^m}$ in $\mathcal{L}(b^{\sigma}_{p,u}, b^{\tau}_{q,v})$, it follows that

$$I \in \mathcal{A}(b^{\sigma}_{p,u}, b^{\tau}_{q,v}). \ \blacksquare$$

Let $\sigma > 0$ and $1 \le p, u \le \infty$. The Besov function space $[B^{\sigma}_{p,u}(0,1), E]$ consists of certain E-valued functions defined on the unit interval $[0,1]$ (see [15, (6.4)]). If E is the scalar field, then we simply write $B^{\sigma}_{p,u}(0,1)$.

For $m > \sigma + 1 - 1/p$, the Ciesielski transform, denoted by C_m, establishes an isomorphism between

$$B^{\sigma}_{p,u}(0,1) \quad \text{and} \quad l_p^m \oplus b_{p,u}^{\sigma-1/p+1/2}.$$

Further information is also given in [15, (6.4)], where the original papers [5] and [16] with the complete proof of this deep result are quoted.

For the embedding operator I_B from $B^{\sigma}_{p,u}(0,1)$ into $B^{\tau}_{q,v}(0,1)'$, which exists if $\sigma + \tau > 1/p + 1/q - 1$, we state

PROPOSITION 3.2. *Let* $\sigma, \tau > 0$, $1 \le p, u \le \infty$ *and* $1 \le q, v < \infty$. *Let*

$$1/t := \begin{cases} 1/p - 1/q' & \text{if } 1 \le p \le q' \le 2, \\ 1/p - 1/2 & \text{if } 1 \le p \le 2 \le q' \le \infty, \\ 0 & \text{if } 2 \le p \le q' \le \infty. \end{cases}$$

Consider the following two cases:

(i) $2 < s \le \infty$, $2 < t \le \infty$, $1/2 - 1/s > 1/t$ *and*

$$\sigma + \tau - 1/p - 1/q + 1 > 1/2 - 1/s - 1/t.$$

(ii) $2 \le s \le \infty$, $1/2 - 1/s \le 1/t$ *and*

$$\sigma + \tau - 1/p - 1/q + 1 > 0.$$

If either (i) *or* (ii) *is satisfied, then*

$$I_B \in \mathcal{M}_{s,2}(B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)').$$

Proof. In (i) and (ii) we have $\sigma + \tau > 1/p + 1/q - 1$. Let $m > \max(\sigma + 1 - 1/p, \tau + 1 - 1/q)$. From [15, (6.4.13)] the embedding I_B is related to embedding operators I_m and I_b acting between sequence spaces by

$$\begin{array}{ccc} B^{\sigma}_{p,u}(0,1) & \xrightarrow{\ I_B\ } & B^{\tau}_{q,v}(0,1)' \\ {\scriptstyle C_m}\big\downarrow & & \big\uparrow{\scriptstyle C'_m} \\ l^m_p \oplus b^{\sigma-1/p+1/2}_{p,u} & \xrightarrow{\ I_m \oplus I_b\ } & (l^m_q)' \oplus (b^{\tau-1/q+1/2}_{q,v})' \end{array}$$

(i) In this case, by Proposition 2.2(i) we have

$$\mu_{s,2}(I_{2^m} : l^{2^m}_p \to l^{2^m}_{q'}) \le c_1 2^{m\alpha}$$

for $m = 0, 1, 2, \ldots$, with $\alpha := 1/2 - 1/s - 1/t$. We have $(b^{\tau-1/q+1/2}_{q,v})' = b^{-\tau+1/q-1/2}_{q',v'}$, and from Lemma 3.1 it follows that

$$J_b \in \mathcal{M}_{s,2}(b^{\sigma-1/p+1/2}_{p,u}, b^{-\tau+1/q-1/2}_{q',v'}),$$

where J_b is the natural embedding from $b^{\sigma-1/p+1/2}_{p,u}$ into $b^{-\tau+1/q-1/2}_{q',v'}$. Hence, in view of the above diagram we obtain

$$I_B \in \mathcal{M}_{s,2}(B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)').$$

(ii) Now, it follows from Proposition 2.2(ii) that

$$\mu_{s,2}(I_{2^m} : l^{2^m}_p \to l^{2^m}_{q'}) \le c_2$$

for $m = 0, 1, 2, \ldots$, and by Lemma 3.1 for the embedding J_b we get

$$J_b \in \mathcal{M}_{s,2}(b^{\sigma-1/p+1/2}_{p,u}, b^{-\tau+1/q-1/2}_{q',v'}).$$

Thus, the preceding diagram also yields

$$I_B \in \mathcal{M}_{s,2}(B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)'). \ \blacksquare$$

4. Integral operators, Besov spaces and Π_2. A kernel K defined on the unit square $[0,1] \times [0,1]$ belongs to

$$[B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)]$$

if the function-valued function

$$K_X : \xi \to K(\xi, \cdot)$$

belongs to $[B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)]$.

We observe that the above type of kernel was introduced by A. Pietsch in [13] (see also [15, (6.4.17)]) in order to establish an important result concerning the distribution of eigenvalues of integral operators.

We formulate

Theorem 4.1. *Let* $\sigma,\tau>0$, $1\le p,u\le\infty$ *and* $1\le q,v<\infty$. *Let* $s:=\max(p,u)$ *and*

$$1/t:=\begin{cases}1/p-1/q' & \text{if } 1\le p\le q'\le 2,\\ 1/p-1/2 & \text{if } 1\le p\le 2\le q'\le\infty,\\ 0 & \text{if } 2\le p\le q'\le\infty.\end{cases}$$

Consider the following two cases:

(i) $2<s\le\infty$, $2<t\le\infty$, $1/2-1/s>1/t$ *and*

$$\sigma+\tau-1/p-1/q+1>1/2-1/s-1/t.$$

(ii) $2\le s\le\infty$, $1/2-1/s\le 1/t$ *and*

$$\sigma+\tau-1/p-1/q+1>0.$$

Suppose that either (i) *or* (ii) *is satisfied. If* $K\in[B^{\sigma}_{p,u}(0,1),B^{\tau}_{q,v}(0,1)]$, *then*

$$T_K: f(\eta)\mapsto\int_0^1 K(\xi,\eta)f(\eta)\,d\eta$$

satisfies $T_K\in\Pi_2(B^{\sigma}_{p,u}(0,1),B^{\sigma}_{p,u}(0,1))$.

Proof. The operator T_K admits the factorization $T_K=S_KI_B$:

$$T_K: B^{\sigma}_{p,u}(0,1)\xrightarrow{I_B}B^{\tau}_{q,v}(0,1)'\xrightarrow{S_K}B^{\sigma}_{p,u}(0,1),$$

where $S_K(a):=\langle K_X(\cdot),a\rangle$, and from [15, (6.4.16)] we get

$$S_K\in\Pi_s(B^{\tau}_{q,v}(0,1)',B^{\sigma}_{p,u}(0,1)).$$

Applying Proposition 3.2, in both cases (i) and (ii), we obtain

$$I_B\in\mathcal{M}_{s,2}(B^{\sigma}_{p,u}(0,1),B^{\tau}_{q,v}(0,1)').$$

Now the formula

$$[\Pi_s,\pi_s]\circ[\mathcal{M}_{s,2},\mu_{s,2}]\subseteq[\Pi_2,\pi_2]$$

(see [14, (20.2.1)]) completes the proof. ■

A variant of the above result is

Theorem 4.2. *Let* $\sigma,\tau>0$, $1\le p,u\le\infty$ *and* $1\le q,v<\infty$ *be such that* $q'\le p$. *Let* $s:=\max(q',u)$. *Suppose that* $2<s\le\infty$ *and*

$$\sigma+\tau-1/p-1/q+1/2+1/s>0.$$

If $K\in[B^{\sigma}_{p,u}(0,1),B^{\tau}_{q,v}(0,1)]$, *then*

$$T_K: f(\eta)\mapsto\int_0^1 K(\xi,\eta)f(\eta)\,d\eta$$

satisfies $T_K\in\Pi_2(B^{\sigma}_{q',u}(0,1),B^{\sigma}_{q',u}(0,1))$.

Proof. Since $1 < q' \le p \le \infty$, from [15, (6.4.4)] we have the (obvious) inclusion

$$[B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)] \subseteq [B^{\sigma}_{q',u}(0,1), B^{\tau}_{q,v}(0,1)],$$

hence $K \in [B^{\sigma}_{q',u}(0,1), B^{\tau}_{q,v}(0,1)]$. Moreover, if $\alpha := 1/2 - 1/s$ then $\sigma + \tau - 1/p - 1/q + 1 > \alpha$, and Theorem 4.1(i) yields the assertion. ∎

Let $(\Pi_2)^2$ be the square $\Pi_2 \circ \Pi_2$ (see [14, (7.1)]). Finally, we get

THEOREM 4.3. *Let $\sigma, \tau > 0$, $1 \le p, u \le 2$ and $1 \le q, v < \infty$. Let*

$$\beta := \begin{cases} 1/2 - 1/p + 1/q' & \text{if } 1 \le q' \le 2, \\ 1 - 1/p & \text{if } 2 \le q' \le \infty. \end{cases}$$

Suppose that $\sigma + \tau - 1/p - 1/q + 1 > \beta$. If $K \in [B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)]$, then

$$T_K : f(\eta) \mapsto \int_0^1 K(\xi,\eta) f(\eta)\, d\eta$$

satisfies $T_K \in (\Pi_2)^2(B^{\sigma}_{p,u}(0,1), B^{\sigma}_{p,u}(0,1))$.

Proof. From [14, (22.4.9)] we have

$$\pi_2(I_{2^m} : l_p^{2^m} \to l_{q'}^{2^m}) = 2^{m\beta}$$

for $m = 0, 1, 2, \ldots$. Therefore, by Lemma 3.1 the natural embedding J_b from $b^{\sigma-1/p+1/2}_{p,u}$ into $b^{-\tau+1/q-1/2}_{q',v'}$ satisfies

$$J_b \in \Pi_2(b^{\sigma-1/p+1/2}_{p,u}, b^{-\tau+1/q-1/2}_{q',v'}).$$

Using the diagram given in the proof of Proposition 3.2 we now obtain

$$I_B \in \Pi_2(B^{\sigma}_{p,u}(0,1), B^{\tau}_{q,v}(0,1)').$$

It remains to recall the factorization $T_K = S_K I_B$ given in the proof of Theorem 4.1, with $S_K \in \Pi_2(B^{\tau}_{q,v}(0,1)', B^{\sigma}_{p,u}(0,1))$. ∎

REMARKS. (i) We recall two important properties of $(\Pi_2)^2$: (a) as proved by H. König [10] (see also [11, (4.a.6)] and [15, (4.2.30)]) this operator ideal admits a spectral trace, and (b) every $(\Pi_2)^2$-operator is nuclear (see [14, (24.6.5)]).

(ii) In [8] and [9] sufficient conditions for kernels of Besov type to generate operators belonging to the ideals Π_1 and $(\Pi_2)^{(a)}_{2,1}$ respectively are established. Here, $(\Pi_2)^{(a)}_{2,1}$ denotes the collection of all operators whose approximation numbers with respect to the 2-summing norm are in $l_{2,1}$; this operator ideal has the above properties (a) and (b).

(iii) There is a translation of the previous results from the continuous into the discrete case: one can obtain the corresponding results for matrix operators of Besov type. For further information on these matrix operators and operator ideals different from Π_2 and $(\Pi_2)^2$, one can see [8], [9], [13] and [15, (5.4)].

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Received 31 October 2001 (4127)