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ON CONVOLUTION SQUARES OF SINGULAR MEASURES

BY

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Abstract. We prove that for every compact, connected group G there is a singular measure μ such that the Fourier series of $\mu * \mu$ converges uniformly on G. Our results extend the earlier results of Saeki and Dooley–Gupta.

1. Introduction. A classical result of Wiener and Wintner [12] (see also [11]) asserts that there exists a singular measure μ on the circle group T such that $\widehat{\mu}(n) = o(|n|^{-1/2+\varepsilon})$ as $n \to \infty$ for every $\varepsilon > 0$. Such a measure μ has the following *property* (P): $\mu^2 = \mu * \mu$ is absolutely continuous and its Radon–Nikodym derivative with respect to Lebesgue measure belongs to $L^p(T)$ for all real numbers $p \geq 1$. Measures with property (P) have been constructed by Hewitt and Zuckerman [6] on any nondiscrete, locally compact, abelian group. Property (P) has also been explored in [1], [2], [6], [7], [9], [10].

Saeki [10] showed that there exists a singular measure μ on the circle group such that $\mu * \mu$ is absolutely continuous and its Radon–Nikodym derivative with respect to Lebesgue measure has a uniformly convergent Fourier series. Such a measure μ clearly has property (P).

In [2], a central, singular, continuous measure μ having property (P) was constructed on any infinite, compact, connected group or infinite, compact Lie group. The authors mentioned that they were unable to construct a singular measure μ such that $\mu * \mu$ has a uniformly convergent Fourier series.

In this article, we construct a singular measure μ on any infinite, compact, connected group or infinite, compact Lie group, with $\mu * \mu$ having a uniformly convergent Fourier series. The proof is similar to Saeki's [10] and we will only provide details of the steps which are different.

2. Main result. Let G be a compact group and $\widehat{G}$ its dual. Let m denote the Haar measure on G, normalized so that $m(G) = 1$. Given $f \in L^1(G)$,

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the Fourier series of f is:

$$f \sim \sum_{\sigma \in \widehat{G}} d_\sigma \operatorname{Tr}(\widehat{f}(\sigma) U^\sigma(x)).$$

For unexplained notation see [4], [5].

Our main results can be stated as follows:

THEOREM 2.1. *Let G be an infinite, compact Lie group and let K be a measurable subset of G having positive measure. Then there exists a singular probability measure μ on G satisfying*:

(i) $\operatorname{supp}(\mu) \subset K$ *and* $m(\operatorname{supp}(\mu)) = 0$;
(ii) *the Fourier series of* $\mu * \mu \equiv \mu^2$ *converges uniformly.*

COROLLARY 2.2. *Let G be an infinite, compact, connected group. Then there is a singular probability measure μ on G such that the Fourier series of $\mu * \mu$ converges uniformly.*

The corollary will be seen to follow from the theorem by an application of the structure theorem for compact, connected groups.

In order to prove the theorem we need some notation and lemmas: For $f \in C(G)$, we consider the norms given by

$$\|f\|_A = \sum_{\sigma \in \widehat{G}} d_\sigma \|\widehat{f}(\sigma)\|_1,$$

$$\|f\|_u = \sup\Big\{\Big\|\sum_{\sigma \in F} d_\sigma \operatorname{Tr}(\widehat{f}(\sigma) U^\sigma(x))\Big\|_\infty : F \text{ is a finite subset of } \widehat{G}\Big\}.$$

Recall that the set of all $f \in C(G)$ with $\|f\|_A < \infty$ (respectively, $\|f\|_u < \infty$) forms a Banach space. For $f \in L^1(G)$, let $f^2 = f * f$ and let $\operatorname{supp}(f)$ denote the closed support of f.

In the lemmas which follow, G will be a compact Lie group and $\mathfrak{g}$ will denote its Lie algebra. Assume the dimension of $\mathfrak{g}$ is n and identify $\mathfrak{g} \simeq \mathbb{R}^n$ as a vector space. We will call $A \subset \mathfrak{g}$ an *n-cell* if under this identification

$$A = \{(x_1, \ldots, x_n) : a \le x_i \le b,\, i = 1, \ldots, n\}$$

for some real numbers a, b. We denote the exponential of A by $\exp(A)$.

We begin with a technical lemma which is needed to prove an analogue of Lemma 3 of [10] for a compact Lie group G.

LEMMA 2.3. *Suppose $A \subset \mathfrak{g}$ is an n-cell and that the exponential map is injective on $A + A$. Let $f \in L^2(G)$ be such that* $\operatorname{supp}(f) \subset \exp(A)$ *and assume $\varepsilon > 0$ is given. Then there exists an integer q and pairwise disjoint sets $B_j \subset G$ for $j = 1, \ldots, q$, such that* $\operatorname{supp}(f) \subset \bigcup_{j=1}^q B_j$, $B_i B_i \cap B_j B_j = \emptyset$

if $i \neq j$, and

$$\int_{B_j} |f|^2\,dx < \varepsilon, \qquad i = 1, \dots, q.$$

Proof. Choose δ_1 such that $\int_B |f|^2\,dx < \varepsilon$ whenever $m(B) < \delta_1$ and choose δ_2 such that if $A' \subset A$ and $m(\exp(A')) < \delta_2$, then $m(A') < \delta_1$ (see [3]). Assume $A = \{(x_1, \dots, x_n) : a \leq x_i \leq b\}$ and suppose $p \in \mathbb{N}$ is given. Set $\tau = (b-a)/p$. For $j_1, \dots, j_n \in \{1, \dots, p\}$, define $A_{j_1,\dots,j_n}$ as the product of intervals

$$[a + (j_1 - 1)\tau, a + j_1\tau) \times \dots \times [a + (j_n - 1)\tau, a + j_n\tau).$$

Choose p such that $m(A_{j_1,\dots,j_n}) < \delta_2$ for all $j_1, \dots, j_n$.

The sets $A_{j_1,\dots,j_n}$ form a disjoint decomposition of A and if $(j_1, \dots, j_n) \neq (k_1, \dots, k_n)$, then

$$(A_{j_1,\dots,j_n} + A_{j_1,\dots,j_n}) \cap (A_{k_1,\dots,k_n} + A_{k_1,\dots,k_n}) = \emptyset.$$

Set $B_{j_1,\dots,j_n} = \exp(A_{j_1,\dots,j_n})$. The choice of δ_2 ensures that $m(B_{j_1,\dots,j_n}) < \delta_1$, hence for all $j_1, \dots, j_n \in \{1, \dots, p\}$,

$$\int_{B_{j_1,\dots,j_n}} |f|^2\,dx < \varepsilon.$$

By [3],

$$B_{j_1,\dots,j_n} B_{j_1,\dots,j_n} \subset \exp(A_{j_1,\dots,j_n} + A_{j_1,\dots,j_n}),$$

and as the exponential map is injective on $A + A$, the sets $B_{j_1,\dots,j_n}$ have the required properties. ∎

Next we will state three lemmas which are analogues of Lemmas 1–3 of Saeki [10]. We will not provide any details of the proofs of the first two lemmas as these are similar to Saeki's proofs. The proof of the third lemma is also similar to Saeki's, but there are some technical differences which we think are worth presenting.

Lemma 2.4. *Given $g \in L^1_+(G)$ and $\varepsilon > 0$, there exists a simple function $h \in L^1_+(G)$ such that*

(i) $\|h\|_1 = \|g\|_1$ *and* $\|\widehat{g} - \widehat{h}\|_\infty < \varepsilon$;
(ii) $\mathrm{supp}(h) \subset \{g \neq 0\}$ *and* $m(\mathrm{supp}(h)) \leq 2^{-1} m(\{g \neq 0\})$;
(iii) $h \leq 2g$ *on* G.

Lemma 2.5. *Let $f_1, \dots, f_p, g \in L^2_+(G)$ and suppose $\delta > 0$ is given. There exists a simple function $h \in L^2_+(G)$ satisfying $\|(g - h) * f_j\|_A < \delta$ and $\|f_j * (g - h)\|_A < \delta$ for all $j_1, \dots, j_n \in \{1, \dots, p\}$, as well as the three properties of the lemma above.*

Lemma 2.6. *Let* $f \in L^2_+(G)$ *with* $\|f\|_1 = 1$ *and assume that* $\operatorname{supp}(f) \subset \exp(A)$ *where* A *is an* n*-cell and the exponential map is injective on* $A + A$*. Let* $\varepsilon > 0$ *be given. There exists a simple function* $g \in L^1_+(G)$ *such that*

(i) $\|g\|_1 = 1$ *and* $\|\widehat{f} - \widehat{g}\|_\infty < \varepsilon$;
(ii) $\operatorname{supp}(g) \subset \{f \neq 0\}$ *and* $m(\operatorname{supp}(g)) \leq 2^{-1} m(\{f \neq 0\})$;
(iii) $\|f * f - g * g\|_u < 16\varepsilon$.

Proof. We begin by applying Lemma 2.3 to obtain pairwise disjoint sets B_j, $j = 1, \dots, q$, such that $\operatorname{supp}(f) \subset \bigcup_{j=1}^q B_j$, $B_i B_i \cap B_j B_j = \emptyset$ and $\int_{B_j} |f|^2 \, dx < \varepsilon$. Let $g_j = f|_{B_j}$.

As the dual of a compact Lie group is countable we may enumerate $\widehat{G}$ as $\{\sigma_n\}_{n=1}^\infty$. Choose N_0 such that for all $j = 1, \dots, q$,

$$\sum_{n \geq N_0} d_{\sigma_n} \operatorname{Tr}(|\widehat{g}_j(\sigma_n)|^2) < \varepsilon / q. \tag{2.1}$$

Inductively apply Lemma 2.5 to obtain simple functions $h_1, \dots, h_q$ and an increasing sequence of positive integers $N_1, \dots, N_q$ such that for each $j = 1, \dots, q$ the following properties are satisfied:

(a) $\|h_j\|_1 = \|g_j\|_1$ and $\|\widehat{g}_j - \widehat{h}_j\|_\infty < \varepsilon / (q N_{j-1} \max\{d_{\sigma_k}^2 : 1 \leq k \leq N_{j-1}\})$;
(b) $\operatorname{supp}(h_j) \subset \{g_j \neq 0\}$ and $m(\operatorname{supp}(h_j)) \leq 2^{-1} m(\{g_j \neq 0\})$;
(c) $h_j \leq 2 g_j$;
(d) for $k = 1, \dots, j-1$,

$$\|(g_j - h_j) * h_k\|_A < \varepsilon / 4q^2, \qquad \|h_k * (g_j - h_j)\|_A < \varepsilon / 4q^2,$$

and for $k = j+1, \dots, q$,

$$\|(g_j - h_j) * g_k\|_A < \varepsilon / 4q^2, \qquad \|g_k * (g_j - h_j)\|_A < \varepsilon / 4q^2;$$

(e) $\sum_{n \geq N_j} d_{\sigma_n} \operatorname{Tr}(|\widehat{h}_j(\sigma_n)|^2) < \varepsilon / q$.

For $j = 0, \dots, q$, set $f_j = (h_1 + \cdots + h_j) + (g_{j+1} + \cdots + g_q)$. As $f \subset \bigcup_{j=1}^q B_j$, we have $f_0 = f = g_1 + \cdots + g_q$. Put $g = f_q = \sum_{i=1}^q h_i$. We will only give the proof of (iii) as (i) and (ii) are easy to verify.

Certainly

$$f * f - g * g = \sum_{j=1}^q \{f_{j-1} * f_{j-1} - f_j * f_j\}$$

and

$$\begin{aligned} f_{j-1} * f_{j-1} - f_j * f_j = {} & [(h_1 + \cdots + h_{j-1}) + (g_j + \cdots + g_q)]^2 \\ & - [(h_1 + \cdots + h_j) + (g_{j+1} + \cdots + g_q)]^2 \end{aligned}$$

$$
\begin{aligned}
&= g_j * g_j - h_j * h_j + (h_1 + \cdots + h_{j-1}) * (g_j - h_j) \\
&\quad + (g_j - h_j) * (h_1 + \cdots + h_{j-1}) \\
&\quad + (g_{j+1} + \cdots + g_q) * (g_j - h_j) + (g_j - h_j) * (g_{j+1} + \cdots + g_q).
\end{aligned}
$$

Set $R_j = f_{j-1}^2 - f_j^2 - (g_j^2 - h_j^2)$.

By (d),

$$\|R_j\|_A \le 2\varepsilon(j-1)/4q^2 + 2\varepsilon(q-j)/4q^2 \le \varepsilon/q,$$

and since the u-norm is majorized by the A-norm we have

$$
\begin{aligned}
\|f * f - g * g\|_u &= \Big\| \sum_{j=1}^{q} (f_{j-1} * f_{j-1} - f_j * f_j) \Big\|_u \\
&\le \Big\| \sum_{j=1}^{q} (g_j * g_j - h_j * h_j) \Big\|_u + \Big\| \sum_{j=1}^{q} R_j \Big\|_u \\
&\le \Big\| \sum_{j=1}^{q} (g_j * g_j - h_j * h_j) \Big\|_u + \varepsilon.
\end{aligned}
$$

If we set

$$M_N = \Big\| \sum_{j=1}^{q} \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}((\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2) U^{\sigma_n}(x)) \Big\|_\infty,$$

then

$$\Big\| \sum_{j=1}^{q} (g_j * g_j - h_j * h_j) \Big\|_u = \sup_N M_N.$$

Thus it will suffice to show that $M_N \le 15\varepsilon$ for each integer N.

We first claim that

$$\Big\| \sum_{j=1}^{k} (g_j^2 - h_j^2) \Big\|_\infty \le 5\varepsilon \quad \text{for } k = 1, \dots, q. \tag{2.2}$$

Indeed, as

$$\operatorname{supp}(g_j * g_j - h_j * h_j) \subset B_j B_j$$

and $B_j B_j \cap B_i B_i = \emptyset$ if $i \ne j$, it follows that

$$
\begin{aligned}
\sup_x \Big| \sum_{j=1}^{k} (g_j * g_j - h_j * h_j)(x) \Big| &= \sup_x \{ |(g_j * g_j - h_j * h_j)(x)| : j = 1, \dots, k \} \\
&\le \sup_{1 \le j \le q} (\|g_j\|_2^2 + \|h_j\|_2^2) \\
&\le 5 \sup\{ \|g_j\|_2^2 : j = 1, \dots, q \} \le 5\varepsilon,
\end{aligned}
$$

proving the claim.

Fix $N \in \mathbb{N}$ and observe that

$$M_N \leq \sum_{j=1}^{q} \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}(|\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2|).$$

If A and B are $d \times d$ matrices, then by writing $A = B + (A - B)$ one can see that

$$\operatorname{Tr}(|A^2 - B^2|) \leq d\|A^2 - B^2\|_\infty \leq d(2\|B\|_\infty \|A - B\|_\infty + \|A - B\|_\infty^2).$$

Consequently, if $N \leq N_0$, and $n \in \{1, \ldots, N\}$, then it follows from (a) that

$$\operatorname{Tr}(|\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2|) \leq d_{\sigma_n} 3\varepsilon/(qN_{j-1} \max\{d_{\sigma_k}^2 : k = 1, \ldots, N_{j-1}\}).$$

Therefore $M_N \leq 3\varepsilon$.

If $N_{k-1} < N \leq N_k$ for some $k = 1, \ldots, q$, then

$$M_N \leq P + Q + R$$

where

$$P = \sup_x \Big| \sum_{j=1}^{k-1} \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}(\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2) U^{\sigma_n}(x) \Big|,$$

$$Q = \sup_x \Big| \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}(\widehat{g}_k(\sigma_n)^2 - \widehat{h}_k(\sigma_n)^2) U^{\sigma_n}(x) \Big|,$$

$$R = \sup_x \Big| \sum_{j=k+1}^{N} \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}(\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2) U^{\sigma_n}(x) \Big|.$$

Since $h_j \leq 2g_j$ and $\int g_j^2 = \int_{B_j} f^2 \leq \varepsilon$,

$$Q \leq \sum_{n=1}^{N} d_{\sigma_n} \operatorname{Tr}(|\widehat{g}_k(\sigma_n)^2| + |\widehat{h}_k(\sigma_n)^2|) \leq \|g_k\|_2^2 + \|h_k\|_2^2 \leq 5\|g_k\|_2^2 \leq 5\varepsilon.$$

Similar arguments to those used for $N \leq N_0$ also show that $R \leq 3\varepsilon$.

Finally, we remark that P is dominated by

$$\begin{aligned}
\Big\| \sum_{j=1}^{k-1} (g_j^2 - h_j^2)(x) \Big\|_\infty + \Big\| \sum_{j=1}^{k-1} \sum_{n=N+1}^{\infty} d_{\sigma_n} \operatorname{Tr}(\widehat{g}_j(\sigma_n)^2 - \widehat{h}_j(\sigma_n)^2) U^{\sigma_n}(x) \Big\|_\infty \\
\leq 5\varepsilon + \sum_{j=1}^{k-1} \sum_{n=N+1}^{\infty} d_{\sigma_n} \operatorname{Tr}(|\widehat{g}_k(\sigma_n)^2| + |\widehat{h}_k(\sigma_n)^2|) \\
\leq 5\varepsilon + (k-1)(\varepsilon/q + \varepsilon/q) \leq 7\varepsilon
\end{aligned}$$

by the choice of N_0 (see (2.1), property (e) and (2.2)). Thus, $M_N \leq 15\varepsilon$ for $N_{k-1} < N \leq N_k$.

If $N > N_m$ then we simply apply the argument used to bound P but taking $k-1=q$. This allows us to conclude that $M_N \le 7\varepsilon$ and completes the proof. ∎

The proof of Theorem 2.1 now follows from the lemmas in the same manner as in [10].

The corollary for general compact, connected groups is based on the following structure theorem which can be found in [8].

Structure theorem for compact, connected groups. *Let T be a compact, connected abelian group, $\{G_\alpha\}_{\alpha\in I}$ a family of compact, simply connected simple Lie groups and K a finite subgroup of the centre of $T\times\prod_{\alpha\in I}G_\alpha$. Then*

$$G = T\times\prod_{\alpha\in I}G_\alpha/K$$

is a compact, connected group and every compact, connected group is of this form.

Proof of Corollary 2.2. Given a compact, connected group G, let

$$G = T\times\prod_{\alpha\in I}G_\alpha/K,$$

where T, I, G_α and K are as in the structure theorem above. If I is an empty set, then the result follows from [10]. Otherwise choose one of the factors G_β and set $H_\beta = P(K)$, where P is the projection of $G = T\times\prod_{\alpha\in I}G_\alpha$ onto G_β. As G_β/H_β is an infinite, compact Lie group, there is a singular measure μ on G_β/H_β with the Fourier series of $\mu*\mu$ converging uniformly. Define $\mu^\sim$ on G_β by

$$\int_G f\,d\mu^\sim = \int_{G_\beta/H_\beta} F(\overline{x})\,d\mu(\overline{x})$$

where $F(\overline{x}) = \{\sum f(y) : y\in G_\beta,\ \overline{y}=\overline{x}\}$.

If μ is concentrated on E, then $\mu^\sim$ is concentrated on $\pi_\beta^{-1}(E)$ (where $\pi_\beta : G_\beta\to G_\beta/H_\beta$ is the quotient map) and hence is singular.

It is easily seen that $\widehat{\mu^\sim}(\sigma)=0$ if $\sigma\in\widehat{G_\beta}$ and $\sigma|_{H_\beta}\ne 1$. Thus the uniform convergence of the Fourier series of $\mu*\mu$ is inherited by $\mu^\sim*\mu^\sim$.

Let $\pi : T\times\prod_{\alpha\in I}G_\alpha \to G$ denote the quotient map. Consider $\nu = \mu^\sim\times m'$, where m' is the Haar measure on $\prod_{\alpha\ne\beta}G_\alpha\times T$, and let $\pi(\nu)$ be the measure on G given by

$$\int_G f\,d\pi(\nu) = \int_{\prod G_\alpha\times T} f\circ\pi\,d\nu.$$

The singularity of $\pi(\nu)$ comes from the singularity of ν and finiteness of K. The Fourier series of $\pi(\nu)*\pi(\nu)$ takes on the same values as that of $\mu^\sim*\mu^\sim$ and hence converges uniformly. ∎

Remark 2.1. Even on SU(2), we are unable to construct a central singular measure μ such that $\mu * \mu$ has a uniformly convergent Fourier series. A related problem is to determine if there is a singular measure μ on $\mathbb{T}$ which satisfies $\mu(-E) = \mu(E)$ with $\mu * \mu$ having a uniformly convergent Fourier series.

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